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A Congruent Property of Gibonacci Number Modulo Prime

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Abstract. Let $a, b \in \mathbb{Z}$ and p be a prime number such that a and b are not divisible by p. In this work, we give a congruent property modulo a prime number p of the gibonacci number defined by $G_n = G_{n-1} + G_{n-2}$ with initial condition $G_1 = a, G_2 = b$. We show that a the gibonacci sequence satisfying $G_{kp-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod{p}$ for all positive integer k and such odd prime $p \neq 5$ if and only if $a \equiv b \pmod{p}$. Moreover, for each odd prime number p, we give a necessary and sufficient condition yielding $G_{kp-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod{p}$. We also find a relation between the sequences in the same equivalent class in modulo 5 constructed by Aoki and Sakai [1] that leads to such congruent property.

1. Introduction

The *Fibonacci sequence* $\{F_n\}_{n\geq 0}$ satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial condition $F_0 = 0, F_1 = 1$ and the *Lucas sequence* $\{L_n\}_{n\geq 0}$ satisfies the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial condition $L_0 = 2, L_1 = 1$. The sequences can be extended to a negative index as follows: for $n \in \mathbb{N}$

$$F_{-n} = (-1)^{n+1} F_n, \tag{1.1}$$

$$L_{-n} = (-1)^n L_n. \tag{1.2}$$

We see that both the Fibonacci and Lucas sequences satisfy the same recurrence relation with different initial conditions. To generalized the mentioned sequences, the *generalized Fibonacci sequence* or *gibonacci sequence* $\{G_n\}_{n>0} = \{G(a, b)\}$ ([2], p.137) is defined to satisfy the recurrence relation $G_n = G_{n-1} + G_{n-2}$, for $n \geq 3$, with initial condition $G_1 = a$ and $G_2 = b$, where $a, b \in \mathbb{Z}$.

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Theorem 1.1 ([2], p.137)**.** *For an integer $n \geq 3$, the n-th gibonacci number satisfies*

$$G_n = G_1 F_{n-2} + G_2 F_{n-1}.$$

Theorem 1.2 ([2], p.137)**.** *For $n \in \mathbb{N}$, we have*

$$G_{-n} = (-1)^{n+1}(G_1 F_{n+2} - G_2 F_{n+1}).$$

The objective of this work is investigating the congruent property of the gibonacci sequence $\{G_n\} = \{G(a,b)\}$ where $a, b \in \mathbb{Z}$ that $p \nmid a$ and $p \nmid b$ analogous to the result from Andrica et. al. [3] appearing in Theorem 1.3. Throughout this article, we let p be an odd prime number and $\left(\frac{p}{5}\right)$ be the Legendre's symbol.

Theorem 1.3. *[3] For a positive integer k and an odd prime number $p \neq 5$, we have*

$$F_{kp-\left(\frac{p}{5}\right)} \equiv F_{k-1} \pmod p,$$
$$L_{kp-\left(\frac{p}{5}\right)} \equiv L_{k-1} \pmod p.$$

We note that Theorem 1.3 is not true when $p = 5$ as $F_{10} \not\equiv F_1 \pmod 5$. As a result, we give a necessary and sufficient condition in terms of the initial condition of the gibonacci sequence and its index k that lead to

$$G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p \tag{1.3}$$

for each prime p characterized by the value of $\left(\frac{p}{5}\right)$. We also give a necessary and sufficient condition resulting in (1.3) when $\left(\frac{p}{5}\right) = -1$ in Theorem 3.1. In Theorem 3.2, we show that if $\left(\frac{p}{5}\right) = 1$, then (1.3) holds for all positive integer k. By combining Theorems 3.1 and 3.2, we show that for a gibonacci sequence $\{G_n\}_{n>0} = \{G(a,b)\}$, (1.3) holds for all $k \in \mathbb{N}$ and for all odd prime number $p \neq 5$ where $p \nmid a$ and $p \nmid b$ if and only if $a \equiv b \pmod p$ in Theorem 3.3.

