International Journal of Analysis and Applications

On Certain Fixed Point Theorems in S_b-Metric Spaces With Applications

N. Mangapathi[1,3,*], **B. Srinuvasa Rao**[2], **K.R.K. Rao**[3], **M.I. Pasha**[1,3]

[1]*Department of Mathematics, B V Raju Institute of Technology, Narsapur, Medak-502313, Telangana, India*

[2]*Department of Mathematics, Dr.B.R.Ambedkar University, Srikakulam, Etcherla-532410, Andhra Pradesh, India*

[3]*Department of Mathematics, Gitam School of Science, GITAM Deemed to be University, Hyderabad, Rudraram-502329, Telangana, India*

[*]*Corresponding author:* nmp.maths@gmail.com

Abstract. In this paper, we introduce the notion of generalized (α, ϕ, ψ)- Geraghty contractive type mappings in the setup of S_b-metric spaces and α-orbital admissible mappings with respect to ϕ. Furthermore, the fixed-point theorems for such mappings in complete S_b-metric spaces are proven without assuming the subadditivity of ψ. Some examples are provided for supporting of our main results. Also, we gave an application to integral equations as well as Homotopy.

1. Introduction

The Banach contraction principle [1] is one of the most significant findings in fixed point theory since it has applications in many areas of mathematics and mathematical sciences. By combining the ideas of S and b-metric spaces, Sedghi et al. [2] created S_b-metric spaces and established common fixed point outcomes in S_b- metric spaces. In order to improve, numerous authors developed numerous findings in S_b-metric spaces (see e.g. [3], [4], [5], [6], [7], [8]).

One of the more intriguing findings is Geraghty's [9] generalisation of the Banach contraction theorem. Multiple researchers have since studied this type of research in different metric spaces (see e.g [10], [11], [12], [13], [14], [15], [16], [17], [18]).

Received: Dec. 9, 2022.

2020 *Mathematics Subject Classification.* 54H25, 47H10, 54E50.

Key words and phrases. (α, ϕ, ψ)-Geraghty type contraction mappings; S_b-metric spaces; α-orbital admissible mappings with respect to ϕ and completeness.

https://doi.org/10.28924/2291-8639-21-2023-18
ISSN: 2291-8639

Triangular α-admissible mappings are a novel idea that Karapinar et al. [15] introduced to study fixed points for such mappings in metric spaces. Three new concepts triangular α-orbital admissible, α -orbital admissible and α-orbital attractive mappings were developed by Popescu [17] in 2014. The idea of triangular α-orbital admissible mappings with respect to η was first suggested in 2016 by Chuadchawna et al. [19]. The concept of generalised $\alpha-\eta-\psi$-Geraghty contractive type mappings and α-orbital attractive mappings with regard to is introduced by Farajzadeh et al. in 2018 [20] in the framework of partial b-metric spaces, which will be effectively applied for establishing our key findings.

In the context of S_b-metric spaces and α-orbital admissible mappings with regard to ϕ, this paper aims to demonstrate unique fixed point theorems for generalised (α, ϕ, ψ)-Geraghty contractive type mapping. Additionally, we may provide relevant applications for homotopy, integral equations, and appropriate examples.

We first review some fundamental findings.

2. Preliminaries

Definition 2.1. *([2]) Let $\mathfrak{P}$ be a non-empty set and $b \geq 1$ be given real number. Suppose that a mapping $S_b : \mathfrak{P}^3 \to [0, \infty)$ be a function satisfying the following properties :*

- (S_b1) $0 < S_b(\sigma, \varsigma, \tau)$ *for all* $\sigma, \varsigma, \tau \in \mathfrak{P}$ *with* $\sigma \neq \varsigma \neq \tau \neq \sigma$*,*
- (S_b2) $S_b(\sigma, \varsigma, \tau) = 0 \Leftrightarrow \sigma = \varsigma = \tau$*,*
- (S_b3) $S_b(\sigma, \varsigma, \tau) \leq b(S_b(\sigma, \sigma, a) + S_b(\varsigma, \varsigma, a) + S_b(\tau, \tau, a))$ *for all* $\sigma, \varsigma, \tau, a \in \mathfrak{P}$.

The function S_b is then referred to as a S_b-metric on $\mathfrak{P}$, and the pair $(\mathfrak{P}, S_b)$ is referred to as a S_b-metric space.

Remark 2.1. *([2]) It should be noted that the S_b-metric space class is effectively larger than the S-metric space class. Each S-metric space is, in fact, a Sb-metric space with $b = 1$.*

Example 2.1. *([2]) Let $(\mathfrak{P}, S)$ be S-metric space and $S_*(\sigma, \varsigma, \tau) = S(\sigma, \varsigma, \tau)^p$, where $p > 1$ is a real number.Then obviously, S_* is a S_b-metric with $b = 2^{2(p-1)}$ but $(\mathfrak{P}, S_*)$ is not necessarily a S-metric space.*

Definition 2.2. *([2]) Let $(\mathfrak{P}, S_b)$ be a S_b-metric space. Then, for $\sigma \in \mathfrak{P}$, $\lambda > 0$ we defined the open ball $B_{S_b}(\sigma, \lambda)$ and closed ball $B_{S_b}[\sigma, \lambda]$ with center σ and radius λ as follows respectively:*

$$B_{S_b}(\sigma, \lambda) = \{\varsigma \in \mathfrak{P} : S_b(\varsigma, \varsigma, \sigma) < \lambda\} \text{ and } B_{S_b}[\sigma, \lambda] = \{\varsigma \in \mathfrak{P} : S_b(\varsigma, \varsigma, \sigma) \leq \lambda\}.$$

Lemma 2.1. *([2]) In the S_b-metric space, we have*

$$S_b(\mu, \mu, \nu) \leq 2bS_b(\mu, \mu, \xi) + b^2 S_b(\xi, \xi, \nu).$$

Definition 2.3. *([2]) Let $\{\sigma_n\}$ be a sequence in S_b-metric space $(\mathfrak{P}, S_b)$ is said to be:*

1. *S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(\sigma_n, \sigma_n, \sigma_m) < \epsilon$ for each $m, n \geq n_0$.*

(2) *S_b-convergent to a point $\sigma \in \mathfrak{P}$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(\sigma_n, \sigma_n, \sigma) < \epsilon$ or $S_b(\sigma, , \sigma, \sigma_n) < \epsilon$ for all $n \geq n_0$ and we denote by $\lim_{n\to\infty} \sigma_n = \sigma$.*

(3) *$(\mathfrak{P}, S_b)$ is S_b-complete if every S_b-Cauchy sequence is S_b-convergent in $\mathfrak{P}$.*

Lemma 2.2. *([2]) If $(\mathfrak{P}, S_b)$ be a S_b-metric space with $b \geq 1$ and suppose that $\{\sigma_n\}$ is a S_b-convergent to σ, then we have*

$$(i)\ \frac{1}{2b} S_b(\varsigma, \varsigma, \sigma) \leq \lim_{n\to\infty} \inf\ S_b(\varsigma, \varsigma, \sigma_n) \leq \lim_{n\to\infty} \sup S_b(\varsigma, \varsigma, \sigma_n) \leq 2b S_b(\varsigma, \varsigma, \sigma)$$

$$(ii)\ \frac{1}{b^2} S_b(\sigma, \sigma, \varsigma) \leq \lim_{n\to\infty} \inf\ S_b(\sigma_n, \sigma_n, \varsigma) \leq \lim_{n\to\infty} \sup\ S_b(\sigma_n, \sigma_n, \varsigma) \leq b^2 S_b(\sigma, \sigma, \varsigma)$$

for all $\varsigma \in \mathfrak{P}$. In particular, if $\sigma = \varsigma$, then we have $\lim_{n\to\infty} S_b(\sigma_n, \sigma_n, \varsigma) = 0$.

We should always consider the following factors in order to obtain our results.

3. Main Results

We say $\mathfrak{F}$ be the class of all functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0$$

Definition 3.1. *Let $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ be a self mapping defined on non-empty set $\mathfrak{P}$ and $\alpha, \phi : \mathfrak{P} \times \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}^+$ be two functions. We say that $\mathcal{G}$ is an α- admissible mapping with respect to ϕ,*

$$\text{if } \alpha(\sigma, \sigma, \varsigma) \geq \phi(\sigma, \sigma, \varsigma) \text{ implies } \alpha(\mathcal{G}\sigma, \mathcal{G}\sigma, \mathcal{G}\varsigma) \geq \phi(\mathcal{G}\sigma, \mathcal{G}\sigma, \mathcal{G}\varsigma) \text{ for all } \sigma, \varsigma \in \mathfrak{P}.$$

We say that $\mathcal{G}$ is an α- admissible mapping if for all $\sigma, \varsigma \in \mathfrak{P}$, $\alpha(\sigma, \sigma, \varsigma) \geq 1$ implies $\alpha(\mathcal{G}\sigma, \mathcal{G}\sigma, \mathcal{G}\varsigma) \geq 1$.

