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Intuitionistic Hesitant Fuzzy UP (BCC)-Filters of UP (BCC)-Algebras

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Abstract. The concepts of intuitionistic hesitant fuzzy UP (BCC)-subalgebras, UP (BCC)-ideals, and UP (BCC)-filters of UP (BCC)-algebras are presented, some of their features are explained, and their extensions are demonstrated using the theory of hesitant fuzzy sets as a foundation. The necessary conditions for those intuitionistic hesitant fuzzy sets are provided and include their relation to their complement. The concept of prime and weakly prime of intuitionistic hesitant fuzzy sets was also introduced and studied. We also talk about the connections between intuitionistic hesitant fuzzy UP (BCC)-subalgebras (UP (BCC)-ideals, UP (BCC)-filters) and their level subsets. The homomorphic pre-images of intuitionistic hesitant fuzzy UP (BCC)-filters in UP (BCC)-algebras are also studied and some related properties are investigated.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [15]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 2, 4]. In 2009 - 2010, Torra and Narukawa [13, 14] introduced the notion of hesitant fuzzy sets, that is a function from a reference set to a power set of the unit interval. The notion of hesitant fuzzy sets is the other generalization of the notion fuzzy sets. The

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hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the notion of hesitant fuzzy sets by Torra and Narukawa [13, 14], several researches were conducted on the generalizations of the notion of hesitant fuzzy sets and application to many logical algebras such as: in 2012, Zhu, Xu and Xia [16] introduced the notion of dual hesitant fuzzy sets, which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin [7] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [9] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. Iampan [6] introduced a new algebraic structure, called a UP-algebra, and Mosrijai et. al. [11] introduced the notion of hesitant fuzzy sets on UP-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy UP-ideals play an important role in studying the many logical algebras. The concepts of UP-algebras (see [6]) and BCC-algebras (see [10]) are the same concept, as shown by Jun et al. [8] in 2022. In this publication and following investigations, our research team will refer to it as BCC rather than UP because of respect for Komori, who first characterized it in 1984.

In this paper, the concepts of intuitionistic hesitant fuzzy BCC-subalgebras, BCC-ideals, and BCC-filters of BCC-algebras are presented, some of their features are explained, and their extensions are demonstrated using the theory of hesitant fuzzy sets as a foundation. The necessary conditions for those intuitionistic hesitant fuzzy sets are provided and include their relation to their complement. The concept of prime and weakly prime of intuitionistic hesitant fuzzy sets was also introduced and studied. We also talk about the connections between intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, BCC-filters) and their level subsets. The homomorphic pre-images of intuitionistic hesitant fuzzy BCC-filters in BCC-algebras are also studied and some related properties are investigated.

2. Preliminaries

The concept of BCC-algebras (see [10]) can be redefined without the condition (2.6) as follows: An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *BCC-algebra* if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0) \tag{2.1}$$

$$(\forall x \in X)(0 \cdot x = x) \tag{2.2}$$

$$(\forall x \in X)(x \cdot 0 = 0) \tag{2.3}$$

$$(\forall x, y \in X)(x \cdot y = 0 = y \cdot x \;\Rightarrow\; x = y) \tag{2.4}$$

After this we assign X instead of a BCC-algebra $(X, \cdot, 0)$ until otherwise specified.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{2.5}$$

In X, the following assertions are valid (see [6]).

$$(\forall x \in X)(x \leq x) \tag{2.6}$$

$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{2.7}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \cdot x \leq z \cdot y) \tag{2.8}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \cdot z \leq x \cdot z) \tag{2.9}$$

$$(\forall x, y, z \in X)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)) \tag{2.10}$$

$$(\forall x, y \in X)(y \cdot x \leq x \Leftrightarrow x = y \cdot x) \tag{2.11}$$

$$(\forall x, y \in X)(x \leq y \cdot y) \tag{2.12}$$

$$(\forall a, x, y, z \in X)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))) \tag{2.13}$$

$$(\forall a, x, y, z \in X)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z) \tag{2.14}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot z) \tag{2.15}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \cdot y) \tag{2.16}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)) \tag{2.17}$$

$$(\forall a, x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)) \tag{2.18}$$

Definition 2.1. *[6] A nonempty subset S of X is called a BCC-subalgebra of X if $x \cdot y \in S$ $\forall x, y \in S$.*

Definition 2.2. *[6] A nonempty subset I of X is called a BCC-ideal of X if*

(1) $0 \in I$,

(2) $(\forall x, y, z \in X)(x \cdot (y \cdot z), y \in I \Rightarrow x \cdot z \in I)$.

Definition 2.3. *[12] A nonempty subset F of X is called a BCC-filter of X if*

(1) $0 \in F$,

(2) $(\forall x, y \in X)(x \cdot y \in F, x \in F \Rightarrow y \in F)$.

Definition 2.4. *[13] A hesitant fuzzy set on a reference set X is defined in term of a function h that when applied to X return a subset of $[0, 1]$, that is, $h : X \to \mathcal{P}([0, 1])$.*

Definition 2.5. *[3] An intuitionistic hesitant fuzzy set on a reference set X is defined in the form $\mathcal{H} = (h, k)$, where h and k are functions that when applied to X return a subset of $[0, 1]$, that is, $h, k : X \to \mathcal{P}([0, 1])$.*

Definition 2.6. *[11] A hesitant fuzzy set h on X is said to be a hesitant fuzzy BCC-filter of X if the following conditions are hold:*

$$(\forall x \in X)(h(0) \supseteq h(x)) \tag{2.19}$$

$$(\forall x, y \in X)(h(y) \supseteq h(x \cdot y) \cap h(x)) \tag{2.20}$$

Definition 2.7. *[13] The complement of a hesitant fuzzy set h in a reference set X is the hesitant fuzzy set $\overline{h}$ defined by $\overline{h}(x) = [0,1] - h(x)$ for all $x \in X$.*

Definition 2.8. *[13] The complement of an intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on a reference set X is the intuitionistic hesitant fuzzy set $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ defined by $\overline{h}(x) = [0,1] - h(x)$ and $\overline{k}(x) = [0,1] - k(x)$ for all $x \in X$.*

3. Intuitionistic hesitant fuzzy BCC-filters

In this section, the concepts of intuitionistic hesitant fuzzy BCC-subalgebras, BCC-ideals, and BCC-filters of BCC-algebras are presented, some of their features are explained.