For the case that $p = 5$, we consider the equivalent class X_5 introduced by Aoki and Sakai [1]. For a prime number p, Aoki and Sakai constructed an equivalent class of the gibonacci sequences

$$X_p = \{\{G_n\} | \{G_n\} \text{ is the gibonacci sequence, where } p \nmid G_1 \text{ and } p \nmid G_2\} / \sim$$

where,

$$\{G_n\} \sim \{G'_n\} \text{ if and only if } G_2 G_1^{-1} \equiv G'_2 G_1'^{-1} \pmod p, \tag{1.4}$$

and G^{-1} is the inverse of G modulo p where $1 \leq G^{-1} < p$. They also showed that

$$X_p = \{\overline{\{G(1,k)\}} | 1 \leq k \leq p-1\}. \tag{1.5}$$

In Theorem 3.4, we consider the representation $\{G(1,h)\}$ of each class in X_5, where $1 \leq h \leq 4$ and give a complete characterization of the initial conditions of a gibonacci sequences and the corresponding indices that (1.3) holds. Later in Theorems 3.5 and 3.6, we give a relation of the sequences in the same class in X_5.

p	$\pi(p)$	$\left(\frac{p}{5}\right)$
3	8	-1
5	20	0
7	16	-1
11	10	1
13	28	-1
17	36	-1
19	18	1
23	48	-1
29	14	1
37	76	-1
43	88	-1

Table 1. List of the Pisono period and the Legendre's symbol of a prime number.

2. Preliminaries

In this section, we give an overview of the related work that will be used to prove the main results.

For any positive integer m, the *Pisano period* [4] modulo m is the period of the Fibonacci number modulo m, denoted by $\pi(m)$. In 2012, Gupta et. al, [5] gave a method to find a period of the Fibonacci number modulo a prime number.

Theorem 2.1. *[4] Let p be a prime number.*

- *If $p \equiv \pm 1 \pmod 5$, then $\pi(p) | (p-1)$.*
- *If $p \equiv \pm 2 \pmod 5$, then $\pi(p) | 2(p+1)$.*

The values of $\pi(p)$ and $\left(\frac{p}{5}\right)$ listed in Table 1 appear in [7] and [8], respectively. In Lemma 3.1, we show that the period of the gibonacci number modulo p is at most $\pi(p)$ which leads to to computation appearing in Table 2.

Lemma 2.1. *[1] Let p be an odd prime number. The following statements are true.*

(1) *If $\left(\frac{p}{5}\right) = 1$, then $p \nmid G_n$ for any $n \in \mathbb{N}$.*
(2) *If $\left(\frac{p}{5}\right) = -1$, then $p | G_n$ for some $n \in \mathbb{N}$.*

The following results are some identities of the Fibonacci and the Lucas sequences that will be used in this work.

Theorem 2.2. *([2], p. 93) For each $n \in \mathbb{N}$,*

$$L_n = F_{n+1} + F_{n-1},$$

$$5F_n = L_{n+1} + L_{n-1}$$

Theorem 2.3. *([2], p. 462) Lucas number is not divisible by 5.*

Theorem 2.4. *[3] For an odd prime number p, a positive integer k and an integer r, the following holds:*

$$2F_{kp+r} \equiv \left(\frac{p}{5}\right) F_k L_r + F_r L_k \pmod{p}, \tag{2.1}$$

$$2L_{kp+r} \equiv 5\left(\frac{p}{5}\right) F_k F_r + L_k L_r \pmod{p}. \tag{2.2}$$

The following corollary is a direct result of Theorem 2.4.

