Int. J. Anal. Appl. (2023), 21:62

International Journal of Analysis and Applications

Strong and Δ-Convergence of a New Iteration for Common Fixed Points of Two Asymptotically Nonexpansive Mappings

J. Robert Dhiliban[1,*], **A. Anthony Eldred**[2]

[1] *Department of Mathematics, Arul Anandar College (Autonomous), Madurai-625514, Tamilnadu, India*

[2] *Department of Mathematics, St. Joseph's College (Autonomous), Affiliated to Bharathidasan University, Tiruchirappalli-620002, Tamilnadu, India*

[*] *Corresponding author:* jrdhiliban@gmail.com

Abstract. The purpose of this paper is to study strong and Δ - convergence of a newly defined iteration to a common fixed point of two asymptotically nonexpansive self mappings in a hyperbolic space framework. We provide an example and a comparison table to support our assertions.

1. Introduction

Globel and Kirk [1] introduced the concept of asymptotically nonexpansive mappings and proved that *every asymptotically nonexpansive self mapping on a non empty closed subset K of a uniformly convex Banach space X posseses a fixed point*. Ever since, many authors (see, [2], [3], [4] and [5]) have established strong and weak convergence theorems for asymptotically nonexpansive mappings based on the modified Mann [6] and Ishikawa [7] iterations. Tan and Xu [8] studied the modified Ishikawa iteration scheme:

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n & = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1 \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ bounded away from 0 and 1.

Received: Apr. 28, 2023.

2020 *Mathematics Subject Classification.* 47H10, 47J25.

Key words and phrases. asymptotically nonexpansive mapping; common fixed points; Δ-convergence; hyperbolic space.

https://doi.org/10.28924/2291-8639-21-2023-62
ISSN: 2291-8639

Aggarwal et al [9] in an attempt to obtain a faster rate of convergence, modified the above iteration process (1.1) as following:

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n \\ y_n & = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1 \end{cases} \tag{1.2}$$

This iteration is called the modified S-iteration process. For further results on Ishikawa iteration process, (refer, [10], [11], [12] and [13]). Recently, iterative approximations are defined and investigated in the framework of hyperbolic spaces. Several authors (refer, [14], [15] and [16]) have put forward different notions of hyperbolic spaces in order to blend convexity and metric structure. The following definition given by Kohlenbach [17] is widely used.

Definition 1.1. *[17] A hyperbolic space is a triplet (X, d, W), where (X, d) is a metric space and $W : X^2 \to [0, 1]$ is a mapping that satisfies the following conditions:*

(1) $d(u, W(x, y, \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y)$
(2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
(3) $W(x, y, \alpha) = W(y, x, (1-\alpha))$
(4) $d(W(x, z, \alpha), W(y, v, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, v)$

for all $x, y, z, u, v \in X$ and $\alpha, \beta \in [0, 1]$.

2. Preliminaries

We recall some definitions and basic concepts which will be useful for our work.

Definition 2.1. *[1] Let (X, d) be a metric space and let K be a closed convex subset of X. A mapping $T : K \to K$ is said to be asymptotically nonexpansive, if there is a sequence of real numbers $\{k_n\} \in [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $d(T^n x, T^n y) \leq k_n d(x, y)$ for all $x, y \in X$ and $\forall\ n \in \mathbb{N}$.*

The concept of an asymptotically nonexpansive mapping is a natural generalization of a nonexpansive mapping $(d(Tx, Ty) \leq d(x, y))$. The set $F(T) = \{Tx = x : x \in K\}$ shall denote the set of all fixed points of any mapping T.

Definition 2.2. *[23] A subset K of a hyperbolic space (X, d, W) is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\forall\ \alpha \in [0, 1]$.*

Definition 2.3. *[24] A hyperbolic space (X, d, W) is said to be uniformly convex if for any $x, y, z \in X$, $r > 0$ and $\epsilon \in (0, 2]$, there is a $\delta \in (0, 1]$ so that $d(W(x, y, \frac{1}{2}), z) \leq (1-\delta)r$ whenever $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq \epsilon r$.*

Definition 2.4. *[25], [26] Consider a bounded sequence $\{x_n\}$ in a hyperbolic space (X, d, W). For any $x \in X$, define, $r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x_n, x)$ and $r(\{x_n\}) = \inf\{r(x, \{x_n\})/x \in X\}$. The* *asymptotic center* $A(\{x_n\})$ *of a bounded sequence* $\{x_n\}$ *is defined as* $A(\{x_n\}) = \{x \in X / r(x, \{x_n\}) \leq r(y, \{x_n\}), \forall\ y \in X\}$.

