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Common Divisor Graph of Finite Groups

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Abstract. The interaction between groups and graphs is the most popular and gainful area of algebraic graph theory. We introduce and scrutinize a graph whose vertex set is a group H and the vertices a and b with different orders are adjacent if $c \notin \{e, a, b\}$ exists such that $|c| | (|a|, |b|)$ or only one of a or b is the identity. We investigate this graph by presenting various examples and demonstrating some important properties and results.

1. Introduction

Any finite group can be modeled using the automorphism group of a connected graph [3]. A group's graphical representation can be created using a collection of generators and relations. Given any group, Cayley graphs are symmetrical graphs that exhibit characteristics related to the group's structure [4]. By connecting two significant areas of mathematics and tying a graph to a group, it is possible to visualize groups. The main analysis models for examining an object's symmetries are groups, and symmetries are frequently connected to graph automorphisms. Due to its numerous applications in areas including operations research, electrical engineering, computer science, and biology, graph theory is one of the most active research areas in mathematics. These two mathematical subfields are significant in contemporary mathematics. While graph theory focuses on the graphs that represent the structure of materials and objects, group theory examines and interprets various groups and their organizational structures. Many important and well-known theorems in mathematics, including group theory, have been proven using the potent combinatorial techniques of graph theory. The study of algebraic structures utilizing graph characteristics has grown in popularity over the past twenty years, producing a wealth of intriguing findings and issues (see [1], [5], and [6]). In this study, we examine a finite group-related graph.

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2. Results

Definition 2.1. *Consider the finite group H. The common divisor graph $CD(H)$ is a graph whose vertex set is H and if $a, b \in H$ with a different order, a is adjacent to b if there exists $c \notin \{e, a, b\}$ such that $|c| \mid (|a|, |b|)$ or only one of a or b is the identity.*

In these new graphs, we give some samples.

Example 2.1. *If $H = D_4$, after which the graph $CD(H)$ is*

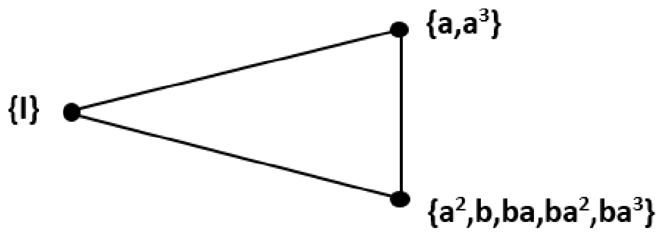

Figure 1. D_4

Example 2.2. *If $H = D_3 \times \mathbb{Z}_5$, after which the common divisor graph $CD(H)$ is*

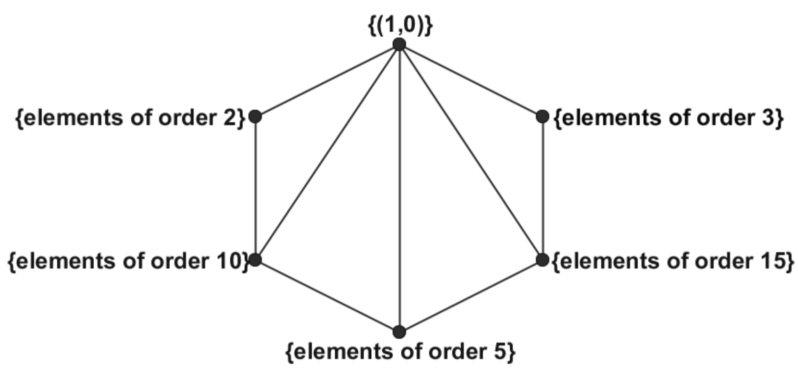

Figure 2. $D_3 \times \mathbb{Z}_5$

Example 2.3. *If $H = \mathbb{Z}_6$, after which the the Figure of $CD(H)$ is*

Example 2.4. *If $H = \mathbb{Z}_2 \times \mathbb{Z}_3$, then the Figure of $CD(H)$ is*