Corollary 2.1. *For a positive integer k and r, we have*

$$F_{5k-r} \equiv 3F_{-r} L_k \pmod{5}, \tag{2.3}$$

$$L_{5k} \equiv L_k \pmod{5}. \tag{2.4}$$

Theorem 2.5. *[3] For an odd prime number p and a positive integer k, we have*

$$F_{kp} \equiv \left(\frac{p}{5}\right) F_k \pmod{p},$$
$$F_p \equiv \left(\frac{p}{5}\right) \pmod{p},$$
$$F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p}.$$

Theorem 2.6. *[6] Let $n, k \in \mathbb{Z}$. If k is an even number, then*

$$F_{n+k} + F_{n-k} = F_n L_k, \tag{2.5}$$

$$F_{n+k} - F_{n-k} = F_k L_n. \tag{2.6}$$

If k is an odd number, then

$$F_{n+k} + F_{n-k} = F_k L_n, \tag{2.7}$$

$$F_{n+k} - F_{n-k} = F_n L_k. \tag{2.8}$$

3. Main Results

The following property of the gibonacci sequence can be obtained directly from Theorem 1.1; however, the authors do not find this result in the literature review.

Lemma 3.1. *Let $\{G_n\}_{n>0} = \{G(a, b)\}$, where $a, b \in \mathbb{Z}$. For $k, r \in \mathbb{Z}$, we have*

$$G_{k\pi(p)+r} \equiv G_r \pmod{p}$$

Proof. By Theorem 1.1, we have that

$$\begin{aligned} G_{k\pi(p)+r} &\equiv G_1F_{k\pi(p)+r-2} + G_2F_{k\pi(p)+r-1} \pmod p \\ &\equiv G_1F_{r-2} + G_2F_{r-1} \pmod p \\ &\equiv G_r \pmod p. \end{aligned}$$

□

Next, we consider each case of an odd prime p characterized by the value of $\left(\frac{p}{5}\right)$ and give a necessary and sufficient condition resulting to $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$.

Theorem 3.1. *Let p be an odd prime number that $\left(\frac{p}{5}\right) = -1$ and $\{G_n\}_{n>0} = \{G(a,b)\}$ be such that a and b are not divisible by p. For $k \in \mathbb{N}$, we have that $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$ if and only if one of the following holds:*

(1) $G_1 \equiv G_2 \pmod p$,
(2) $L_{k-1} \equiv 0 \pmod p$.

Proof. By Theorems 1.1, 1.3, 2.4, 2.5 and 2.6, we have

$$\begin{aligned} G_{pk-\left(\frac{p}{5}\right)} &= G_{pk+1} \\ &= aF_{pk-1} + bF_{pk} \\ &\equiv 2^{-1}a(-F_kL_{-1} + F_{-1}L_k) - bF_k \pmod p \\ &\equiv aF_{k+1} - bF_k \pmod p. \end{aligned} \tag{3.1}$$

It follows from Theorem 2.2 that

$$\begin{aligned} G_{pk-\left(\frac{p}{5}\right)} - G_{k-1} &\equiv a(F_{k+1} - F_{k-3}) - b(F_k - F_{k-2}) \pmod p \\ &\equiv (a-b)L_{k-1} \pmod p. \end{aligned}$$

Hence, $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$ if and only if $a \equiv b \pmod p$ or $L_{k-1} \equiv 0 \pmod p$. □

By Theorem 3.1, the listed p and k in Table 2 yield $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$.

Corollary 3.1. *Let p be an odd prime number where $\left(\frac{p}{5}\right) = -1$ and $\{G_n\}_{n>0} = \{G(a,b)\}$ where a and b are integers that are not divisible by p. Then $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$ for all $k \in \mathbb{N}$ if and only if $a \equiv b \pmod p$.*

We note that, by (3.1), if $\{G_n\}_{n>0} = \{G(1,h)\}$ where $1 \le h \le p-1$, then

$$G_{pk-\left(\frac{p}{5}\right)} \equiv (F_{k+1} + F_{k-1}) - (F_{k-1} + hF_k) \equiv L_k - G_{k+1} \pmod p. \tag{3.2}$$