It is well known that in uniformly convex Banach spaces, bounded sequences have unique asymptotic centers. The following Lemma proved by Leustean [27] guarantees that complete uniformly convex hyperbolic spaces also enjoy this property.

Lemma 2.1. *[27] Let* (X, d, W) *be a complete uniformly convex hyperbolic space. Then every bounded sequence* $\{x_n\}$ *in* X *has a unique asymptotic center.*

Definition 2.5. *[28] A sequence* $\{x_n\}$ *in a hyperbolic space* (X, d, W) *is said to* Δ*-converge to a point* $x \in X$*, if every subsequence* $\{z_n\}$ *of* $\{x_n\}$ *has* x *as its unique asymptotic center.*

Lemma 2.2. *[29] Let* K *be a nonempty closed convex subset of a uniformly convex hyperbolic space* (X, d, W) *and let* $\{x_n\}$ *be a bounded sequence in* K *such that* $A(\{x_n\}) = \{z\}$ *and* $r(\{x_n\}) = \omega$*. If* $\{z_m\}$ *is a sequence in* K *such that* $\lim\limits_{m\to\infty} r(z_m, \{x_n\}) = \omega$*, then* $\lim\limits_{m\to\infty} z_m = z$*.*

Lemma 2.3. *[29] Let* (X, d, W) *be a uniformly convex hyperbolic space. Let* $x \in X$ *and let* $\{t_n\}$ *be a sequence in* $(0, 1)$ *such that* $\delta \leq t_n \leq 1 - \delta$ *for all* $n \in \mathbb{N}$ *and for some* $\delta > 0$*. If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\lim\limits_{n\to\infty} \sup d(x_n, x) \leq c$*,* $\lim\limits_{n\to\infty} \sup d(y_n, x) \leq c$ *and* $\lim\limits_{n\to\infty} d(W(x_n, y_n, t_n), x) = c$ *for some* $c \geq 0$*, then* $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$*.*

Lemma 2.4. *[3] Let* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative numbers such that*

$$\delta_{n+1} \leq \alpha_n \delta_n + \beta_n \quad \forall \quad n \in \mathbb{N}.$$

If $\alpha_n \geq 1 \quad \forall \quad n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} (\alpha_n - 1) < \infty$ *and* $\beta_n < \infty$*, then* $\lim\limits_{n\to\infty} \delta_n$ *exists.*

Uniformly convex Banach spaces and CAT(0) spaces are some of the known examples of hyperbolic spaces. Sahin and Basarir [18] studied the following iterative process in a hyperbolic space setting and established some convergence results under suitable conditions:

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = W(T^n x_n, T^n y_n, \alpha_n) \\ y_n & = W(x_n, T^n x_n, \beta_n), \quad n \geq 1 \end{cases} \tag{2.1}$$

Ishikawa type iteration is also employed to study the convergence of common fixed points of two asymptotically nonexpansive mappings. In a Banach space framework, Das and Debata [19] initiated the study of two mapping iterative procedure. The authors in [20] and [21] have studied the following iteration for the convergence of common fixed points:

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = W(x_n, S^n y_n, \alpha_n) \\ y_n & = W(x_n, T^n x_n, \beta_n), \quad n \geq 1 \end{cases} \tag{2.2}$$

where S and T are asymptotically nonexpansive mappings with atleast one common fixed point and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Recently, Saluja [22] modified the iterative procedure introduced by Khan et al [13] in hyperbolic spaces to obtain a faster iterative procedure:

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = W(T^n x_n, S^n y_n, \alpha_n) \\ y_n & = W(x_n, T^n x_n, \beta_n), \quad n \geq 1 \end{cases} \tag{2.3}$$

where S and T are asymptotically nonexpansive mappings with atleast one common fixed point and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

The purpose of this paper is to introduce and study a new iterative procedure (3.1) even in Banach spaces to approximate the common fixed points of two asymptotically nonexpansive mappings. We prove strong and Δ - convergence of such an iteration in the general nonlinear framework of hyperbolic spaces.