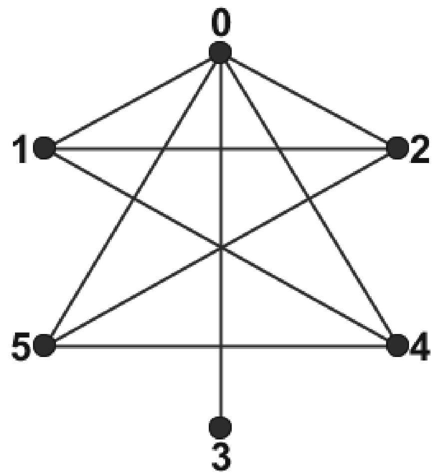

Figure 3. $\mathbb{Z}_6$

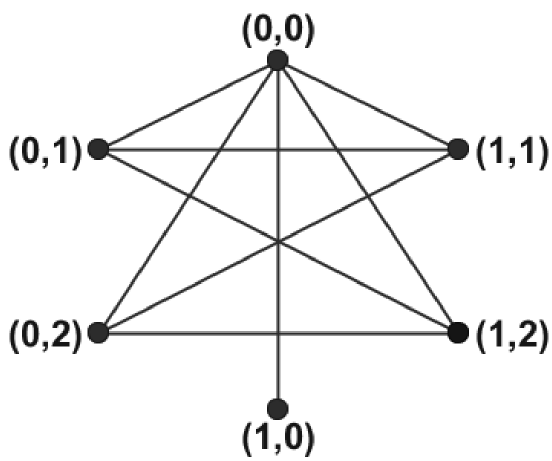

Figure 4. $\mathbb{Z}_2 \times \mathbb{Z}_3$

The graphs in Examples 2.3 and 2.4 are isomorphic, as you may have noticed.

Proposition 2.1. *Let H and K be isomorphic groups. Then $CD(H) \cong CD(K)$ as graphs.*

Proof. Assume that $f : H \to K$ is an isomorphism. Since the isomorphism of finite groups preserve the order of elements, so this isomorphism induces an isomorphism $\alpha : CD(H) \to CD(K)$ where $\alpha(a) = f(a)$ for all $a \in H$. □

A graph is said to be multipartite if its graph vertices may be partitioned into several of these separate sets, in the situation that there are no adjacent vertices in the same set.

Proposition 2.2. *Suppose H is a finite group. Following that, $CD(H)$ is a multipartite graph.*

Proof. Suppose H is a finite group. It is possible to partition the group H into subsets, with the elements in each subset having the same order and the elements in different subsets having a different order. Then there aren't any adjacent elements in the same set. As a result, $CD(H)$ is a multipartite graph.

□

Every pair of vertices from several independent sets must be connected by an edge for a multipartite graph to be considered complete. In the following examples, $CD(H)$ for a given group H are complete multipartite graphs.