Hence, if $\left(\frac{p}{5}\right) = -1$, then

$$G_{pk-\left(\frac{p}{5}\right)} + G_{k-\left(\frac{p}{5}\right)} \equiv L_k \pmod p. \tag{3.3}$$

Prime p	$k \pmod{\pi(p)}$
3	$3 \pmod 8$ $7 \pmod 8$
7	$5 \pmod{16}$ $13 \pmod{16}$
13	-
17	-
23	$13 \pmod{48}$ $37 \pmod{48}$
37	-
43	$23 \pmod{88}$ $67 \pmod{88}$

Table 2. List of p and k where $\left(\frac{p}{5}\right) = -1$ and $p|L_{k-1}$.

Theorem 3.2. *Let p be an odd prime number and $\{G_n\}_{n>0} = \{G(a,b)\}$ where a and b are integers that are not divisible by p. If $\left(\frac{p}{5}\right) = 1$, then $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$, for all $k \in \mathbb{N}$.*

Proof. By Theorem 1.1, 2.4 and 2.6, it follows that

$$\begin{aligned} G_{pk-\left(\frac{p}{5}\right)} &= G_{pk-1} \\ &= aF_{pk-3} + bF_{pk-2} \\ &\equiv 2^{-1}a(F_kL_{-3} + F_{-3}L_k) + 2^{-1}b(F_kL_{-2} + F_{-2}L_k) \pmod p \\ &\equiv aF_{k-3} + bF_{k-2} \pmod p \\ &\equiv G_{k-1} \pmod p. \end{aligned}$$

This completes the proof. □

Corollary 3.2. *If p is an odd prime number such that $\left(\frac{p}{5}\right) = 1$, then $G_{p^ik-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$, for all $i, k \in \mathbb{N}$.*

The following theorem is a direct result of Theorems 3.1 and 3.2.

Theorem 3.3. *Let $\{G_n\}_{n>0} = \{G(a,b)\}$ be such that $a, b \in \mathbb{Z}$. Then $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod p$ for all positive integers k and odd prime numbers $p \neq 5$ such that $p \nmid a$, $p \nmid b$ if and only if $a \equiv b \mod p$.*

Next, we consider the case that $p = 5$. Firstly, in Theorem 3.4, we give a complete characterization of the initial condition of the gibonacci sequence $\{G_n\}_{n>0} = \{G(1,h)\}$ where $1 \leq h \leq 4$ and the values of k where (1.3) holds. Then, we give a relation of such congruent property of the sequences in the same equivalent class in X_5.

Theorem 3.4. *Let $\{G_n\}_{n>0} = \{G(1,h)\}$ be such that $1 \le h \le 4$. If $p = 5$ and $k \in \mathbb{N}$, then $G_{pk-\left(\frac{p}{5}\right)} \equiv G_{k-1} \pmod{p}$ if and only if one of the following holds:*

(1) $\{G_n\} = \{G(1,4)\}$ *and* $k \equiv 0 \pmod 5$

(2) $\{G_n\} = \{G(1,1)\}$ *and* $k \equiv 1 \pmod 5$

(3) $\{G_n\} = \{G(1,2)\}$ *and* $k \equiv 3 \pmod 5$.

Proof. Since $p = 5$, we have $\left(\frac{p}{5}\right) = 0$. By Theorem 1.1, we have

$$G_{pk-\left(\frac{p}{5}\right)} = G_{5k} = F_{5k-2} + hF_{5k-1}. \tag{3.4}$$

Let $q, r \in \mathbb{Z}$ be such that $k = 5q + r$, where $0 \le r \le 4$. By Corollary 2.1 and (3.4),

$$\begin{aligned}
G_{5k} &\equiv F_{5k-2} + hF_{5k-1} \pmod 5\\
&\equiv 3\left(F_{-2}L_{5k} + hF_{-1}L_{5k}\right) \pmod 5\\
&\equiv 3(-L_{5k} + hL_{5k}) \pmod 5\\
&\equiv 3L_k(-1+h) \pmod 5\\
&\equiv 3(-1+h)L_{5q+r} \pmod 5\\
&\equiv 3(-1+h)2^{-1}L_qL_r \pmod 5, \qquad \text{by Theorem 2.4,}\\
&\equiv (1-h)L_qL_r \pmod 5.
\end{aligned}$$