Definition 3.1. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on X is called an intuitionistic hesitant fuzzy BCC-subalgebra of X if it satisfies the following property:*

$$(\forall x, y \in X)\begin{pmatrix} h(x \cdot y) \supseteq h(x) \cap h(y) \\ k(x \cdot y) \subseteq k(x) \cup k(y) \end{pmatrix} \tag{3.1}$$

Definition 3.2. *The characteristic intuitionistic hesitant fuzzy set of a subset A of a set X is defined to be the structure $\chi_A = (h_{\chi_A}, k_{\chi_A})$, where*

$$h_{\chi_A}(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{cases} \text{ and } k_{\chi_A}(x) = \begin{cases} \emptyset & \text{if } x \in A \\ [0,1] & \text{otherwise.} \end{cases}$$

Lemma 3.1. *The constant 0 of X is in a nonempty subset B of X if and only if $h_{\chi_B}(0) \supseteq h_{\chi_B}(x)$ and $k_{\chi_B}(0) \subseteq k_{\chi_B}(x)$ for all $x \in X$.*

Proof. If $0 \in B$, then $h_B(0) = [0,1]$. Thus $h_B(0) = [0,1] \supseteq h_B(x)$ for all $x \in X$. Also, $k_B(0) = \emptyset$. Then $k_B(0) = \emptyset \subseteq k_B(x)$ for all $x \in X$.

Conversely, assume that $h_B(0) \supseteq h_B(x)$ and $k_B(0) \subseteq k_B(x)$ for all $x \in X$. Since B is a nonempty subset of X, we have $a \in B$ for some $a \in X$. Then $h_B(0) \supseteq h_B(a) = [0,1]$, so $h_B(0_Y) = [0,1]$. Hence, $0 \in B$. □

Definition 3.3. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on X is said to be an intuitionistic hesitant fuzzy BCC-filter of X if the following conditions are hold:*

$$(\forall x \in X)\begin{pmatrix} h(0) \supseteq h(x) \\ k(0) \subseteq k(x) \end{pmatrix} \tag{3.2}$$

$$(\forall x, y \in X)\begin{pmatrix} h(y) \supseteq h(x \cdot y) \cap h(x) \\ k(y) \subseteq k(x \cdot y) \cup k(x) \end{pmatrix} \tag{3.3}$$

Example 3.1. *Let* $X = \{0, 1, 2, 3\}$ *with the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	0
3	0	1	2	0

Then X *is a BCC-algebra. We define an intuitionistic hesitant fuzzy set* $\mathcal{H} = (h, k)$ *on* X *as follows:*

$$h(0) = [0, 1], h(1) = \{0.1\}, h(2) = \emptyset, h(3) = \{0.2, 0.3\},$$

$$k(0) = \emptyset, k(1) = \{0.1, 0.2\}, k(2) = [0, 1], k(3) = \{0.3\}$$

Then $\mathcal{H}$ *is an intuitionistic hesitant fuzzy BCC-subalgebra of* X*.*

Definition 3.4. *An intuitionistic hesitant fuzzy set* $\mathcal{H} = (h, k)$ *on* X *is said to be an intuitionistic hesitant fuzzy BCC-ideal of* X *if* (3.2) *and the following condition are hold:*

$$(\forall x, y, z \in X) \begin{pmatrix} h(x \cdot y) \supseteq h(x \cdot (y \cdot z)) \cap h(y) \\ k(x \cdot y) \subseteq k(x \cdot (y \cdot z)) \cup k(y) \end{pmatrix} \tag{3.4}$$

Theorem 3.1. *Every intuitionistic hesitant fuzzy BCC-ideal of* X *is an intuitionistic hesitant fuzzy BCC-filter.*

Proof. Let $\mathcal{H} = (h, k)$ be an intuitionistic hesitant fuzzy BCC-ideal of X. Then (3.2) holds. Let $x, y \in X$. Then

$$h(y) = h(0 \cdot y) \supseteq h(0 \cdot (x \cdot y)) \cap h(x) = h(x \cdot y) \cap h(x),$$

$$k(y) = k(0 \cdot y) \subseteq k(0 \cdot (x \cdot y)) \cup k(x) = k(x \cdot y) \cup k(x).$$

Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X. □

The following example shows that the converse of Theorem 3.1 is not true in general.

Example 3.2. *Let* $X = \{0, 1, 2, 3\}$ *with the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then X *is a BCC-algebra. We define an intuitionistic hesitant fuzzy set* $\mathcal{H} = (h, k)$ *on* X *as follows:*

$$h(0) = [0, 1], h(1) = \{0.7\}, h(2) = \emptyset, h(3) = \{0.2, 0.5\},$$

$$k(0) = \emptyset, k(1) = \{0.1, 0.2\}, k(2) = \{0.1, 0.2, 0.3\}, k(3) = [0, 1]$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X but not an intuitionistic hesitant fuzzy BCC-ideal of X.

Theorem 3.2. *Every intuitionistic hesitant fuzzy BCC-filter of X is an intuitionistic hesitant fuzzy BCC-subalgebra.*

Proof. Let $\mathcal{H} = (h, k)$ be an intuitionistic hesitant fuzzy BCC-filter of X. Then for all $x, y \in X$,

$$h(x \cdot y) \supseteq h(y \cdot (x \cdot y)) \cap h(y) = h(0) \cap h(y) = h(y) \supseteq h(x) \cap h(y),$$

$$k(x \cdot y) \subseteq k(y \cdot (x \cdot y)) \cup k(y) = k(0) \cup k(y) = k(y) \subseteq k(x) \cup k(y).$$

Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X. □

The following example shows that the converse of Theorem 3.2 is not true in general.