Example 2.5. *If $H = \mathbb{Z}_9$, then the graph $CD(H)$ is*

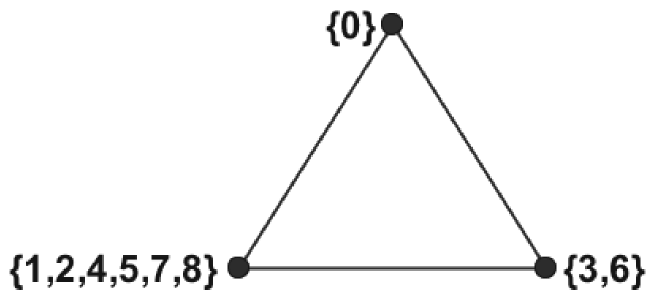

FIGURE 5. $\mathbb{Z}_9$

Example 2.6. *If $H = \mathbb{Z}_{27}$, then the graph $CD(H)$ is*

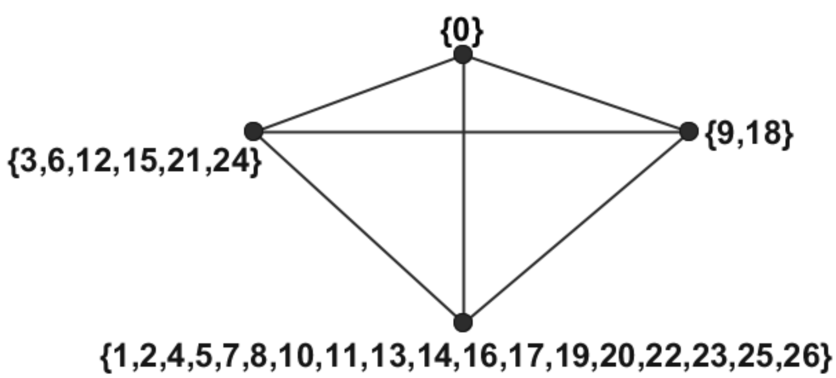

FIGURE 6. $\mathbb{Z}_{27}$

Proposition 2.3. *Let H be a finite group. Then the common divisor graph $CD(H)$ is a non-star complete multipartite graph if and only if either H is 2-group with more than one element of order 2 or H is p-group with p is an odd prime.*

Proof. Let $CD(H)$ be a multipatite non-star complete graph. Since elements of co-prime order are not adjacent, it is a contradiction if the order of H includes two distinct prime divisors, p and q, leading to elements of prime order p and q, respectively. H is therefore a p-group. Assume that $p = 2$ and that H only contains one element of order 2, in which case this element is adjacent to none of the other elements of H, which is contradictory. As a result, H contains many elements of order 2. Because of this, H is either a 2-group with multiple elements of order 2 or a p-group with p being an odd prime.

On the other hand, suppose that H is a p-group of order p^n, where p is an odd prime and n is a positive integer. If $a \in H$ has an order p^r, then the order p^r elements are multiple. In fact, since $a \in H$ is of p^r, $|a^2| = p^r$. The situation where $|H| = 2^n$ and H contains several elements of order 2. If $a \in H\backslash\{e\}$ is of order 2^r for some positive integer $1 < r \leq n$, then $a^3 \neq a$ and $|a^3| = 2^r$. Therefore, $CD(H)$ is a complete multipartite graph. □

Take note that every non-cyclic abelian group has more than one member of order 2 according to the classification theory of finite abelian groups. Consequently, we have the following:

Corollary 2.1. *The multipatite graph $CD(H)$ is complete if H is a noncyclic abelian group of order 2^n.*

Example 2.7. *Consider the Quaternion group $\mathcal{Q}$, $\mathcal{Q} = <\{a, b|a^4 = e, a^2 = b^2, ba = a^{-1}b\}>$. $\mathcal{Q}$ is a nonabelian group which has only one elements of order 2 (a^2). Thus, the common divisor graph of $\mathcal{Q}$ is a star graph.*

Theorem 2.1. *Consider the finite group H. Consequently, the following statements are valid:*

(1) *If all elements in H are of the same order (moreover, H is an elementary abelian group), then $CD(H)$ is a star graph.*
(2) *In the case where H is a 2-group and there is only one element of order 2 and no elements of order higher than 4, $CD(H)$ is a star graph.*
(3) *For any non-cyclic group H of order pq, the graph $CD(H)$ is a star.*

Proof.
(1) If every element in H is in the same order, then $CD(H)$ is a star graph since there are no edges connecting any two different elements $a, b \in H\backslash\{e\}$ and $CD(H)$.
(2) If H is a 2-group with one element of order 2 and no elements of order more than 4, the element of order 2 is adjacent only to e. Moreover, the elements of order 4 are adjacent to e and not adjacent to the element of order 2. Hence, $CD(H)$ is a star graph.
(3) Every element in $H\backslash\{e\}$ is either of order p or q if H is an order pq noncyclic. Consequently, $CD(H)$ is a star graph.