Definition 3.2. *Let $\mathfrak{P}$ be a non-empty set. $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ be a self mapping and $\alpha : \mathfrak{P} \times \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}^+$. We say that $\mathcal{G}$ is a triangular α- admissible mapping, if*

(a) *$\mathcal{G}$ is an α- admissible mapping;*

(b) *$\alpha(\sigma, \sigma, \varsigma) \geq 1$ and $\alpha(\varsigma, \varsigma, \tau) \geq 1$ implies $\alpha(\sigma, \sigma, \tau) \geq 1$ for all $\sigma, \varsigma, \tau \in \mathfrak{P}$.*

Definition 3.3. *Let $\mathfrak{P}$ be a non-empty set. $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ be a self mapping and $\alpha, \phi : \mathfrak{P} \times \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}^+$ be two functions. We say that $\mathcal{G}$ is an α- orbital admissible mapping with respect to ϕ,*

$$\text{if } \alpha(\sigma, \sigma, \mathcal{G}\sigma) \geq \phi(\sigma, \sigma, \mathcal{G}\sigma) \text{ implies } \alpha(\mathcal{G}\sigma, \mathcal{G}\sigma, \mathcal{G}^2\sigma) \geq \phi(\mathcal{G}\sigma, \mathcal{G}\sigma, \mathcal{G}^2\sigma) \text{ for all } \sigma \in \mathfrak{P}.$$

Definition 3.4. *Let $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ be a self mapping defined on nonempty set $\mathfrak{P}$ and $\alpha, \phi : \mathfrak{P} \times \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}^+$. We say that $\mathcal{G}$ is a triangular α-orbital admissible mapping with respect to ϕ, if*

(a) *$\mathcal{G}$ is an α-orbital admissible mapping with respect to ϕ;*

(b) *$\alpha(\sigma, \sigma, \varsigma) \geq \phi(\sigma, \sigma, \varsigma)$ and $\alpha(\varsigma, \varsigma, \mathcal{G}\varsigma) \geq \phi(\varsigma, \varsigma, \mathcal{G}\varsigma)$ imply $\alpha(\sigma, \sigma, \mathcal{G}\varsigma) \geq \phi(\sigma, \sigma, \mathcal{G}\varsigma)$ for all $\sigma, \varsigma \in \mathfrak{P}$.*

Definition 3.5. *Let* $\Omega = \{\Gamma/\Gamma : [0,\infty) \to [0,\infty)\}$ *be a family of functions that satisfy the following properties;*

(i) Γ *is a continuously nondecreasing map;*

(ii) $\Gamma(t) = 0$ *if and only if* $t = 0$*;*

(iii) $\Gamma(t)$ *is subadditive,* $\Gamma(p+q) \leq \Gamma(p) + \Gamma(q)$.

Definition 3.6. *Let* $(\mathfrak{P}, S_b)$ *be an* S_b*-metric space, a mapping* $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ *is said to be a generalized* (α, ϕ, Γ)*-Geraghty contractive type mapping if there exist* $\Gamma \in \Omega$*,* $\alpha, \phi : \mathfrak{P} \times \mathfrak{P} \times \mathfrak{P} \to [0,\infty)$ *and* $\beta \in \mathfrak{F}$ *such that*

$$\alpha(\sigma,\sigma,\varsigma) \geq \phi(\sigma,\sigma,\varsigma) \text{ implies } \Gamma\left((\frac{1+b^3}{2})S_b(\mathcal{G}\sigma,\mathcal{G}\sigma,\mathcal{G}\varsigma)\right) \leq \beta\left(\Gamma(M_b^{\mathcal{G}}(\sigma,\sigma,\varsigma))\right)\Gamma(M_b^{\mathcal{G}}(\sigma,\sigma,\varsigma)) \tag{3.1}$$

where $M_b^{\mathcal{G}}(\sigma,\sigma,\varsigma) = \max\left\{\begin{array}{c} S_b(\sigma,\sigma,\varsigma), S_b(\sigma,\sigma,\mathcal{G}\sigma), \\ S_b(\varsigma,\varsigma,\mathcal{G}\varsigma), \frac{S_b(\sigma,\sigma,\mathcal{G}\varsigma)+S_b(\varsigma,\varsigma,\mathcal{G}\sigma)}{4b^3} \end{array}\right\} \forall\, \sigma,\varsigma \in \mathfrak{P}$

Lemma 3.1. *Let* $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ *be a triangular* α*-orbital admissible mapping with respect to* ϕ*. Assume that there exists* $\sigma_1 \in \mathfrak{P}$ *such that* $\alpha(\sigma_1, \mathcal{G}\sigma_1) \geq \phi(\sigma_1, \mathcal{G}\sigma_1)$*. Define a sequence* $\{\sigma_n\}$ *by* $\sigma_{n+1} = \mathcal{G}\sigma_n$*. Then we have* $\alpha(\sigma_n, \sigma_m) \geq \phi(\sigma_n, \sigma_m)$ *for all* $m, n \in N$ *with* $n < m$*.*

Proof. Since $\alpha(\sigma_1,\sigma_1,\mathcal{G}\sigma_1) \geq \phi(\sigma_1,\sigma_1,\mathcal{G}\sigma_1)$ and $\mathcal{G}$ is α-orbital admissible with respect to ϕ, we obtain that

$$\alpha(\sigma_2,\sigma_2,\sigma_3) = \alpha(\mathcal{G}\sigma_1,\mathcal{G}\sigma_1,\mathcal{G}(\mathcal{G}\sigma_1)) \geq \phi(\mathcal{G}\sigma_1,\mathcal{G}\sigma_1,\mathcal{G}(\mathcal{G}\sigma_1)) = \phi(\sigma_2,\sigma_2,\sigma_3).$$

By continuing the process as above, we have $\alpha(\sigma_n,\sigma_n,\sigma_{n+1}) \geq \Gamma(\sigma_n,\sigma_n,\sigma_{n+1})$ for all $n \in N$. Suppose that

$$\alpha(\sigma_n,\sigma_n,\sigma_m) \geq \phi(\sigma_n,\sigma_n,\sigma_m) \tag{3.2}$$

and we will prove that $\alpha(\sigma_n,\sigma_n,\sigma_{m+1}) \geq \phi(\sigma_n,\sigma_n,\sigma_{m+1})$, where $m > n$.
Since $\alpha(\sigma_m,\sigma_m,\sigma_{m+1}) \geq \phi(\sigma_m,\sigma_m,\sigma_{m+1})$, we obtain that

$$\alpha(\sigma_m,\sigma_m,\mathcal{G}\sigma_m) = \alpha(\sigma_m,\sigma_m,\sigma_{m+1}) \geq \phi(\sigma_m,\sigma_m,\sigma_{m+1}) = \phi(\sigma_m,\sigma_m,\mathcal{G}\sigma_m). \tag{3.3}$$

By (3.2), (3.3) and triangular α-orbital admissibility of $\mathcal{G}$, we have
$\alpha(\sigma_n,\sigma_n,\mathcal{G}\sigma_m) \geq \phi(\sigma_n,\sigma_n,\mathcal{G}\sigma_m)$. This implies that $\alpha(\sigma_n,\sigma_n,\sigma_{m+1}) \geq \phi(\sigma_n,\sigma_n,\sigma_{m+1})$.
Hence, $\alpha(\sigma_n,\sigma_m) \geq \phi(\sigma_n,\sigma_m)$ for all $m, n \in N$ with $n < m$. □

Theorem 3.1. *Let* $(\mathfrak{P}, S_b)$ *be a complete* S_b*-metric space with coefficient* $b \geq 1$*. Let* $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ *be an be a generalized* (α, ϕ, Γ)*-Geraghty contractive type mapping. Assume the following conditions are hold:*

(i) $\mathcal{G}$ *is a triangular* α*-orbital admissible mapping w.r.t* ϕ*;*

(ii) *there exists* $\sigma_1 \in \mathfrak{P}$ *such that* $\alpha(\sigma_1,\sigma_1,\mathcal{G}\sigma_1) \geq \phi(\sigma_1,\sigma_1,\mathcal{G}\sigma_1)$*;*

(*iii*) *if* $\{\sigma_n\}$ *is a* S_b*-convergent sequence to* ν *in* $\mathfrak{P}$ *and* $\alpha(\sigma_n, \sigma_n, \sigma_{n+1}) \geq \phi(\sigma_n, \sigma_n, \sigma_{n+1})$ *for each* $n \in N$ *then* $\alpha(\nu, \nu, \nu) \geq \phi(\nu, \nu, \nu)$*;*

(*iv*) $\mathcal{G}$ *is continuous.*

Then $\mathcal{G}$ *has a fixed point.*

Proof. Let $\sigma_1 \in \mathfrak{P}$ such that $\alpha(\sigma_1, \sigma_1, \mathcal{G}\sigma_1) \geq \phi(\sigma_1, \sigma_1, \mathcal{G}\sigma_1)$. Define the sequence $\{\sigma_n\}$ in $\mathfrak{P}$ by $\sigma_{n+1} = T\sigma_n$ for all $n \in N$. By Lemma(2.1), we get that

$$\alpha(\sigma_n, \sigma_n, \sigma_{n+1}) \geq \phi(\sigma_n, \sigma_n, \sigma_{n+1}) \text{ for all } n \in N. \tag{3.4}$$

If $\sigma_n = \sigma_{n+1}$ for some $n \in N$, then σ_n is a fixed point of $\mathcal{G}$. Assume that $\sigma_n \neq \sigma_{n+1}$ for all $n \in N$. The sequence $\{S_b(\sigma_n, \sigma_n, \sigma_{n+1})\}$ is first shown to be non-increasing and to trend to 0 as $n \to \infty$.