3. Main Results

In this section, we introduce and study a new iterative scheme to approximate common fixed points of two asymptotically nonexpansive mappings in a hyperbolic space.

Let (X, d, W) be a uniformly convex hyperbolic space. Let K be a non-empty subset of X. Let S and T be two asymptotically nonexpansive self mappings on K. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ such that, $\delta \leq \alpha_n, \beta_n \leq 1 - \delta$, for all $n \in \mathbb{N}$ and for some $\delta > 0$.

We define the following iteration:

$$\begin{cases} x_1 & = x \in K \\ x_{n+1} & = W(S^n x_n, T^n y_n, \alpha_n) \\ y_n & = W(x_n, S^n(T^n x_n), \beta_n), \quad n \geq 1 \end{cases} \tag{3.1}$$

Lemma 3.1. *Let K be a non-empty subset of a uniformly convex hyperbolic space X. Let S and T be asymptotically nonexpansive self mappings on K with a common sequence of real numbers $k_n \geq 1$ satisfying $\sum(k_n^2 - 1) < \infty$. Let F denote the set of all common fixed points of S and T. i.e., $F = F(S) \cap F(T)$. Let $p \in F$. If $\{x_n\}$ and $\{y_n\}$ are sequences as defined in (3.1), then $\lim_{n\to\infty} d(x_n, p)$ and $\lim_{n\to\infty} d(y_n, p)$ exist and*

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(y_n, p).$$

Proof. Since $p \in F(S) \cap F(T)$,

$$\begin{aligned} d(x_{n+1}, p) &= d(W(S^n x_n, T^n y_n, \alpha_n), p) \\ &\leq (1 - \alpha_n) d(S^n x_n, p) + \alpha_n d(T^n y_n, p) \end{aligned}$$

$$\leq (1-\alpha_n)k_n d(x_n,p) + \alpha_n k_n d(y_n,p) \tag{3.2}$$

$$\begin{aligned} d(y_n,p) &= d(W(x_n, S^n(T^n x_n), \beta_n), p) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d(S^n(T^n x_n), p) \\ &= (1-\beta_n)d(x_n,p) + \beta_n d(S^n(T^n x_n), S^n(T^n p)) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n k_n d(T^n x_n, T^n p) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n k_n^2 d(x_n,p) \\ &= d(x_n,p)\left[(1-\beta_n) + \beta_n k_n^2\right] \end{aligned} \tag{3.3}$$

By substituting (3.3) in (3.2), we get,

$$\begin{aligned} d(x_{n+1},p) &\leq (1-\alpha_n)k_n d(x_n,p) + \alpha_n k_n\left[(1-\beta_n) + \beta_n k_n^2\right] \ d(x_n,p) \\ &= \left[(1-\alpha_n)k_n + \alpha_n k_n((1-\beta_n) + \beta_n k_n^2)\right] \ d(x_n,p) \\ &= \left[k_n - \alpha_n k_n \beta_n + \alpha_n \beta_n k_n^3\right] \ d(x_n,p) \\ &= \left[1 + (k_n - 1) - \alpha_n \beta_n k_n + \alpha_n \beta_n k_n^3\right] \ d(x_n,p) \\ &= \left[1 + (k_n - 1) + (k_n^2 - 1)\alpha_n \beta_n k_n\right] \ d(x_n,p) \end{aligned} \tag{3.4}$$

Hence, $d(x_{n+1},p) \leq \left[1 + (k_n - 1) + (k_n^2 - 1)\alpha_n \beta_n k_n\right] \ d(x_n,p)$

By Lemma 2.4, $\lim_{n\to\infty} d(x_n,p)$ exists.