Example 3.3. *Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:*

·	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	1	2	0

Then X is a BCC-algebra. We define an intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on X as follows:

$$h(0) = \{0.1, 0.2, 0.3\}, h(1) = \{0.1\}, h(2) = \{0.2\}, h(3) = \emptyset,$$

$$k(0) = \emptyset, k(1) = \{0.1\}, k(2) = \{0.1, 0.2\}, k(3) = [0, 1]$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-subalgebra of X but not an intuitionistic hesitant fuzzy BCC-filter of X.

Theorem 3.3. *A nonempty subset F of X is a BCC-filter of X if and only if the characteristic intuitionistic hesitant fuzzy set $\chi_F = (h_{\chi_F}, k_{\chi_F})$ is an intuitionistic hesitant fuzzy BCC-filter of X.*

Proof. Assume that F is a BCC-filter of X. Since $0 \in F$, it follows from Lemma 3.1 that $h_{\chi_F}(0) \supseteq h_{\chi_F}(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1 : If $x, y \in F$, then $h_{\chi_F}(x) = [0, 1]$ and $h_{\chi_F}(y) = [0, 1]$. Hence, $h_{\chi_F}(y) = [0, 1] \supseteq h_{\chi_F}(x \cdot y) = h_{\chi_F}(x \cdot y) \cap h_{\chi_F}(x)$. Also, $k_{\chi_F}(x) = \emptyset$ and $k_{\chi_F}(y) = \emptyset$. Hence, $k_{\chi_F}(y) = \emptyset \subseteq k_{\chi_F}(x \cdot y) = k_{\chi_F}(x \cdot y) \cup k_{\chi_F}(x)$.

Case 2 : If $x \notin F$ and $y \in F$, then $h_{\chi_F}(x) = \emptyset$ and $h_{\chi_F}(y) = [0, 1]$. Thus $h_{\chi_F}(y) = [0, 1] \supseteq \emptyset = h_{\chi_F}(x \cdot y) \cap h_{\chi_F}(x)$. Also $k_{\chi_F}(x) = [0, 1]$ and $k_{\chi_F}(y) = \emptyset$. Thus $k_{\chi_F}(y) = \emptyset \subseteq [0, 1] = k_{\chi_F}(x \cdot y) \cup k_{\chi_F}(x)$.

Case 3 : If $x \in F$ and $y \notin F$, then $h_{\chi_F}(x) = [0, 1]$ and $h_{\chi_F}(y) = \emptyset$. Since F is a BCC-filter of X, we have $x \cdot y \notin F$ or $x \notin F$. But $x \in F$, so $x \cdot y \notin F$. Then $h_{\chi_F}(x \cdot y) = \emptyset$. Thus

$h_{\chi_F}(y) = \emptyset \supseteq \emptyset = h_{\chi_F}(x \cdot y) \cap h_{\chi_F}(x)$. Also, $k_{\chi_F}(x) = \emptyset$, $k_{\chi_F}(y) = [0, 1]$ and $k_{\chi_F}(x \cdot y) = [0, 1]$. Thus $k_{\chi_F}(y) = [0, 1] \subseteq [0, 1] = k_{\chi_F}(x \cdot y) \cup k_{\chi_F}(x)$.

Case 4 : If $x \notin F$ and $y \notin F$, then $h_{\chi_F}(x) = \emptyset$ and $h_{\chi_F}(y) = \emptyset$. Thus $h_{\chi_F}(y) = \emptyset \subseteq \emptyset = h_{\chi_F}(x \cdot y) \cap h_{\chi_F}(x)$. Also, $k_{\chi_F}(x) = [0, 1]$ and $k_{\chi_F}(y) = [0, 1]$. Thus $k_{\chi_F}(y) = [0, 1] \subseteq [0, 1] = k_{\chi_F}(x \cdot y) \cup k_{\chi_F}(x)$.

Hence, $\chi_F = (h_{\chi_F}, k_{\chi_F})$ is an intuitionistic hesitant fuzzy BCC-filter of X.

Conversely, assume that $\chi_F = (h_{\chi_F}, k_{\chi_F})$ is an intuitionistic hesitant fuzzy BCC-filter of X. Since $h_{\chi_F}(0) \supseteq h_{\chi_F}(x)$ for all $x \in X$, it follows from Lemma 3.1 that $0 \in F$. Next, let $x, y \in X$ be such that $x \cdot y \in F$ and $x \in F$. Then $h_{\chi_F}(x \cdot y) = [0, 1]$ and $h_{\chi_F}(x) = [0, 1]$. Thus $h_{\chi_F}(y) \supseteq h_{\chi_F}(x \cdot y) \cap h_{\chi_F}(x) = [0, 1]$, so $h_{\chi_F}(y) = [0, 1]$. Therefore, $y \in F$ and so F is a BCC-filter of X. □

Definition 3.5. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on X is called a prime intuitionistic hesitant fuzzy set on X if it satisfies the following property:*

$$(\forall x, y \in X)\left(\begin{array}{l} h(x \cdot y) \subseteq h(x) \cup h(y) \\ k(x \cdot y) \supseteq k(x) \cap k(y) \end{array}\right) \tag{3.5}$$

Definition 3.6. *[5] A nonempty subset B of X is called a prime subset of X if it satisfies the following property:*

$$(\forall x, y \in X)(x \cdot y \in B \Rightarrow x \in B \text{ or } y \in B)$$

Theorem 3.4. *A nonempty subset B of X is a prime subset of X if and only if the characteristic intuitionistic hesitant fuzzy set χ_B is a prime intuitionistic hesitant fuzzy set on X.*

Proof. Assume that B is a prime subset of X and let $x, y \in X$.