□

Theorem 2.2. *let H be a finite group.Then $CD(H)$ is a complete graph if and only if $H \cong Z_2$.*

Proof. Suppose that $CD(H)$ is a complete graph and $a \in H\backslash\{e\}$ with $|a| = r$, $r > 2$, so $(r, r-1) = 1$ and $|a| = |a^{r-1}|$. Thus a is not adjacent to a^{r-1} if they are different and $a = a^{r-1}$ which implies that $a^{r-1} = e$ which is a contradiction with $|a| = r, r > 2$. Hence $|a| = 2$ for every $a \in H\backslash\{e\}$ and all elements in $H\backslash\{e\}$ are of the same order.Therefore, $|H| = 2$ and $H \cong Z_2$. □

Proposition 2.4. *Let H be a finite group. Then $CD(H)$ is a path if and only if $H \cong \mathbb{Z}_2$ or $H \cong \mathbb{Z}_3$.*

Proof. suppose that H is a finite group with $CD(H)$ is a path and $|H| > 3$, since e is adjacent to every element in $H \backslash \{e\}$, so $CD(H)$is not a path. Thus $|H| = 2$ or $|H| = 3$ and $H \cong \mathbb{Z}_2$ or $H \cong \mathbb{Z}_3$. The converse is true.

□

Definition 2.2. *A set B of vertices in a graph with no adjacent vertices is known as an independent set or stable set. In other words, it is a collection B of vertices where there is no edge connecting any two vertices in B. Each edge in the graph, equivalently, has just one possible endpoint in B.*

An independent set's size is determined by how many vertices it has.

Definition 2.3. *An independent set with the largest size feasible for the given graph G is referred to as a "maximum independent set." The symbol for this size, which is also known as the independence number of G, is typically $\alpha(G)$.*

The following results will discuss the maximum independent set of common divisor graphs of a finite group H, denoted by $\alpha(CD(H))$.

Proposition 2.5. *Consider H is a finite abelian group, the set of all elements of prime order is a maximal independent set of the common divisor graph of H.*

Proof. Consider H is a finite abelian group, it is clear that any two different elements with prime order are not adjacent. Assume that a has a composite order, $|a| = p_1 p_2 r$ where $p_1 < p_2$ are different primes and r is a positive integer. Thus, there are at least two elements of order p_2 and they are adjacent to a. Therefore, the set of all elements of prime order is a maximal independent set of the common divisor graph of H. □

Definition 2.4. *If each vertex that is not in D is adjacent to at least one member of D, then the subset D of V is a dominating set for the graph $G = (V, E)$. The domination number $\gamma(G)$ for G is the number of vertices in the smallest dominating set.*

In the following results, we obtain the values of the domination number of common divisor graph for finite group H rings, represented by $\gamma(CD(H))$.

Proposition 2.6. *For any finite group H, we have $\gamma(CD(H)) = 1$*

Proof. From the definition of common divisor graph, the identity element e is adjacent to every element in $H - \{e\}$. Therefore, $\gamma(CD(H)) = 1$. □