By using (3.4), for each $n \in N$, we have

$$\begin{aligned} \Gamma\left((\frac{1+b^3}{2})S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2})\right) &= \Gamma\left((\frac{1+b^3}{2})S_b(\mathcal{G}\sigma_n, \mathcal{G}\sigma_n, \mathcal{G}\sigma_{n+1})\right) \\ &\leq \beta\left(\Gamma(M_b^{\mathcal{G}}(\sigma_n, \sigma_n, \sigma_{n+1}))\right)\Gamma(M_b^{\mathcal{G}}(\sigma_n, \sigma_n, \sigma_{n+1})) \\ &< \Gamma(M_b^{\mathcal{G}}(\sigma_n, \sigma_n, \sigma_{n+1})) \end{aligned} \tag{3.5}$$

where,

$$\begin{aligned} M_b^{\mathcal{G}}(\sigma_n, \sigma_n, \sigma_{n+1}) &= \max\left\{ \begin{array}{c} S_b(\sigma_n, \sigma_n, \sigma_{n+1}), S_b(\sigma_n, \sigma_n, \mathcal{G}\sigma_n), \\ S_b(\sigma_{n+1}, \sigma_{n+1}, \mathcal{G}\sigma_{n+1}), \frac{S_b(\sigma_n, \sigma_n, \mathcal{G}\sigma_{n+1}) + S_b(\sigma_{n+1}, \sigma_{n+1}, \mathcal{G}\sigma_n)}{4b^3} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} S_b(\sigma_n, \sigma_n, \sigma_{n+1}), S_b(\sigma_n, \sigma_n, \sigma_{n+1}), \\ S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}), \frac{S_b(\sigma_n, \sigma_n, \sigma_{n+2}) + S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+1})}{4b^3} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} S_b(\sigma_n, \sigma_n, \sigma_{n+1}), S_b(\sigma_n, \sigma_n, \sigma_{n+1}), \\ S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}), \frac{2bS_b(\sigma_n, \sigma_n, \sigma_{n+1}) + b^2 S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2})}{4b^3} \end{array} \right\} \\ &= \max\left\{ S_b(\sigma_n, \sigma_n, \sigma_{n+1}), S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) \right\}. \end{aligned}$$

If $\max\left\{ S_b(\sigma_n, \sigma_n, \sigma_{n+1}), S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) \right\} = S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2})$ then
$\Gamma\left((\frac{1+b^3}{2})S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2})\right) < \Gamma(S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}))$ implies that
$(\frac{1+b^3}{2})S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) < S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2})$ which is contradiction.
Hance, $\max\left\{ S_b(\sigma_n, \sigma_n, \sigma_{n+1}), S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) \right\} = S_b(\sigma_n, \sigma_n, \sigma_{n+1})$.
It follows that $0 < S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) \leq S_b(\sigma_n, \sigma_n, \sigma_{n+1})$. Hence the sequence $\{S_b(\sigma_n, \sigma_n, \sigma_{n+1})\}$ is nonnegative non-increasing and bounded below. Thus there exist some $\xi \geq 0$ such that $\lim_{n\to\infty} S_b(\sigma_n, \sigma_n, \sigma_{n+1}) = \xi$. Suppose that $\xi > 0$.

By using (3.5), we have

$$\begin{aligned}\frac{\Gamma\left(S_b(\sigma_{n+1},\sigma_{n+1},\sigma_{n+2})\right)}{\Gamma\left(S_b(\sigma_n,\sigma_n,\sigma_{n+1})\right)} &\leq \frac{\Gamma\left((\frac{1+b^3}{2})S_b(\sigma_{n+1},\sigma_{n+1},\sigma_{n+2})\right)}{\Gamma\left(S_b(\sigma_n,\sigma_n,\sigma_{n+1})\right)} \\ &\leq \beta\left(\Gamma(M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1}))\right) < 1,\end{aligned}$$

for all $n \in N$. On letting $n \to \infty$ in above inequality, we have

$$\lim_{n\to\infty} \beta\left(\Gamma(M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1}))\right) = 1.$$

Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} \Gamma(M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1})) = 0$ and so $\xi = \lim_{n\to\infty} S_b(\sigma_n,\sigma_n,\sigma_{n+1}) = 0$. We now demonstrate that the $(\mathfrak{P}, S_b)$ Cauchy sequence is $\{\sigma_n\}$. On the other hand, we assume that $\{\sigma_n\}$ is not Cauchy. In this case, a monotonically rising sequence of the natural numbers $\{m_k\}$ and $\{n_k\}$ exists, where $n_k > m_k$.

$$S_b\left(\sigma_{m_k},\sigma_{m_k},\sigma_{n_k}\right) \geq \epsilon \tag{3.6}$$

and

$$S_b\left(\sigma_{m_k},\sigma_{m_k},\sigma_{n_k-1}\right) < \epsilon. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\epsilon \leq S_b\left(\sigma_{m_k},\sigma_{m_k},\sigma_{n_k}\right) \leq 2bS_b\left(\sigma_{m_k},\sigma_{m_k},\sigma_{m_k+1}\right) + b^2 S_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{n_k}\right).$$

So that $(\frac{1+b^3}{2b^2})\epsilon \leq (\frac{1+b^3}{b})S_b\left(\sigma_{m_k},\sigma_{m_k},\sigma_{m_k+1}\right) + (\frac{1+b^3}{2})S_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{n_k}\right)$.
We obtain that by applying Γ on both sides and letting $k \to \infty$.

$$\begin{aligned}\Gamma((\frac{1+b^3}{2b^2})\epsilon) &\leq \lim_{k\to\infty} \Gamma\left((\frac{1+b^3}{2})S_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{n_k}\right)\right) \\ &\leq \lim_{k\to\infty} \Gamma\left((\frac{1+b^3}{2})S_b\left(\mathcal{G}\sigma_{m_k},\mathcal{G}\sigma_{m_k},\mathcal{G}\sigma_{n_k-1}\right)\right).\end{aligned} \tag{3.8}$$

By applying the triangular inequality, we get that

$$\begin{aligned}S_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{n_k}\right) &\leq 2bS_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{m_k}\right) + b^2 S_b\left(\sigma_{m_k},\sigma_{m_k},\sigma_{n_k}\right). \\ &\leq 2bS_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{m_k}\right) + 2b^3 S_b\left(\sigma_{m_k},\sigma_{m_k},\sigma_{n_k-1}\right) \\ &\qquad + b^3 S_b\left(\sigma_{n_k},\sigma_{n_k},\sigma_{n_k-1}\right) \\ &\leq 2bS_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{m_k}\right) + 2b^3\epsilon + b^3 S_b\left(\sigma_{n_k},\sigma_{n_k},\sigma_{n_k-1}\right).\end{aligned}$$

In the above inequality, we get that by taking the limit as $k \to \infty$.

$$\lim_{k\to\infty} S_b\left(\sigma_{m_k+1},\sigma_{m_k+1},\sigma_{n_k}\right) \leq 2b^3\epsilon. \tag{3.9}$$