Case 1 : If $x \cdot y \in B$, then $h_{\chi_B}(x \cdot y) = [0, 1]$. Since B is a prime subset of X, we have $x \in B$ or $y \in B$. Then $h_{\chi_B}(x) = [0, 1]$ or $h_{\chi_B}(y) = [0, 1]$, so $h_{\chi_B}(x) \cup h_{\chi_B}(y) = [0, 1]$. Hence, $h_{\chi_B}(x \cdot y) = [0, 1] \subseteq [0, 1] = h_{\chi_B}(x) \cup h_{\chi_B}(y)$. Also, $k_{\chi_B}(x \cdot y) = \emptyset \supseteq k_{\chi_B}(x) \cap k_{\chi_B}(y)$.

Case 2 : If $x \cdot y \notin B$, then $h_{\chi_B}(x \cdot y) = \emptyset \subseteq h_{\chi_B}(x) \cup h_{\chi_B}(y)$. Also, $k_{\chi_B}(x \cdot y) = [0, 1] \supseteq k_{\chi_B}(x) \cap k_{\chi_B}(y)$.

Hence, χ_B is a prime intuitionistic hesitant fuzzy set on X.

Conversely, assume that $\chi_B = (h_{\chi_B}, k_{\chi_B})$ is a prime intuitionistic hesitant fuzzy set on X. Let $x, y \in X$ be such that $x \cdot y \in B$. Then $h_{\chi_B}(x \cdot y) = [0, 1]$, so $[0, 1] = h_{\chi_B}(x \cdot y) \subseteq h_{\chi_B}(x) \cup h_{\chi_B}(y)$. Thus $h_{\chi_B}(x) \cup h_{\chi_B}(y) = [0, 1]$, so $h_{\chi_B}(x) = [0, 1]$ or $h_{\chi_B}(y) = [0, 1]$. Hence, $x \in B$ or $y \in B$ and so B is a prime subset of X. □

Theorem 3.5. *Let $\mathcal{H} = (h, k)$ be an intuitionistic hesitant fuzzy set on X. Then the following statements are equivalent:*

(1) $\mathcal{H}$ is a prime intuitionistic hesitant fuzzy BCC-filter of X,

(2) $\mathcal{H}$ is a constant intuitionistic hesitant fuzzy set on X.

Proof. Assume that $\mathcal{H}$ is a prime intuitionistic hesitant fuzzy BCC-filter of X. Then $h(0) \supseteq h(x)$ and $k(0) \subseteq k(x)$ for all $x \in X$. By (2.6), we have $h(0) = h(x \cdot x) \subseteq h(x) \cup h(x) = h(x)$ and $k(0) = k(x \cdot x) \supseteq k(x) \cup k(x) = k(x)$ for all $x \in X$ and so $h(x) = h(0)$ and $k(x) = k(0)$ for all $x \in X$. Hence, $\mathcal{H}$ is a constant intuitionistic hesitant fuzzy set on X.

Conversely, assume that $\mathcal{H}$ is a constant intuitionistic hesitant fuzzy set on X. Hence, we can easily show that $\mathcal{H}$ is a prime intuitionistic hesitant fuzzy BCC-filter of X. □

Definition 3.7. *[5] A nonempty subset B of X is called a weakly prime subset of X if it satisfies the following property:*

$$(\forall x, y \in X, x \neq y)(x \cdot y \in B \Rightarrow x \in B \text{ or } y \in B)$$

Definition 3.8. *[5] A BCC-filter B of X is called a weakly prime BCC-filter of X if B is a weakly prime subset of X.*

Definition 3.9. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on X is called a weakly prime intuitionistic hesitant fuzzy set on X if it satisfies the following property:*

$$(\forall x, y \in X, x \neq y) \begin{pmatrix} h(x \cdot y) \subseteq h(x) \cup h(y) \\ k(x \cdot y) \supseteq k(x) \cap k(y) \end{pmatrix} \tag{3.6}$$

Definition 3.10. *An intuitionistic hesitant fuzzy BCC-filter $\mathcal{H} = (h, k)$ of X is called a weakly prime intuitionistic hesitant fuzzy BCC-filter of X if $\mathcal{H}$ is a weakly prime intuitionistic hesitant fuzzy set on X.*

Theorem 3.6. *A nonempty subset B of X is a weakly prime subset of X if and only if the characteristic intuitionistic hesitant fuzzy set χ_B is a weakly prime intuitionistic hesitant fuzzy set on X.*

Proof. Assume that B is a weakly prime subset of X and let $x, y \in X$ be such that $x \neq y$.

Case 1 : If $x \cdot y \in B$, then $h_{\chi_B}(x \cdot y) = [0, 1]$. Since B is a weakly prime subset of X, we have $x \in B$ or $y \in B$. Then $h_{\chi_B}(x) = [0, 1]$ or $h_{\chi_B}(y) = [0, 1]$, so $h_{\chi_B}(x) \cup h_{\chi_B}(y) = [0, 1]$. Hence, $h_{\chi_B}(x \cdot y) = [0, 1] \subseteq [0, 1] = h_{\chi_B}(x) \cup h_{\chi_B}(y)$. Also, $k_{\chi_B}(x) = \emptyset$ or $k_{\chi_B}(y) = \emptyset$, so $k_{\chi_B}(x) \cap k_{\chi_B}(y) = \emptyset$. Hence, $k_{\chi_B}(x \cdot y) = \emptyset \supseteq \emptyset = k_{\chi_B}(x) \cap k_{\chi_B}(y)$.

Case 2 : If $x \cdot y \notin B$, then $h_{\chi_B}(x \cdot y) = \emptyset \subseteq h_{\chi_B}(x) \cup h_{\chi_B}(y)$. Also, $k_{\chi_B}(x \cdot y) = [0, 1] \supseteq k_{\chi_B}(x) \cap k_{\chi_B}(y)$.

Hence, χ_B is a weakly prime intuitionistic hesitant fuzzy set on X.