Similarly

$$\begin{aligned}
G_{k-1} &\equiv F_{k-3} + hF_{k-2} \pmod 5\\
&\equiv F_{5q+r-3} + hF_{5q+r-2} \pmod 5\\
&\equiv 2^{-1}\left(F_{r-3}L_{5q} + hF_{r-2}L_{5q}\right) \pmod 5\\
&\equiv 3L_q(F_{r-3} + hF_{r-2}) \pmod 5.
\end{aligned}$$

Thus $G_{5k} \equiv G_{k-1} \pmod 5$ if and only if

$$(1-h)L_qL_r \equiv 3L_q(F_{r-3} + hF_{r-2}) \pmod 5.$$

By Theorem 2.3, we have

$$(1-h)L_r \equiv 3(F_{r-3} + hF_{r-2}) \pmod 5. \tag{3.5}$$

If $r = 4$, then (3.5) does not hold. By a direct computation $5 \nmid (3F_{r-2} + L_r)$ for all $r \in \{0,1,2,3\}$, we have

$$h \equiv (3F_{r-2} + L_r)^{-1}(L_r - 3F_{r-3}) \pmod 5. \tag{3.6}$$

If $r = 2$, then $h \equiv 0 \pmod 5$ contradiction. By a direct computation, the above equation holds if and only if $(r,h) \in \{(0,4),(1,1),(3,2)\}$. This completes the proof. $\square$

For $\{G_n\}_{n>0} = \{G(a, b)\}$ where $a, b \in \mathbb{Z}$ that a and b are not divisible by p, let

$$\delta(a) = \begin{cases} 0 & \text{if } a \equiv 1, 4 \pmod 5 \\ -1 & \text{if } a \equiv 2 \pmod 5 \\ 1 & \text{if } a \equiv 3 \pmod 5. \end{cases}$$

Theorem 3.5. *Let $\{G_n\}_{n>0} = \{G(1, h)\}$ and $\{G'_n\}_{n>0} = \{G(a, b)\}$, where $1 \le h \le 4$ and $a, b \in \mathbb{Z}$ be such that $\{G_n\} \sim \{G'_n\}$. Then $G'_k \equiv a^{-1}G_k + \delta(a)F_{k-2} \pmod 5$ for all $k \in \mathbb{N}$.*

Proof. Since $\{G_n\} \sim \{G'_n\}$, we have $b \equiv G'_2G_1^{-1} \equiv G_2(G'_1)^{-1} \equiv ha^{-1} \pmod 5$. For the case that $k = 1, 2$, we can compute the result directly. For $k > 2$, we have

$$\begin{aligned} G'_k &\equiv aF_{k-2} + bF_{k-1} \pmod 5 \\ &\equiv aF_{k-2} + ha^{-1}F_{k-1} \pmod 5. \end{aligned}$$

If $a \equiv 1 \pmod 5$ or $a \equiv 4 \pmod 5$, then $a \equiv a^{-1} \pmod 5$. Hence,

$$G'_k \equiv a^{-1}G_k \pmod 5. \tag{3.7}$$

Otherwise,

$$G'_k \equiv \begin{cases} a^{-1}G_k - F_{k-2} & \text{if } G'_1 \equiv 2 \pmod 5, \\ a^{-1}G_k + F_{k-2} & \text{if } G'_1 \equiv 3 \pmod 5. \end{cases}$$

Therefore, we have $G'_k \equiv a^{-1}G_k + \delta(a)F_{k-2} \pmod 5$ for all positive integer k. $\square$