We obtain (3.5) because $\mathcal{G}$ is a triangular α-orbital admissible mapping with respect to ϕ. and $\alpha(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}) \geq \phi(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1})$. By using (3.1), we have

$$\Gamma\left((\frac{1+b^3}{2}) S_b(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}) \right) \leq \beta\left(\Gamma(M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}))\right) \Gamma(M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1})) \tag{3.10}$$

where,

$$\begin{aligned} & M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}) \\ & = \max \left\{ \begin{array}{c} S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}\right), S_b\left(\sigma_{m_k}, \sigma_{m_k}, \mathcal{G}\sigma_{m_k}\right), \\ S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \mathcal{G}\sigma_{n_k-1}\right), \frac{S_b\left(\sigma_{m_k}, \sigma_{m_k}, \mathcal{G}\sigma_{n_k-1}\right) + S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \mathcal{G}\sigma_{m_k}\right)}{4b^3} \end{array} \right\} \\ & \leq \max \left\{ \begin{array}{c} S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}\right), S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{m_k+1}\right), S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right), \\ \frac{2bS_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}\right) + b^2 S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right) + S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{m_k+1}\right)}{4b^3} \end{array} \right\} \\ & \leq \max \left\{ \begin{array}{c} S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}\right), S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{m_k+1}\right), S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right), \\ \frac{2bS_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}\right) + b^2 S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right)}{4b^3} \\ + \frac{2bS_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right) + bS_b\left(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}\right)}{4b^3} \end{array} \right\} \end{aligned}$$

Using (3.7) and (3.9) and treating the limit of the inequality above as $k \to \infty$, this results.

$$\begin{aligned} & \lim_{k\to\infty} M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}) \\ \leq & \lim_{k\to\infty} \max \left\{ \begin{array}{c} S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}\right), S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{m_k+1}\right), S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right), \\ \frac{2bS_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}\right) + b^2 S_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right)}{4b^3} \\ + \frac{2bS_b\left(\sigma_{n_k-1}, \sigma_{n_k-1}, \sigma_{n_k}\right) + bS_b\left(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}\right)}{4b^3} \end{array} \right\} \\ \leq & \max \left\{ \epsilon, 0, 0, \epsilon(\tfrac{1+b^3}{2b^2}) \right\} = \epsilon(\frac{1+b^3}{2b^2}) \end{aligned} \tag{3.11}$$

By taking the limit in (3.10) as $k \to \infty$ and using (3.8) and (3.11), we have

$$\begin{aligned} \Gamma((\frac{1+b^3}{2b^2})\epsilon) & \leq \Gamma\left(\lim_{k\to\infty} (\frac{1+b^3}{2}) S_b\left(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}\right) \right). \\ & \leq \beta\left(\Gamma(\lim_{k\to\infty} M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1})) \right) \Gamma(\lim_{k\to\infty} M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1})) \\ & \leq \beta\left(\Gamma(\lim_{k\to\infty} M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1})) \right) \Gamma(\epsilon(\frac{1+b^3}{2b^2})) \end{aligned}$$

This implies that $\frac{\Gamma((\frac{1+b^3}{2b^2})\epsilon)}{\Gamma((\frac{1+b^3}{2b^2})\epsilon)} \leq \beta\left(\Gamma\left(\lim_{k\to\infty} M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1})\right)\right)$.

Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} \beta\left(\Gamma(\lim_{k\to\infty} M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}))\right) = 1$.

It follows that $\Gamma(\lim_{k\to\infty} M_b^{\mathcal{G}}(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1})) = 0$. By using (3.10) we obtain $\lim_{n\to\infty} S_b(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}) = 0$. which contradicts to(3.9).In the S_b-metric space $(\mathfrak{P}, S_b)$, the sequence $\{\sigma_n\}$ is a S_b-Cauchy sequence. The sequence $\{\sigma_n\} \to \nu \in (\mathfrak{P}, S_b)$ emerges from the completeness of $(\mathfrak{P}, S_b)$. We begin by presuming that $\mathcal{G}$ is continuous. Therefore, we have $\nu = \lim_{n\to\infty} \sigma_{n+1} = \lim_{n\to\infty} \mathcal{G}\sigma_n = \mathcal{G} \lim_{n\to\infty} \sigma_n = \mathcal{G}\nu$. Since $\{\sigma_n\}$ is a S_b-convergent sequence to ν in $\mathfrak{P}$ and $\alpha(\nu,\nu,\nu) \geq \phi(\nu,\nu,\nu)$. Then to prove $\nu = \mathcal{G}\nu$. Suppose that $\nu \neq \mathcal{G}\nu$. From (3.1), we have

$$\begin{aligned}
\Gamma\left(S_b(\nu,\nu,\mathcal{G}\nu)\right) &\leq \Gamma\left((\frac{1+b^3}{2})S_b(\nu,\nu,\mathcal{G}\nu)\right) = \Gamma\left((\frac{1+b^3}{2})\lim_{n\to\infty} S_b(\sigma_{n+1},\sigma_{n+1},\mathcal{G}\sigma_{n+1})\right) \\
&= \Gamma\left((\frac{1+b^3}{2})\lim_{n\to\infty} S_b(\mathcal{G}\sigma_n,\mathcal{G}\sigma_n,\mathcal{G}\sigma_{n+1})\right) \\
&\leq \beta\left(\Gamma(\lim_{n\to\infty} M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1}))\right)\Gamma(\lim_{n\to\infty} M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1}))
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\lim_{n\to\infty} M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1}) &= \lim_{n\to\infty} \max\left\{\begin{array}{c} S_b\left(\sigma_n,\sigma_n,\sigma_{n+1}\right), S_b\left(\sigma_n,\sigma_n,\mathcal{G}\sigma_n\right), \\ S_b\left(\sigma_{n+1},\sigma_{n+1},\mathcal{G}\sigma_{n+1}\right), \\ \frac{S_b(\sigma_n,\sigma_n,\mathcal{G}\sigma_{n+1})+S_b(\sigma_{n+1},\sigma_{n+1},\mathcal{G}\sigma_n)}{4b^3} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} S_b\left(\nu,\nu,\nu\right), S_b\left(\nu,\nu,\mathcal{G}\nu\right), \\ S_b\left(\nu,\nu,\mathcal{G}\nu\right), \frac{S_b(\nu,\nu,\mathcal{G}\nu)}{2b^3} \end{array}\right\} = S_b\left(\nu,\nu,\mathcal{G}\nu\right).
\end{aligned}$$

By taking limit as $n \to \infty$ in (3.12), we have

$$\begin{aligned}
\Gamma\left(S_b(\nu,\nu,\mathcal{G}\nu)\right) &\leq \beta\left(\Gamma(\lim_{n\to\infty} M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1}))\right)\Gamma(\lim_{n\to\infty} M_b^{\mathcal{G}}(\sigma_n,\sigma_n,\sigma_{n+1})) \\
&\leq \beta\left(\Gamma(S_b\left(\nu,\nu,\mathcal{G}\nu\right))\right)\Gamma(S_b\left(\nu,\nu,\mathcal{G}\nu\right))
\end{aligned}$$

we can deduce that $\frac{\Gamma(S_b(\nu,\nu,\mathcal{G}\nu))}{\Gamma(S_b(\nu,\nu,\mathcal{G}\nu))} \leq \beta\left(\Gamma(S_b\left(\nu,\nu,\mathcal{G}\nu\right))\right)$
We obtain that $\lim_{n\to\infty} \beta\left(\Gamma(S_b\left(\nu,\nu,\mathcal{G}\nu\right))\right) = 1$. Therefore, $S_b\left(\nu,\nu,\mathcal{G}\nu\right) = 0$ implies $\mathcal{G}\nu = \nu$. and thus ν is a fixed point of $\mathcal{G}$. Assume further that ν and ν^* are two fixed points of $\mathcal{G}$ such that $\nu \neq \nu^*$. Consider

$$\Gamma\left(S_b(\nu,\nu,\nu^*)\right) \leq \Gamma\left((\frac{1+b^3}{2})S_b(\mathcal{G}\nu,\mathcal{G}\nu,\mathcal{G}\nu^*)\right) \leq \beta\left(\Gamma(M_b^{\mathcal{G}}(\nu,\nu,\nu^*))\right)\Gamma(M_b^{\mathcal{G}}(\nu,\nu,\nu^*)) \tag{3.13}$$

where

$$\begin{aligned}
M_b^{\mathcal{G}}(\nu,\nu,\nu^*) &= \max\left\{\begin{array}{c} S_b\left(\nu,\nu,\nu^*\right), S_b\left(\nu,\nu,\mathcal{G}\nu\right), \\ S_b\left(\nu^*,\nu^*,\mathcal{G}\nu^*\right), \frac{S_b(\nu,\nu,\mathcal{G}\nu^*)+S_b(\nu^*,\nu^*,\mathcal{G}\nu)}{4b^3} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} S_b\left(\nu,\nu,\nu^*\right), S_b\left(\nu,\nu,\nu\right), \\ S_b\left(\nu^*,\nu^*,\nu^*\right), \frac{S_b(\nu,\nu,\nu^*)+2bS_b(\nu^*,\nu^*,\nu^*)+S_b(\nu,\nu,\nu^*)}{4b^3} \end{array}\right\} = S_b\left(\nu,\nu,\nu^*\right)
\end{aligned}$$