Conversely, assume that h_{χ_B} is a weakly prime intuitionistic hesitant fuzzy set on X. Let $x, y \in X$ be such that $x \cdot y \in B$ and $x \neq y$. Then $h_{\chi_B}(x \cdot y) = [0, 1]$, so $[0, 1] = h_{\chi_B}(x \cdot y) \subseteq h_{\chi_B}(x) \cup h_{\chi_B}(y)$. Thus $h_{\chi_B}(x) \cup h_{\chi_B}(y) = [0, 1]$, so $h_{\chi_B}(x) = [0, 1]$ or $h_{\chi_B}(y) = [0, 1]$. Hence, $x \in B$ or $y \in B$ and so B is a weakly prime subset of X. □

Theorem 3.7. *A nonempty subset F of X is a weakly prime BCC-filter of X if and only if the characteristic intuitionistic hesitant fuzzy set χ_F is a weakly prime intuitionistic hesitant fuzzy BCC-filter of X.*

Proof. It is straightforward by Theorems 3.3 and 3.6. □

Theorem 3.8. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X if and only if the hesitant fuzzy sets h and $\overline{k}$ are hesitant fuzzy BCC-filters of X.*

Proof. Assume that $\mathcal{H} = (h, k)$ is an intuitionistic fuzzy BCC-filter of X. Then for any $x, y \in X$, we have $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Hence, h is a hesitant fuzzy BCC-filter of X. Now for any $x, y \in X$, we have $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Then $\overline{k}(0) = [0, 1] - k(0) \supseteq [0, 1] - k(x) = \overline{k}(x)$ and

$$\begin{aligned}
\overline{k}(y) &= [0,1] - k(y) \\
&\supseteq [0,1] - (k(x \cdot y) \cup k(x)) \\
&= [0,1] - k(x \cdot y) \cap [0,1] - k(x) \\
&= \overline{k}(x \cdot y) \cap \overline{k}(x).
\end{aligned}$$

Hence, $\overline{k}$ is a hesitant fuzzy BCC-filter of X.

Conversely, assume that the hesitant fuzzy sets h and $\overline{k}$ are hesitant fuzzy BCC-filters of X. Then for any $x, y \in X$, we have $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Now for any $x, y \in X$, we have $\overline{k}(0) \supseteq \overline{k}(x)$ and $\overline{k}(y) \supseteq \overline{k}(x \cdot y) \cap \overline{k}(x)$. Then $[0, 1] - k(0) \supseteq [0, 1] - k(x)$ and so $k(0) \subseteq k(x)$. Now,

$$\begin{aligned}
[0,1] - k(y) &\supseteq [0,1] - k(x \cdot y) \cap [0,1] - k(x) \\
&= [0,1] - (k(x \cdot y) \cup k(x)), \\
k(y) &\subseteq k(x \cdot y) \cup k(x).
\end{aligned}$$

Hence, $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X. □

Theorem 3.9. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X if and only if the intuitionistic hesitant fuzzy set $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy BCC-filter of X.*

Proof. Assume that $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X. Then for any $x, y, z \in X$, $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Hence, for any $x, y, z \in X$, $\overline{h}(0) = [0, 1] - h(0) \subseteq [0, 1] - h(x) = \overline{h}(x)$ and

$$\begin{aligned}
\overline{h}(y) &= [0,1] - h(y) \\
&\subseteq [0,1] - (h(x \cdot y) \cap h(x)) \\
&= [0,1] - h(x \cdot y) \cup [0,1] - h(x) \\
&= \overline{h}(x \cdot y) \cup \overline{h}(x).
\end{aligned}$$

Now, for any $x, y, z \in X$, $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Hence, for any $x, y, z \in X$, $\overline{k}(0) = [0,1] - k(0) \supseteq [0,1] - k(x) = \overline{k}(x)$ and

$$\begin{aligned} \overline{k}(y) &= [0,1] - k(y) \\ &\supseteq [0,1] - (k(x \cdot y) \cup k(x)) \\ &= [0,1] - k(x \cdot y) \cap [0,1] - k(x) \\ &= \overline{k}(x \cdot y) \cap \overline{k}(x). \end{aligned}$$

Hence, $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy BCC-filter of X.

Conversely, assume that the intuitionistic hesitant fuzzy set $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy BCC-filter of X. Then for any $x, y, z \in X$, $\overline{k}(0) \supseteq \overline{k}(x)$ and $\overline{k}(y) \supseteq \overline{k}(x \cdot y) \cap \overline{k}(x)$. Then $[0,1] - k(0) \supseteq [0,1] - k(x)$ and $[0,1] - k(y) \supseteq [0,1] - (k(x \cdot y) \cup k(x))$, so $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Now, for any $x, y, z \in X$, we have $\overline{h}(0) \subseteq \overline{h}(x)$ and $\overline{h}(y) \subseteq \overline{h}(x \cdot y) \cup \overline{h}(x)$. Then $[0,1] - h(0) \subseteq [0,1] - h(x)$ and $[0,1] - h(y) \supseteq [0,1] - (h(x \cdot y) \cup h(x))$, so $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Hence, $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X. □

Definition 3.11. *Let $\mathcal{H} = (h, k)$ be an intuitionistic hesitant fuzzy set on X. The intuitionistic hesitant fuzzy sets $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are defined as $\oplus\mathcal{H} = (h, \overline{h})$ and $\otimes\mathcal{H} = (\overline{k}, k)$.*

Theorem 3.10. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X, then the sets $X_h = \{x \in X \mid h(x) = h(0)\}$ and $X_k = \{x \in X \mid k(x) = k(0)\}$ are BCC-filters of X.*