Theorem 3.6. *Let $\{G_n\}_{n>0} = \{G(1, h)\}$ where $1 \le h \le 4$ and k be a positive integer satisfying $G_{5k} \equiv G_{k-1} \pmod 5$. Let $\{G'_n\}_{n>0} = \{G(a, b)\}$ be such that a and b are integers that are not divisible by 5. If $\{G_n\} \sim \{G'_n\}$ in X_5, then $G'_{5k} \equiv G'_{k-1} \pmod 5$ if and only if $a \equiv a^{-1} \pmod 5$.*

Proof. Let $\delta = \delta(a)$. So

$$\begin{aligned} G'_{5k} - G'_{k-1} &\equiv \left(a^{-1}G_{5k} + \delta F_{5k-2}\right) - \left(a^{-1}G_{k-1} + \delta F_{k-3}\right) \pmod 5 \\ &\equiv a^{-1}(G_{5k} - G_{k-1}) + \delta(F_{5k-2} - F_{k-3}) \pmod 5 \\ &\equiv a^{-1}(G_{5k} - G_{k-1}) + \delta(3F_{-2}L_k - F_{k-3}) \pmod 5 \\ &\equiv a^{-1}(G_{5k} - G_{k-1}) + \delta(3(F_{k-2} - F_{k+2}) - F_{k-3}) \pmod 5 \\ &\equiv a^{-1}(G_{5k} - G_{k-1}) + \delta(3((F_k - F_{k-1}) - (F_{k+1} + F_k)) - F_{k-3}) \pmod 5 \\ &\equiv a^{-1}(G_{5k} - G_{k-1}) + \delta(2(F_{k-1} + F_{k+1}) - (F_{k-1} - F_{k-2})) \pmod 5 \\ &\equiv a^{-1}(G_{5k} - G_{k-1}) + \delta F_{k+3} \pmod 5. \end{aligned}$$

Since $G_{5k} \equiv G_{k-1} \pmod 5$, it follows that $G'_{5k} \equiv G'_{k-1} \pmod 5$ if and only if $5|\delta F_{k+3}$. So $\delta = 0$ or $k \equiv 2 \pmod 5$. By Theorem 3.4, since $G_{5k} \equiv G_{k-1} \pmod 5$, we have that $k \not\equiv 2 \pmod 5$. Hence $\delta = 0$ and it follows that $a \equiv a^{-1} \pmod 5$. So the equation holds if and only if $a \equiv a^{-1} \pmod 5$. $\square$

By Theorem 3.4 and 3.6, we have the following corollaries.

Corollary 3.3. *Let* $\{G_n\} = \{G(1,h)\}$ *where* $1 \leq h \leq 4$. *Let* $\{G'_n\} = \{G(a,b)\}$ *be such that* $\{G_n\} \sim \{G'_n\}$, *where* $a, b \in \mathbb{Z}$ *and* $G'_1 \equiv 1, 4 \pmod 5$. *If* $(h,r) \in \{(4,0),(1,1),(2,3)\}$ *where* $k \equiv r \pmod 5$ *and* $0 \leq r \leq 4$, *then we have* $G'_{5k} \equiv G'_{k-1} \pmod 5$.

Corollary 3.4. *Let* $\{G_n\} = \{G(1,h)\}$ *where* $1 \leq h \leq 4$. *Let* $\{G'_n\} = \{G(a,b)\}$ *be such that* $\{G_n\} \sim \{G'_n\}$, *where where* $a, b \in \mathbb{Z}$ *and* $G'_1 \equiv 2, 3 \pmod 5$. *If* $(h,r) \in \{(4,0),(1,1),(2,3)\}$ *where* $k \equiv r \pmod 5$ *and* $0 \leq r \leq 4$, *then we have* $G'_{5k} \not\equiv G'_{k-1} \pmod 5$.

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