Example 2.8. *If $H = \mathbb{Z}_3$, the graph $CD(H)$ is then is a path.*

Example 2.9. *If $H = \mathbb{Z}_4$, then the graph $CD(H)$ is a star graph.*

Example 2.10. *If $H = S_3$, then the graph $CD(H)$ is a star graph.*

Example 2.11. *If $H = A_4$, then the graph $CD(H)$ is a star graph.*

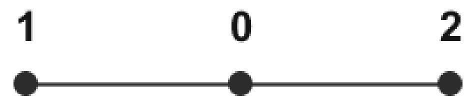

Figure 7. $\mathbb{Z}_3$

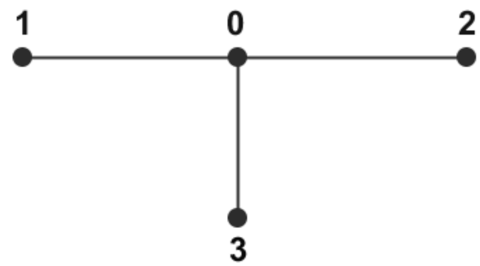

Figure 8. $\mathbb{Z}_4$

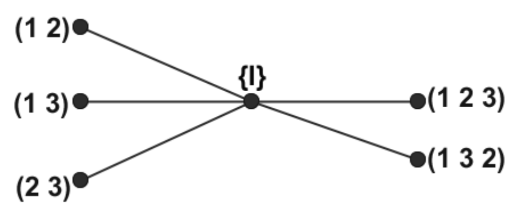

Figure 9. S_3

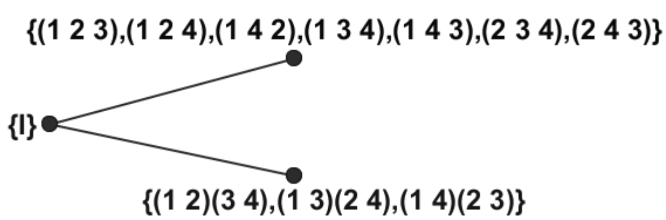

Figure 10. A_4

Example 2.12. *In the case where $H = D_3$, the graph $CD(H)$ is a star graph.*

Proposition 2.7. *Consider the finite group H. Consequently, the following claims are true:*

(1) *Daim $CD(H) \leq 2$.*
(2) *Girth $CD(H) = 3$, whenever $CD(H)$ is not a star graph.*

Proof. (1) Suppose H is a finite group. Since e is adjacent to every element in $H \backslash \{e\}$, so daim $(CD(H)) \leq 2$.

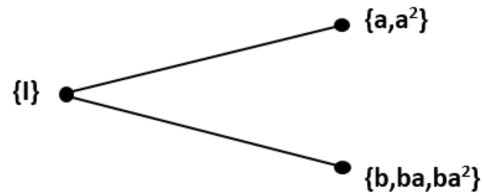

Figure 11. D_3

(2) Assuming that $CD(H)$ is not a star graph, it follows that, there exist $a \neq e \neq b$ such that a is adjacent to b. Hence we obtain a triangle $e - a - b - e$ and girth $CD(H)) = 3$.

□

Example 2.13. *If $H = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, Consequently, the graph CD(H) is a star graph.*

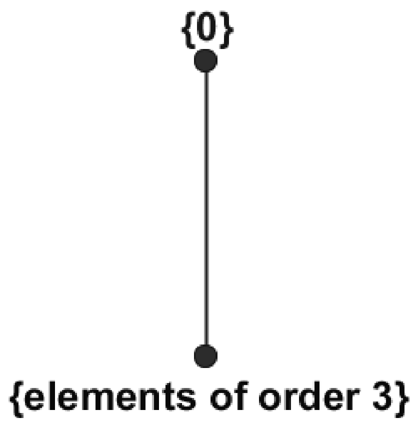

Figure 12. $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

Example 2.14. *If $H = \mathbb{Z}_9 \times \mathbb{Z}_3$, then the graph $CD(H)$ appears as follows:*

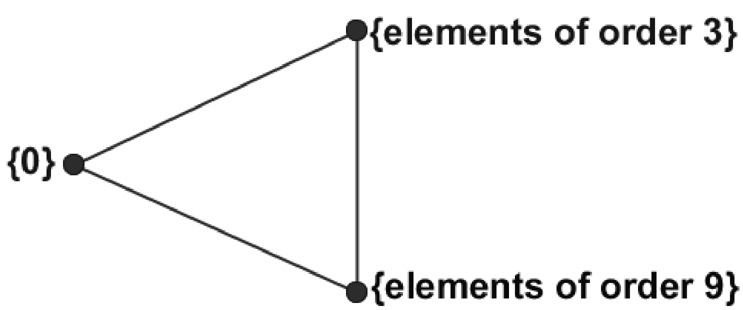

Figure 13. $\mathbb{Z}_9 \times \mathbb{Z}_3$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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