Proof. Clearly, $0 \in X_h \cap X_k$. Let $x, y \in X$ be such that $x \cdot y, x \in X_h$. Then $h(x \cdot y) = h(0)$ and $h(x) = h(0)$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X, by (3.3), $h(y) \supseteq h(x \cdot y) \cap h(x) = h(0)$, whence $h(y) = h(0)$, by (3.2). This means that $y \in X_h$. Hence, X_h is a BCC-filter of X. Let $x, y \in X$ be such that $x \cdot y, x \in X_k$. Then $k(x \cdot y) = k(0)$ and $k(x) = k(0)$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X, by (3.3), $k(y) \subseteq k(x \cdot y) \cup k(x) = k(0)$, whence $k(y) = k(0)$, by (3.2). This means that $y \in X_k$. Hence, X_k is a BCC-filter of X. □

Theorem 3.11. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X if and only if the intuitionistic hesitant fuzzy sets $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy BCC-filters of X.*

Proof. Assume that $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X. Let $x \in X$. Then $\overline{h}(0) = [0,1] - h(0) \subseteq [0,1] - h(x) = \overline{h}(x)$. Let $x, y \in X$. Then

$$\begin{aligned} \overline{h}(y) &= [0,1] - h(y) \\ &\subseteq [0,1] - (h(x \cdot y) \cap h(x)) \\ &= ([0,1] - h(x \cdot y)) \cup ([0,1] - h(x)) \\ &= \overline{h}(x \cdot y) \cup \overline{h}(x). \end{aligned}$$

Hence, $\oplus\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X.

Let $x \in X$. Then $\overline{k}(0) = [0,1] - k(0) \supseteq [0,1] - k(x) = \overline{k}(x)$. Let $x, y \in X$. Then

$$\begin{aligned} \overline{k}(y) &= [0,1] - k(y) \\ &\supseteq [0,1] - (k(x \cdot y) \cup k(x)) \\ &= ([0,1] - k(x \cdot y)) \cap ([0,1] - k(x)) \\ &= \overline{k}(x \cdot y) \cap \overline{k}(x). \end{aligned}$$

Hence, $\otimes\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X.

Conversely, assume that $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy BCC-filters of X. Then for any $x, y \in X$, we have $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$ and $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X. □

Lemma 3.2. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X, then*

$$(\forall x, y, z \in X)\left(z \leq x \cdot y \Rightarrow \begin{cases} h(y) \supseteq h(x) \cap h(z) \\ k(y) \subseteq k(x) \cup k(z) \end{cases} \right). \tag{3.7}$$

Proof. Let $x, y, z \in X$ be such that $z \leq x \cdot y$. Then $z \cdot (x \cdot y) = 0$ and so

$$\begin{aligned} h(y) &\supseteq h(x \cdot y) \cap h(x) \\ &\supseteq h(z \cdot (x \cdot y)) \cap h(z) \cap h(x) \\ &= h(0) \cap h(z) \cap h(x) \\ &= h(x) \cap h(z), \end{aligned}$$

$$\begin{aligned} k(y) &\subseteq k(x \cdot y) \cup k(x) \\ &\subseteq k(z \cdot (x \cdot y)) \cup k(z) \cup k(x) \\ &= k(0) \cup k(z) \cup k(x) \\ &= k(x) \cup k(z). \end{aligned}$$

□

Lemma 3.3. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X, then*

$$(\forall x, y \in X)\left(x \leq y \Rightarrow \begin{cases} h(y) \supseteq h(x) \\ k(y) \subseteq k(x) \end{cases} \right). \tag{3.8}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 0$ and so

$$h(y) \supseteq h(x \cdot y) \cap h(x) = h(0) \cap h(x) = h(x),$$

$$k(y) \subseteq k(x \cdot y) \cup k(x) = k(0) \cup k(x) = k(x).$$

□

Lemma 3.4. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of X, then*

$$(\forall x, y, z \in X)\left(\begin{array}{l} h(y \cdot z) \cap h(x \cdot y) \subseteq h(x \cdot z) \\ k(y \cdot z) \cup k(x \cdot y) \supseteq k(x \cdot z) \end{array} \right). \tag{3.9}$$

Proof. Let $x, y, z \in X$. By (2.1), we have $(y \cdot z) \leq (x \cdot y) \cdot (x \cdot z)$. Then it follows from Lemma 3.2 that

$$h(y \cdot z) \cap h(x \cdot y) \subseteq h(x \cdot z),$$
$$k(y \cdot z) \cup k(x \cdot y) \supseteq k(x \cdot z).$$

□

Definition 3.12. *Let $h : X \to \mathcal{P}([0, 1])$. For any $\pi \in \mathcal{P}([0, 1])$, the sets $U(h, \pi) = \{x \in X \mid h(x) \supseteq \pi\}$ and $U^+(h, \pi) = \{x \in X \mid h(x) \supset \pi\}$ are called an upper π-level subset and an upper π-strong level subset of h, respectively. The sets $L(h, \pi) = \{x \in X \mid h(x) \subseteq \pi\}$ and $L^-(h, \pi) = \{x \in X \mid h(x) \subset \pi\}$ are called a lower π-level subset and a lower π-strong level subset of h, respectively. The set $E(h, \pi) = \{x \in X \mid h(x) = \pi\}$ is called an equal π-level subset of h. Then $U(h, \pi) = U^+(h, \pi) \cup E(h, \pi)$ and $L(h, \pi) = L^-(h, \pi) \cup E(h, \pi)$.*

Theorem 3.12. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on X is an intuitionistic hesitant fuzzy BCC-filter of X if and only if for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi)$ and $L(k, \pi)$ of X are BCC-filters.*