Using by (3.13), we have $\Gamma\left(S_b(\nu,\nu,\nu^*)\right) \le \beta\left(\Gamma(S_b\left(\nu,\nu,\nu^*\right))\right)\Gamma(S_b\left(\nu,\nu,\nu^*\right))$.
we can deduce that $\frac{\Gamma(S_b(\nu,\nu,\nu^*))}{\Gamma(S_b(\nu,\nu,\nu^*))} \le \beta\left(\Gamma(S_b\left(\nu,\nu,\nu^*\right))\right)$.
We obtain that $\lim_{n\to\infty}\beta\left(\Gamma(S_b\left(\nu,\nu,\nu^*\right))\right) = 1$.
Therefore, $S_b\left(\nu,\nu,\nu^*\right) = 0$ implies $\nu = \nu^*$. Consequently, ν is a unique fixed point of $\mathcal{G}$. □

Example 3.1. *Let $S_b : \mathfrak{P}^3 \to \mathbb{R}^+$ be defined as $S_b(\mu,\nu,\xi) = (|\nu+\xi-2\mu| + |\nu-\xi|)^2$ where $\mathfrak{P} = [0,\infty)$. It is obvious that $(\mathfrak{P}, S_b)$ is a complete with $b = 4$. Define $\mathcal{G} : \mathfrak{P} \to \mathfrak{P}$ by $\mathcal{G}(\mu) = \frac{\mu}{4^3}$ and $\Gamma : [0,\infty) \to [0,\infty)$ and $\beta : [0,\infty) \to [0,1)$ as $\Gamma(t) = t$,* $\beta(t) = \begin{cases} \frac{e^{-\frac{4^5}{3969}t}}{1+\frac{4^5}{3969}t}, & t \in (0,\infty) \\ 0, & t = 0 \end{cases}$
also define $\alpha, \phi : \mathfrak{P}\times\mathfrak{P}\times\mathfrak{P} \to \mathbb{R}^+$ $\alpha(\mu,\mu,\nu) = \begin{cases} 4, & (\mu,\mu,\nu) \in [0,1] \\ 0, & Otherwise \end{cases}$

$\phi(\mu,\mu,\nu) = \begin{cases} 1, & (\mu,\mu,\nu) \in [0,1] \\ 0, & Otherwise \end{cases}$

Let $\alpha(\mu,\mu,\mathcal{G}\mu) \ge \phi(\mu,\mu,\mathcal{G}\mu)$. Thus $\mu, \mathcal{G}\mu \in [0,1]$ and so $\mathcal{G}^2\mu = \mathcal{G}(\mathcal{G}\mu) \in [0,1]$ which implies that $\alpha(\mathcal{G}\mu,\mathcal{G}\mu,\mathcal{G}^2\mu) \ge (\mathcal{G}\mu,\mathcal{G}\mu,\mathcal{G}^2\mu)$ that is $\mathcal{G}$ is α-orbital admissible with respect to ϕ. Now, let $\alpha(\mu,\mu,\nu) \ge \phi(\mu,\mu,\nu)$ and $\alpha(\nu,\nu,\mathcal{G}\nu) \ge \phi(\nu,\nu,\mathcal{G}\nu)$, we get that $\mu,\nu,\mathcal{G}\nu \in [0,1]$ and so $\alpha(\mu,\mu,\mathcal{G}\nu) \ge \phi(\mu,\mu,\mathcal{G}\nu)$. Therefore $\mathcal{G}$ is triangular α-orbital admissible with respect to ϕ. Let $\{\mu_n\}$ be a sequence such that $\{\mu_n\}$ is S_b-convergent to χ
and $\alpha(\mu_n,\mu_n,\mu_{n+1}) \ge \phi(\mu_n,\mu_n,\mu_{n+1})$ for all $n \in N$. Then $\{\mu_n\} \in [0,1]$ for any $n \in N$ and so $\chi \in [0,1]$ which we have, $\alpha(\chi,\chi,\chi) \ge \phi(\chi,\chi,\chi)$ and obviously the function $\mathcal{G}$ is continuous. Following that, we show that $\mathcal{G}$ is a generalised (α,ϕ,Γ)-Geraghty contraction type mapping. Let $\mu,\nu \in \mathfrak{P}$ with $\alpha(\mu,\mu,\nu) \ge \phi(\mu,\mu,\nu)$. Thus $\mu,\nu \in [0,1]$.We can assume without losing generality that $0 \le \nu \le \mu \le 1$.
Therefore,

$$S_b(\mathcal{G}\mu,\mathcal{G}\mu,\mathcal{G}\nu) = (|\mathcal{G}\mu+\mathcal{G}\nu-2\mathcal{G}\mu| + |\mathcal{G}\mu-\mathcal{G}\nu|)^2 = \left(2\left|\frac{\mu}{4^3}-\frac{\nu}{4^3}\right|\right)^2 = \frac{1}{4^6}S_b(\mu,\mu,\nu)$$

and
$$M_b^{\mathcal{G}}(\mu,\mu,\nu) = \max\left\{\begin{array}{c} (2|\mu-\nu|)^2, \frac{3969}{4^5}\mu^2, \frac{3969}{4^5}\nu^2 \\ (\frac{(|4^3\mu-\nu|)^2+(|\mu-4^3\nu|)^2}{4^6 b^3} \end{array}\right\} = \frac{3969}{4^5}\mu^2$$
Since $\frac{65}{2\times4^6} \le \frac{1}{2e} \le \frac{e^{-\mu^2}}{1+\mu^2}$ so that $\frac{65}{2\times4^6}\frac{3969}{4^5}\mu^2 \le \frac{e^{-\mu^2}}{1+\mu^2}\frac{3969}{4^5}\mu^2$

$$\begin{aligned}
\Gamma\left((\tfrac{1+b^3}{2})S_b(\mathcal{G}\mu,\mathcal{G}\mu,\mathcal{G}\nu)\right) &= \Gamma\left(\tfrac{65}{2\times4^6}S_b(\mu,\mu,\nu)\right) = \tfrac{65}{2\times4^6}S_b(\mu,\mu,\nu) \\
&\le \tfrac{65}{2\times4^6}\tfrac{3969}{4^5}\mu^2 \le \tfrac{e^{-\mu^2}}{1+\mu^2}\tfrac{3969}{4^5}\mu^2 \le \beta\left(\Gamma(\tfrac{3969}{4^5}\mu^2)\right)\Gamma(\tfrac{3969}{4^5}\mu^2) \\
&\le \beta\left(\Gamma(M_b^{\mathcal{G}}(\mu,\mu,\nu))\right)\Gamma(M_b^{\mathcal{G}}(\mu,\mu,\nu))
\end{aligned}$$

As a result, all of Theorem (3.1)'s requirements are satisfied, and 0 is the only fixed point of $\mathcal{G}$.

4. Application to Integral Equations

As an application of Theorem (3.1), we will look at the existence of a unique solution to an initial value problem in this section.

Theorem 4.1. *Consider the I. V. P.*

$$\sigma'(t) = \mathcal{G}(t, \sigma(t)), \ t \in I = [0, 1], \ \sigma(0) = \sigma_0 \tag{4.1}$$

where $\mathcal{G} : I \times \mathbf{R} \to \mathbf{R}$ is a continuous function and $\sigma_0 \in \mathbb{R}$.
Let $\Gamma : [0, \infty) \to [0, \infty), \beta : [0, \infty) \to [0, 1)$ be a two functions defined as $\Gamma(t) = t$ and $\beta(t) = \frac{1}{3}$. Also examine the following conditions,

(*i*) *If there exist a function $\alpha, \phi : \mathbf{R}^3 \to \mathbf{R}$ such that there is an $\sigma_1 \in C(I)$, for all $t \in I$,we've*

$$\alpha\left(\sigma_1(t), \sigma_1(t), \int_0^t \mathcal{G}(s, \sigma_1(s))ds\right) \geq \phi\left(\sigma_1(t), \sigma_1(t), \int_0^t \mathcal{G}(s, \sigma_1(s))ds\right).$$

(*ii*) *$\forall\ t \in I$, and $\forall\ x, y \in C(I)$,*

$$\begin{aligned}
&\alpha\left(\sigma(t), \sigma(t), \varsigma(t)\right) \geq \alpha\left(\sigma(t), \sigma(t), \varsigma(t)\right) \Rightarrow \\
&\alpha\left(\tfrac{\sigma_0}{3b^3} + \int_0^t \mathcal{G}(s, \sigma(s))ds, \tfrac{\sigma_0}{3b^3} + \int_0^t \mathcal{G}(s, \sigma(s))ds, \tfrac{\varsigma_0}{3b^3} + \int_0^t \mathcal{G}(s, \varsigma(s))ds\right) \\
&\geq \phi\left(\tfrac{\sigma_0}{3b^3} + \int_0^t \mathcal{G}(s, \sigma(s))ds, \tfrac{\sigma_0}{3b^3} + \int_0^t \mathcal{G}(s, \sigma(s))ds, \tfrac{\varsigma_0}{3b^3} + \int_0^t \mathcal{G}(s, \varsigma(s))ds\right).
\end{aligned}$$