Proof. Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X. Let $\pi \in \mathcal{P}([0, 1])$ be such that $U(h, \pi) \neq \emptyset$ and let $x \in U(h, \pi)$. Then $h(x) \supseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X, we have $h(0) \supseteq h(x) \supseteq \pi$. Thus $0 \in U(h, \pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in U(h, \pi)$. Then $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X, we have $h(y) \supseteq h(x \cdot y) \cap h(x) \supseteq \pi$. So $y \in U(h, \pi)$. Let $\pi \in \mathcal{P}([0, 1])$ be such that $L(k, \pi) \neq \emptyset$ and let $x \in L(k, \pi)$. Then $k(x) \subseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X, we have $k(0) \subseteq k(x) \subseteq \pi$. Thus $0 \in L(k, \pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in L(k, \pi)$. Then $k(x) \subseteq \pi$ and $k(x \cdot y) \subseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X, we have $k(y) \subseteq k(x \cdot y) \cup k(x) \subseteq \pi$. So $y \in L(k, \pi)$. Hence, $U(h, \pi)$ and $L(k, \pi)$ are BCC-filters of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi)$ and $L(k, \pi)$ of X are BCC-filters. Let $x \in X$. Then $h(x) \in \mathcal{P}([0, 1])$. Choose $\pi = h(x) \in \mathcal{P}([0, 1])$. Then $h(x) \supseteq \pi$. Thus $x \in U(h, \pi)$. By assumption, we have $U(h, \pi)$ is a BCC-filter of X and thus $0 \in U(h, \pi)$. So $h(0) \supseteq \pi = h(x)$. Let $x, y \in X$. Then $h(x), h(x \cdot y) \in \mathcal{P}([0, 1])$. Choose $\pi = h(x) \cap h(x \cdot y) \in \mathcal{P}([0, 1])$. Then $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. Since $x, x \cdot y \in U(h, \pi) \neq \emptyset$. By assumption, we have $U(h, \pi)$ is a BCC-filter of X and then $y \in U(h, \pi)$. Thus $h(y) \supseteq \pi = h(x) \cap h(x \cdot y)$. Let $x \in X$. Then $k(x) \in \mathcal{P}([0, 1])$. Choose $\pi_1 = k(x) \in \mathcal{P}([0, 1])$. Then $k(x) \subseteq \pi_1$. Thus $x \in L(k, \pi_1)$. By assumption, we have $L(k, \pi_1)$ is a BCC-filter of X and thus $0 \in L(k, \pi_1)$. So $k(0) \subseteq \pi_1 = k(x)$. Let $x, y \in X$. Then $k(x), k(x \cdot y) \in \mathcal{P}([0, 1])$. Choose $\pi_1 = k(x) \cup k(x \cdot y) \in \mathcal{P}([0, 1])$. Then $k(x) \subseteq \pi_1$ and $k(x \cdot y) \subseteq \pi_1$. Since $x, x \cdot y \in L(k, \pi_1) \neq \emptyset$. By assumption, we have $L(k, \pi_1)$ is a BCC-filter of X and then $y \in L(k, \pi_1)$. Thus $k(y) \subseteq \pi_1 = k(x) \cup k(x \cdot y)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy BCC-filter of X. □

The following theorem can be proved similarly to Theorem 3.12.

Theorem 3.13. *An intuitionistic hesitant fuzzy set $\mathcal{H} = (h, k)$ on X is an intuitionistic hesitant fuzzy BCC-subalgebra (BCC-ideal) of X if and only if for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h,\pi)$ and $L(k,\pi)$ of X are BCC-subalgebras (BCC-ideals).*

Definition 3.13. *Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of intuitionistic hesitant fuzzy sets on a reference set X. We define the intuitionistic hesitant fuzzy set $\bigcap\limits_{\alpha\in\Delta} \mathcal{H}_\alpha = (\bigcap\limits_{\alpha\in\Delta} h_\alpha, \bigcup\limits_{\alpha\in\Delta} k_\alpha)$ by $(\bigcap\limits_{\alpha\in\Delta} h_\alpha)(x) = \bigcap\limits_{\alpha\in\Delta} h_\alpha(x)$ and $(\bigcup\limits_{\alpha\in\Delta} k_\alpha)(x) = \bigcup\limits_{\alpha\in\Delta} k_\alpha(x)$ for all $x \in X$, which is called the intuitionistic hesitant intersection of intuitionistic hesitant fuzzy sets.*

Proposition 3.1. *If $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ is a family of intuitionistic hesitant fuzzy BCC-filters of X, then $\bigcap\limits_{\alpha\in\Delta} \mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy BCC-filter of X.*

Proof. Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of intuitionistic hesitant fuzzy BCC-filter of X. Let $x \in X$. Then

$$(\bigcap_{\alpha\in\Delta} h_\alpha)(0) = \bigcap_{\alpha\in\Delta} h_\alpha(0) \supseteq \bigcap_{\alpha\in\Delta} h_\alpha(x) = (\bigcap_{\alpha\in\Delta} h_\alpha)(x),$$

$$(\bigcup_{\alpha\in\Delta} k_\alpha)(0) = \bigcup_{\alpha\in\Delta} k_\alpha(0) \subseteq \bigcup_{\alpha\in\Delta} k_\alpha(x) = (\bigcup_{\alpha\in\Delta} k_\alpha)(x).$$

Let $x, y \in X$. Then

$$\begin{aligned}
(\bigcap_{\alpha\in\Delta} h_\alpha)(y) &= \bigcap_{\alpha\in\Delta} h_\alpha(y) \\
&\supseteq \bigcap_{\alpha\in\Delta} (h_\alpha(x\cdot y) \cap h_\alpha(x)) \\
&= (\bigcap_{\alpha\in\Delta} h_\alpha(x\cdot y)) \cap (\bigcap_{\alpha\in\Delta} h_\alpha(x)) \\
&= (\bigcap_{\alpha\in\Delta} h_\alpha)(x\cdot y) \cap (\bigcap_{\alpha\in\Delta} h_\alpha)(x),
\end{aligned}$$

$$\begin{aligned}
(\bigcup_{\alpha\in\Delta} k_\alpha)(y) &= \bigcup_{\alpha\in\Delta} k_\alpha(y) \\
&\subseteq \bigcup_{\alpha\in\Delta} (k_\alpha(x\cdot y) \cup k_\alpha(x)) \\
&= \bigcup_{\alpha\in\Delta} k_\alpha(x\cdot y) \cup \bigcup_{\alpha\in\Delta} k_\alpha(x) \\
&= (\bigcup_{\alpha\in\Delta} k_\alpha)(x\cdot y) \cup (\bigcup_{\alpha\in\Delta} k_\alpha)(x).
\end{aligned}$$