(*iii*) *for any point σ of a sequence $\{\sigma_n\}$ of points in $C(I)$ with $\alpha\left(\sigma_n, \sigma_n, \sigma_{n+1}\right) \geq \phi\left(\sigma_n, \sigma_n, \sigma_{n+1}\right)$, $\lim_{n\to\infty} \inf \alpha\left(\sigma_n, \sigma_n, \sigma\right) \geq \lim_{n\to\infty} \inf \phi\left(\sigma_n, \sigma_n, \sigma\right)$. Then (4.1) has unique solution .*

Proof. The integral equation of I. V. P. (4.1) is

$$\sigma(t) = \sigma_0 + 3(\frac{1+b^3}{2}) \int_0^t \mathcal{G}(s, \sigma(s))ds.$$

Let $\mathfrak{P} = C\,(I)$ and $S_b(\sigma, \varsigma, \tau) = (|\varsigma + \tau - 2\sigma| + |\varsigma - \tau|)^2$ for $\sigma, \varsigma, \varsigma \in \mathfrak{P}$. Then $(\mathfrak{P}, S_b)$ is a complete, also define $T : \mathfrak{P} \to \mathfrak{P}$ by

$$T(\sigma)(t) \;=\; \frac{2\sigma_0}{3(1+b^3)} + \int_0^t \mathcal{G}(s, \sigma(s))ds. \tag{4.2}$$

Now

$$\Gamma\left((\frac{1+b^3}{2})S_b(T\sigma(t), T\sigma(t), T\varsigma(t))\right) = (\frac{1+b^3}{2})\left\{|\ T\sigma(t) + T\varsigma(t) - 2T\sigma(t)\ | + |\ T\sigma(t) - T\varsigma(t)\ |\right\}^2$$

$$= \frac{8(1+b^3)}{9(1+b^3)^2} \left| \sigma_0 + 3(\tfrac{1+b^3}{2}) \int_0^t \mathcal{G}(s,\sigma(s))ds - \varsigma_0 - 3(\tfrac{1+b^3}{2}) \int_0^t \mathcal{G}(s,\varsigma(s))ds \right|^2$$

$$= \frac{8}{9(1+b^3)} \mid \sigma(t) - \varsigma(t) \mid^2 = \frac{8}{9(1+b^3)} S_b(\sigma,\sigma,\varsigma) \leq \frac{1}{3} S_b(\sigma,\sigma,\varsigma)$$

$$\leq \beta\left(\Gamma(S_b(\sigma,\sigma,\varsigma))\right)\Gamma(S_b(\sigma,\sigma,\varsigma)) \leq \beta\left(\Gamma(M_b^T(\sigma,\sigma,\varsigma))\right)\Gamma(M_b^T(\sigma,\sigma,\varsigma))$$

Thus we have $\Gamma\left((\frac{1+b^3}{2})S_b(T\sigma(t),T\sigma(t),T\varsigma(t))\right) \leq \beta\left(\Gamma(M_b^T(\sigma,\sigma,\varsigma))\right)\Gamma(M_b^T(\sigma,\sigma,\varsigma)) \forall \sigma,\varsigma \in \mathfrak{P}$
Let us define $\alpha : \mathfrak{P}\times\mathfrak{P}\times\mathfrak{P} \to [0,\infty)$ and $\phi : \mathfrak{P}\times\mathfrak{P}\times\mathfrak{P} \to [0,\infty)$ by

$$\alpha(\sigma,\sigma,\varsigma) = \begin{cases} 6, & \sigma,\varsigma \in [0,1] \\ 0, & \sigma,\varsigma \in (1,\infty) \end{cases}, \ \phi(\sigma,\sigma,\varsigma) = \begin{cases} 2, & \sigma,\varsigma \in [0,1] \\ 1, & \sigma,\varsigma \in (1,\infty) \end{cases}$$

Then obviously, T is triangular α-orbital admissible with respect to ϕ. Let $\sigma,\varsigma \in \mathfrak{P}$, if $\alpha(\sigma,\sigma,\varsigma) = 6$ and $\phi(\sigma,\sigma,\varsigma) = 2$, then $\alpha(\sigma(t),\sigma(t),\varsigma(t)) \geq \phi(\sigma(t),\sigma(t),\varsigma(t))$. From (ii) we have $\alpha(T\sigma(t),T\sigma(t),T\varsigma(t)) \geq \phi(T\sigma(t),T\sigma(t),T\varsigma(t))$ and so $\alpha(T\sigma,T\sigma,T\varsigma) = 6$ and $\phi(T\sigma,T\sigma,T\varsigma) = 2$. Thus T is is triangular α-orbital admissible with respect to ϕ. From (i), there exist $\sigma_1,\varsigma_1 \in \mathfrak{P}$ such that $\alpha(\sigma_1,\sigma_1,T\sigma_1) = 6$ and $\phi(\varsigma_1,\varsigma_1,T\varsigma_1) = 2$. By (iii), we have that for any point σ of a sequence $\{\sigma_n\}$ of points in $C(I)$ with $\alpha(\sigma_n,\sigma_n,\sigma_{n+1}) = 6$, $\lim_{n\to\infty} \inf \alpha(\sigma_n,\sigma_n,\sigma) = 6$ and $\phi(\sigma_n,\sigma_n,\sigma_{n+1}) = 2$, $\lim_{n\to\infty} \inf \phi(\sigma_n,\sigma_n,\sigma) = 2$. Therefore, for all $\sigma,\varsigma \in \mathfrak{P}$ and $t \in I$, we have

$$\alpha(\sigma(t),\sigma(t),\varsigma(t)) \geq \phi(\sigma(t),\sigma(t),\varsigma(t)) \implies \Gamma\left((\tfrac{1+b^3}{2})S_b(T\sigma(t),T\sigma(t),T\varsigma(t))\right) \leq \beta\left(\Gamma(M_b^T(\sigma,\sigma,\varsigma))\right)\Gamma(M_b^T(\sigma,\sigma,\varsigma))$$

Theorem (3.1) states that T has a unique solution in $\mathfrak{P}$.

□

5. Application to Homotopy

The existence of a unique homotopy theory solution is investigated in this section.

Theorem 5.1. *Let $(\mathfrak{P}, S_b)$ be a complete S_b-metric space, $\mathcal{U}$ and $\overline{\mathcal{U}}$ be a open and closed subset of $\mathfrak{P}$ such that $\mathcal{U} \subseteq \overline{\mathcal{U}}$. Suppose $\alpha,\phi : \mathfrak{P}\times\mathfrak{P}\times\mathfrak{P} \to [0,\infty)$, $\mathfrak{H}_\mathfrak{b} : \overline{\mathcal{U}} \times [0,1] \to \mathfrak{P}$ is a triangular α-orbital admissible operator with respect to ϕ and $\beta \in \mathfrak{F}$ satisfying the following conditions:*

(τ_0) *$\sigma \neq \mathfrak{H}_\mathfrak{b}(\sigma,\kappa)$, for each $\sigma \in \partial\mathcal{U}$ and $\kappa \in [0,1]$ (Here $\partial\mathcal{U}$ is boundary of $\mathcal{U}$ in $\mathfrak{P}$)*

(τ_1) *for all $\sigma,\varsigma \in \overline{\mathcal{U}}$ and $\kappa \in [0,1]$, $\alpha(\sigma,\sigma,\mathfrak{H}_\mathfrak{b}(\sigma,\kappa)) \geq \phi(\sigma,\sigma,\mathfrak{H}_\mathfrak{b}(\sigma,\kappa))$ implies*

$$\Gamma\left((\frac{1+b^3}{2})S_b(\mathfrak{H}_\mathfrak{b}(\sigma,\kappa),\mathfrak{H}_\mathfrak{b}(\sigma,\kappa),\mathfrak{H}_\mathfrak{b}(\varsigma,\kappa))\right) \leq \beta\left(\Gamma(S_b(\sigma,\sigma,\varsigma))\right)\Gamma(S_b(\sigma,\sigma,\varsigma))$$

(τ_2) *$\exists\ M_b \geq 0 \ni S_b(\mathfrak{H}_\mathfrak{b}(\sigma,\kappa),\mathfrak{H}_\mathfrak{b}(\sigma,\kappa),\mathfrak{H}_\mathfrak{b}(\sigma,\zeta)) \leq M_b|\kappa-\zeta|$ for every $\sigma \in \overline{\mathcal{U}}$and $\kappa,\zeta \in [0,1]$.*

Then $\mathfrak{H}_\mathfrak{b}(.,0)$ has a fixed point $\iff$ $\mathfrak{H}_\mathfrak{b}(.,1)$ has a fixed point.

Proof. Let $A = \{\kappa \in [0,1] : \sigma = \mathfrak{H}_{\mathfrak{b}}(\sigma, \kappa)$ for some $\sigma \in \mathcal{U}\}$. We have that $0 \in A$ since $\mathfrak{H}_{\mathfrak{b}}(.,0)$ has a fixed point in $\mathcal{U}$. As a result, the set A is not empty. By demonstrating that A is both open and closed in $[0,1]$, we shall establish that $A = [0,1]$. As a result, in $\mathcal{U}$, $H_b(,1)$ has a fixed point.First, we demonstrate that A is a closed set in $[0,1]$. Let $\{\kappa_n\}_{n=1}^{\infty} \subseteq A$ with $\kappa_n \to \kappa \in [0,1]$ as $n \to \infty$. We have to demonstrate that $\kappa \in A$. Since $\kappa_n \in A$ for $n = 0,1,2,\cdots$. there exists $\sigma_n \in \mathcal{U}$ with $\sigma_{n+1} = \mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n)$. Since $\mathfrak{H}_{\mathfrak{b}}$ is a triangular α-orbital admissible operator with respect to ϕ.
$\alpha(\sigma_0, \sigma_0, \mathfrak{H}_{\mathfrak{b}}(\sigma_0, \kappa_0)) \geq \phi(\sigma_0, \sigma_0, \mathfrak{H}_{\mathfrak{b}}(\sigma_0, \kappa_0))$. We can deduce from Lemma (2.1) that

$$\alpha(\sigma_n, \sigma_n, \sigma_{n+1}) \geq \phi(\sigma_n, \sigma_n, \sigma_{n+1}) \text{ for all } n \geq 0$$

Consider,

$$\begin{aligned}
S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) &= S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_{n+1})\right) \\
&\leq 2bS_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_n)\right) \\
&\qquad + b^2 S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_{n+1})\right) \\
&\leq 2bS_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_n)\right) + b^2 M_b|\kappa_n - \kappa_{n+1}|
\end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) \leq \lim_{n\to\infty} 2bS_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_n)\right)$$

. We get Γ since it is continuous and non-decreasing.

$$\begin{aligned}
&\lim_{n\to\infty} \Gamma\left((\frac{1+b^3}{4b}) S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) \right) \\
= &\lim_{n\to\infty} \Gamma\left((\frac{1+b^3}{2}) S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_n, \kappa_n), \mathfrak{H}_{\mathfrak{b}}(\sigma_{n+1}, \kappa_n)\right) \right) \\
\leq &\lim_{n\to\infty} \beta\left(\Gamma(S_b(\sigma_n, \sigma_n, \sigma_{n+1}))\right) \Gamma(S_b(\sigma_n, \sigma_n, \sigma_{n+1}))
\end{aligned}$$

Therefore,

$$\frac{\lim_{n\to\infty} \Gamma\left((\frac{1+b^3}{4b}) S_b(\sigma_{n+1}, \sigma_{n+1}, \sigma_{n+2}) \right)}{\lim_{n\to\infty} \Gamma(S_b(\sigma_n, \sigma_n, \sigma_{n+1}))} \leq \lim_{n\to\infty} \beta\left(\Gamma(S_b(\sigma_n, \sigma_n, \varsigma_{n+1}))\right) < 1.$$

In above inequality, we have $\lim_{n\to\infty} \beta\left(\Gamma(S_b(\sigma_n, \sigma_n, \sigma_{n+1}))\right) = 1$. Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} \Gamma(S_b(\sigma_n, \sigma_n, \sigma_{n+1})) = 0$ and so $\lim_{n\to\infty} S_b(\sigma_n, \sigma_n, \sigma_{n+1}) = 0$. It is now time to demonstrate $\{\sigma_n\}$, a S_b-Cauchy sequence in $(\mathfrak{P}, S_b)$. On the other hand, suppose $\{\sigma_n\}$ is not a S_b-Cauchy sequence. There is a monotone increasing sequence with $\epsilon > 0$ and Natural numbers with the property that $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$.

$$S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k}\right) \geq \epsilon \tag{5.1}$$

and

$$S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k - 1}\right) < \epsilon. \tag{5.2}$$

From (5.1) and (5.2), we have

$$\epsilon \leq S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k}\right) \leq 2bS_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{m_k+1}\right) + b^2 S_b\left(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}\right).$$

So that $(\frac{1+b^3}{2b^2})\epsilon \leq (\frac{1+b^3}{b})S_b\left(\sigma_{m_k}, \sigma_{m_k}, \sigma_{m_k+1}\right) + (\frac{1+b^3}{2})S_b\left(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}\right)$.
We get that by setting $k \to \infty$ and Γ is applied on both sides,

$$\begin{aligned}
\Gamma((\frac{1+b^3}{2b^2})\epsilon) &\leq \lim_{k\to\infty} \Gamma\left((\frac{1+b^3}{2})S_b\left(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}\right)\right)\\
&\leq \lim_{n\to\infty} \beta\left(\Gamma(S_b(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}))\right)\Gamma(S_b(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}))\\
&\leq \lim_{n\to\infty} \beta\left(\Gamma(S_b(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}))\right)\Gamma((\frac{1+b^3}{2b^2})\epsilon).
\end{aligned}$$

That is

$$1 \leq \lim_{n\to\infty} \beta\left(\Gamma(S_b(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}))\right) \Rightarrow \lim_{n\to\infty} \beta\left(\Gamma(S_b(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}))\right) = 1$$

. This leads to the result $\lim_{n\to\infty} S_b(\sigma_{m_k}, \sigma_{m_k}, \sigma_{n_k-1}) = 0$. and hence, $\lim_{n\to\infty} S_b(\sigma_{m_k+1}, \sigma_{m_k+1}, \sigma_{n_k}) = 0$.It contradicts itself. In the S_b-metric space $(\mathfrak{P}, S_b)$, the sequence $\{\sigma_n\}$ is a S_b-Cauchy sequence.The sequence $\{\sigma_n\} \to \nu \in (\mathfrak{P}, S_b)$ emerges from the completeness of $(\mathfrak{P}, S_b)$. $\lim_{n\to\infty} \sigma_{n+1} = \nu = \lim_{n\to\infty} \sigma_n$. Since $\{\sigma_n\}$ is a S_b-convergent sequence to ν in X and $\alpha(\nu, \nu, \nu) \geq \phi(\nu, \nu, \nu)$. Then to prove $\nu = \mathfrak{H}_{\mathfrak{b}}(\nu, \kappa)$. Now

$$\begin{aligned}
\Gamma\left(\frac{1}{2b}S_b(\mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \nu)\right) &\leq \lim_{n\to\infty} \inf \Gamma\left(S_b(\mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \mathfrak{H}_{\mathfrak{b}}(\sigma_n,\kappa))\right)\\
&\leq \lim_{n\to\infty} \inf \Gamma\left((\frac{1+b^3}{2})S_b(\mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \mathfrak{H}_{\mathfrak{b}}(\sigma_n,\kappa))\right)\\
&\leq \lim_{n\to\infty} \beta\left(\Gamma(S_b(\nu,\nu,\sigma_n))\right)\Gamma(S_b(\nu,\nu,\sigma_n)).
\end{aligned}$$

So that

$$\frac{\Gamma\left(\frac{1}{2b}S_b(\mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \nu)\right)}{\lim_{n\to\infty}\Gamma(S_b(\nu,\nu,\sigma_n))} \leq \lim_{n\to\infty} \beta\left(\Gamma(S_b(\nu,\nu,\sigma_n))\right)$$

That is $1 \leq \lim_{n\to\infty} \beta\left(\Gamma(S_b(\nu,\nu,\sigma_n))\right)$ implies $\lim_{n\to\infty} \beta\left(\Gamma(S_b(\nu,\nu,\sigma_n))\right) = 1$.

As a result, we obtain $\lim_{n\to\infty} \Gamma(S_b(\nu,\nu,\sigma_n)) = 0$ and hence $S_b(\mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \mathfrak{H}_{\mathfrak{b}}(\nu,\kappa), \nu) = 0$. Thus, it follows $\nu = \mathfrak{H}_{\mathfrak{b}}(\nu,\kappa)$. Thus $\kappa \in A$. Clearly, $[0,1]$ closes A. Let $\kappa_0 \in A$.Consequently, there is $\sigma_0 \in U$ such that $\sigma_0 = \mathfrak{H}_{\mathfrak{b}}(\sigma_0, \kappa_0)$. Due to the fact that $\mathcal{U}$is open, $r > 0$ exists such that $B_{S_b}(\sigma_0, r) \subseteq \mathcal{U}$. Choose $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ such that $|\kappa - \kappa_0| \leq \frac{1}{M^n} < \epsilon$.