Hence, $\bigcap\limits_{\alpha\in\Delta} \mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy BCC-filter of X. □

Definition 3.14. *Let $A = (h_A, k_A)$ and $B = (h_B, k_B)$ be intuitionistic hesitant fuzzy sets on sets X and Y, respectively. The Cartesian product $A \times B = (h, k)$ defined by $h(x, y) = h_A(x) \cap h_B(y)$ and $k(x, y) = k_A(x) \cup k_B(y)$, where $h : X \times Y \to \mathcal{P}([0,1])$ and $k : X \times Y \to \mathcal{P}([0,1])$ for all $x \in X$ and $y \in Y$.*

Remark 3.1. *Let $(X, \cdot, 0_X)$ and $(Y, \star, 0_Y)$ be BCC-algebras. Then $(X \times Y, \diamond, (0_X, 0_Y))$ is a BCC-algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.*

Proposition 3.2. *If $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are two intuitionistic hesitant fuzzy BCC-filters of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is also an intuitionistic hesitant fuzzy BCC-filter of $X \times Y$.*

Proof. Let $(x, y) \in X \times Y$. Then

$$\begin{aligned} h(0_X, 0_Y) &= h_A(0_X) \cap h_B(0_Y) \\ &\supseteq h_A(x) \cap h_B(y) \\ &= h(x, y), \end{aligned}$$

$$\begin{aligned} k(0_X, 0_Y) &= k_A(0_X) \cup k_B(0_Y) \\ &\subseteq k_A(x) \cup k_B(y) \\ &= k(x, y). \end{aligned}$$

Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned} h(y_1, y_2) &= h_A(y_1) \cap h_B(y_2) \\ &\supseteq (h_A(x_1 \cdot y_1) \cap h_A(x_1)) \cap (h_B(x_2 \star y_2) \cap h_B(x_2)) \\ &= h_A(x_1 \cdot y_1) \cap h_B(x_2 \star y_2) \cap h_A(x_1) \cap h_B(x_2) \\ &= h(x_1 \cdot y_1, x_2 \star y_2) \cap h(x_1, x_2) \\ &= h((x_1, x_2) \diamond (y_1, y_2)) \cap h(x_1, x_2), \end{aligned}$$

$$\begin{aligned} k(y_1, y_2) &= k_A(y_1) \cup k_B(y_2) \\ &\subseteq (k_A(x_1 \cdot y_1) \cup k_A(x_1)) \cup (k_B(x_2 \star y_2) \cup k_B(x_2)) \\ &= k_A(x_1 \cdot y_1) \cup k_B(x_2 \star y_2) \cup k_A(x_1) \cup k_B(x_2) \\ &= k(x_1 \cdot y_1, x_2 \star y_2) \cup k(x_1, x_2) \\ &= k((x_1, x_2) \diamond (y_1, y_2)) \cup k(x_1, x_2). \end{aligned}$$

Hence, $A \times B$ is an intuitionistic hesitant fuzzy BCC-filter of $X \times Y$. □

Theorem 3.14. *Two intuitionistic hesitant fuzzy sets $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are intuitionistic hesitant fuzzy BCC-filters of BCC-algebras X and Y, respectively if and only if the intuitionistic hesitant fuzzy sets $\oplus(A \times B)$ and $\otimes(A \times B)$ are intuitionistic hesitant fuzzy BCC-filters of $X \times Y$.*

Proof. It follows from Proposition 3.2 and Theorem 3.11. □

A mapping $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ of BCC-algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCC-algebras, then $f(0_X) = 0_Y$.

Definition 3.15. *Let f be a function from a nonempty set X to a nonempty set Y. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy set on Y, then the intuitionistic hesitant fuzzy set $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ in X is called the pre-image of $\mathcal{H}$ under f.*

Theorem 3.15. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy BCC-filter of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ is an intuitionistic hesitant fuzzy BCC-filter of X.*

Proof. By assumption, $h(f(0_X)) = h(0_Y) \supseteq h(y)$ for every $y \in Y$. In particular, $(h \circ f)(0_X) = h(f(0_X)) \supseteq h(f(x)) = (h \circ f)(x)$ for all $x \in X$. Also, $k(f(0_X)) = k(0_Y) \subseteq k(y)$ for every $y \in Y$. In particular, $(k \circ f)(0_X) = k(f(0_X)) \subseteq k(f(x)) = (k \circ f)(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned}(h \circ f)(y) &= h(f(y)) \\ &\supseteq h(f(x) \star f(y)) \cap h(f(x)) \\ &= h(f(x \cdot y)) \cap h(f(x)) \\ &= (h \circ f)(x \cdot y) \cap (h \circ f)(x),\end{aligned}$$

$$\begin{aligned}(k \circ f)(y) &= k(f(y)) \\ &\subseteq k(f(x) \star f(y)) \cup k(f(x)) \\ &= k(f(x \cdot y)) \cup k(f(x)) \\ &= (k \circ f)(x \cdot y) \cup (k \circ f)(x).\end{aligned}$$

Hence, $f^{-1}(\mathcal{H})$ is an intuitionistic hesitant fuzzy BCC-filter of X. □

4. Conclusion

In the present paper, we have introduced the concepts of intuitionistic hesitant fuzzy BCC-subalgebras, BCC-ideals, and BCC-filters of BCC-algebras. The relationship between intuitionistic hesitant fuzzy BCC-subalgebras (BCC-ideals, BCC-filters) and their level subsets is described. Moreover, the homomorphic pre-images of intuitionistic hesitant fuzzy BCC-filters in BCC-algebras are also studied and some related properties are investigated.

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