Then, for $\overline{B_p(\sigma_0, r)} = \{\sigma \in \mathfrak{P} : S_b(\sigma, \sigma, \sigma_0) \leq r + b^2 S_b(\sigma_0, \sigma_0, \sigma_0)\}$. Now

$$\begin{aligned}
S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa), \mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa), \sigma_0)\right) &= S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa), \mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa), \mathfrak{H}_{\mathfrak{b}}(\sigma_0,\kappa_0)\right)\\
&\leq 2bS_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa), \mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa), \mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0)\right)\\
&\quad + b^2 S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0), \mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0), \mathfrak{H}_{\mathfrak{b}}(\sigma_0,\kappa_0)\right)\\
&\leq 2bM|\kappa - \kappa_0| + b^2 S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0), \mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0), \mathfrak{H}_{\mathfrak{b}}(\sigma_0,\kappa_0)\right)
\end{aligned}$$

Upon letting $n \to \infty$ and applying Γ to both sides,

$$\begin{aligned}\Gamma\left(S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa),\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa),\sigma_0\right)\right) &\leq \Gamma\left(b^2 S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0),\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0),\mathfrak{H}_{\mathfrak{b}}(\sigma_0,\kappa_0)\right)\right)\\ &\leq \Gamma\left(\left(\frac{1+b^3}{2}\right)S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0),\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa_0),\mathfrak{H}_{\mathfrak{b}}(\sigma_0,\kappa_0)\right)\right)\\ &\leq \beta\left(\Gamma(S_b(\sigma,\sigma,\sigma_0))\right)\Gamma(S_b(\sigma,\sigma,\sigma_0)) \leq \Gamma(S_b(\sigma,\sigma,\sigma_0)).\end{aligned}$$

Therefore, $S_b\left(\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa),\mathfrak{H}_{\mathfrak{b}}(\sigma,\kappa),\sigma_0\right) \leq S_b(\sigma,\sigma,\sigma_0) \leq r + b^2 S_b(\sigma_0,\sigma_0,\sigma_0)$. Thus for each fixed $\kappa \in (\kappa_0-\epsilon,\kappa_0+\epsilon)$, $\mathfrak{H}_{\mathfrak{b}}(.;\kappa) : \overline{B_p(\sigma_0,r)} \to \overline{B_p(\sigma_0,r)}$. Thus, Theorem (5.1)'s criteria are met in full. Consequently, it may be said that $\mathfrak{H}_{\mathfrak{b}}(.;\kappa)$ has a fixed point in $\overline{\mathcal{U}}$. But this must be in $\mathcal{U}$. Therefore, $\kappa \in A$ for $\kappa \in (\kappa_0-\epsilon,\kappa_0+\epsilon)$. Hence $(\kappa_0-\epsilon,\kappa_0+\epsilon) \subseteq A$. In $[0,1]$, A is clearly open. The converse can be proven using a similar method. □

Conclusion: Using generalised (α,ϕ,Γ)-Geraghty contractive type fixed point theorems in the setup of S_b-metric spaces through α-orbital admissible mappings with respect to ϕ, we conclude several applications to homotopy theory and integral equations in this study.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Banach, Sur les Operations dans les Ensembles Abstraits et Leur Applications aux Equations Integrales, Fund. Math. 3 (1922), 133-181. https://eudml.org/doc/213289.

[2] S. Radenovic, S. Sedghi, A. Gholidahneh, T. Dosenovic, J. Esfahani, Common fixed point of four maps in -Metric spaces, J. Linear Topol. Algebra, 5 (2016), 93-104.

[3] N. Souayaha, N. Mlaikib, A fixed point theorem in S_b-metric spaces, J. Math. Computer Sci. 16 (2016), 131-139. https://doi.org/10.22436/jmcs.016.02.01.

[4] M.V.R. Kameswari, K. Sujatha, D.M.K Kiran, Fixed Point Results in S_b-Metric Spaces via (α,ψ,ϕ)- Generalized Weakly Contractive Maps, Glob. J. Pure Appl. Math. 15 (2019), 411-428.

[5] N. Souayah, A Fixed Point in Partial S_b-Metric Spaces, An. Univ. "Ovidius" Constanta - Ser. Mat. 24 (2016), 351-362. https://doi.org/10.1515/auom-2016-0062.

[6] Y. Rohen, T. Dosenovic, S. Radenovic, A Note on the Paper "A Fixed Point Theorems in S_b-Metric Spaces", Filomat. 31 (2017), 3335-3346. https://doi.org/10.2298/fil1711335r.

[7] B. Srinuvasa, G.N. Venkata Ki, M. Sarwar, N. Konda Redd, Fixed Point Theorems in Ordered S_b-Metric Spaces by Using (α,β)-Admissible Geraghty Contraction and Applications, J. Appl. Sci. 18 (2017), 9-18. https://doi.org/10.3923/jas.2018.9.18.

[8] G. Kishore, K. Rao, D. Panthi, B. Srinuvasa Rao, S. Satyanaraya, Some Applications via Fixed Point Results in Partially Ordered S_b-Metric Spaces, Fixed Point Theory Appl. 2017 (2016), 10. https://doi.org/10.1186/s13663-017-0603-2.

[9] M.A. Geraghty, On Contractive Mappings, Proc. Amer. Math. Soc. 40 (1973), 604-608.

[10] A. Amini-Harandi, H. Emami, A Fixed Point Theorem for Contraction Type Maps in Partially Ordered Metric Spaces and Application to Ordinary Differential Equations, Nonlinear Anal.: Theory Methods Appl. 72 (2010), 2238-2242. https://doi.org/10.1016/j.na.2009.10.023.

[11] M.M.M. Jaradat, Z. Mustafa, A.H. Ansari, P.S. Kumari, D. Dolicanin-Djekic, H.M. Jaradat, Some Fixed Point Results for $F_{\alpha-\omega,\varphi}$-Generalized Cyclic Contractions on Metric-Like Space With Applications to Graphs and Integral Equations, J. Math. Anal. 8 (2017), 28-45.

[12] J.R. Roshan, V. Parvaneh, S. Radenovic, M. Rajovic, Some Coincidence Point Results for Generalized (α, β)-Weakly Contractions in Ordered b-Metric Spaces, Fixed Point Theory Appl. 2015 (2015), 68. https://doi.org/10.1186/s13663-015-0313-6.

[13] M. Dinarvand, Some Fixed Point Results for Admissible Geraghty Contraction Type Mappings in Fuzzy Metric Spaces, Iran. J. Fuzzy Syst. 14 (2017), 161-177.

[14] S.H. Cho, J.S. Bae, E. Karapinar, Fixed Point Theorems for α-Geraghty Contraction Type Maps in Metric Spaces, Fixed Point Theory Appl. 2013 (2013), 329. https://doi.org/10.1186/1687-1812-2013-329.

[15] E. Karapinar, $\alpha - \psi$-Geraghty Contraction Type Mappings and Some Related Fixed Point Results, Filomat. 28 (2014), 37-48. https://www.jstor.org/stable/24896705.

[16] A. Mukheimer, $\alpha - \psi - \varphi$-Contractive Mappings in Ordered Partial b-Metric Spaces, J. Nonlinear Sci. Appl. 7 (2014), 168-179.

[17] O. Popescu, Some New Fixed Point Theorems for α-Geraghty Contraction Type Maps in Metric Spaces, Fixed Point Theory Appl. 2014 (2014), 190. https://doi.org/10.1186/1687-1812-2014-190.

[18] P. Salimi, A. Latif, N. Hussain, Modified $\alpha - \psi$-Contractive Mappings With Applications, Fixed Point Theory Appl. 2013 (2013), 151. https://doi.org/10.1186/1687-1812-2013-151.

[19] P. Chuadchawna, A. Kaewcharoen, S. Plubtieng, Fixed Point Theorems for Generalized $\alpha - \eta - \psi$-Geraghty Contraction Type Mappings in $\alpha - \eta$-Complete Metric Spaces, J. Nonlinear Sci. Appl. 09 (2016), 471-485. https://doi.org/10.22436/jnsa.009.02.13.

[20] A. Farajzadeh, C. Noytaptim, A. Kaewcharoen, Some Fixed Point Theorems for Generalized $\alpha - \eta - \psi$-Geraghty Contractive Type Mappings in Partial b-Metric Spaces, J. Inform. Math. Sci. 10 (2018), 455–478.