$$\text{Let } \lim_{n\to\infty} d(x_n,p) = c. \tag{3.5}$$

From (3.2), we have,

$$d(y_n,p) \leq \left[(1-\beta_n) + \beta_n k_n^2\right] d(x_n,p)$$

$$\text{Hence, } \lim_{n\to\infty} \sup d(y_n,p) \leq \lim_{n\to\infty} \sup d(x_n,p)$$

$$\text{i.e., } \lim_{n\to\infty} \sup d(y_n,p) \leq c. \tag{3.6}$$

Now consider,

$$\begin{aligned} d(x_{n+1},p) &= d(W(S^n x_n, T^n y_n, \alpha_n), p) \\ &\leq (1-\alpha_n)k_n d(x_n,p) + \alpha_n k_n d(y_n,p) \\ &= \left[1 + (k_n - 1) + (k_n^2 - 1)\alpha_n \beta_n k_n\right] d(x_n,p) \end{aligned}$$

By (3.5), we have, $\lim_{n\to\infty} \sup d(x_{n+1},p) = c$ and $\lim_{n\to\infty} \sup d(x_n,p) = c$. Hence, from (3.2) and (3.4),

$$\lim_{n\to\infty} \sup \left[(1-\alpha_n)k_n d(x_n,p) + \alpha_n k_n d(y_n,p)\right] = c.$$

i.e,

$$\lim_{n\to\infty} \sup \left[k_n d(x_n, p) - k_n\alpha_n d(x_n, p) + \alpha_n k_n d(y_n, p)\right] = c.$$

Since, $\lim_{n\to\infty} \sup k_n = 1$, we have,

$$c + \lim_{n\to\infty} \sup \alpha_n k_n \left[d(y_n, p) - d(x_n, p)\right] = c \implies \lim_{n\to\infty} \sup \alpha_n k_n \left[d(y_n, p) - d(x_n, p)\right] = 0.$$

Since, $\lim_{n\to\infty} \sup \alpha_n k_n > 0$, this will imply that,

$$\lim_{n\to\infty} \sup \left[d(y_n, p) - d(x_n, p)\right] = 0.$$

Therefore,

$$\lim_{n\to\infty} \sup d(y_n, p) = c.$$

Similarly, we can show that, $\lim_{n\to\infty} \inf d(y_n, p) = c$.

Hence,

$$\lim_{n\to\infty} d(y_n, p) = c \tag{3.7}$$

□

Lemma 3.2. *Let K be a non-empty subset of a uniformly convex hyperbolic space X. Let S and T be asymptotically nonexpansive self mappings on K with a common sequence of real numbers $k_n \geq 1$ satisfying $\sum(k_n^2 - 1) < \infty$. If $\{x_n\}$ is a sequence as defined in (3.1) and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Let F denote the set of all common fixed points of S and T.
i.e., $F = F(S) \cap F(T)$. Let $p \in F$. Now since, $\lim_{n\to\infty} k_n = 1$, from (3.5), we have,

$$\lim_{n\to\infty} \sup d(T^n y_n, p) \leq \lim_{n\to\infty} \sup d(x_n, p) = c.$$

Similarly,

$$\lim_{n\to\infty} \sup d(S^n x_n, p) \leq c. \tag{3.8}$$

$$\begin{aligned} \text{Now, } d(x_{n+1}, p) &= d(W(S^n x_n, T^n y_n, \alpha_n), p) \\ &\leq [1 + (k_n - 1) + (k_n^2 - 1)\alpha_n\beta_n k_n] d(x_n, p). \end{aligned}$$

From (3.5), we have, $d(W(S^n x_n, T^n y_n, \alpha_n), p) = c$.

By Lemma 2.3, we have,

$$\lim_{n\to\infty} d(S^n x_n, T^n y_n) = 0. \tag{3.9}$$

Now consider,

$$\begin{aligned} d(y_n, p) &= d(W(x_n, S^n(T^n x_n), \beta_n), p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(S^n(T^n x_n), p) \\ &= d(x_n, p)\big[(1-\beta_n) + \beta_n k_n^2\big]. \end{aligned}$$

Since, $\lim_{n\to\infty} \sup d(y_n, p) = c$ and $\lim_{n\to\infty} \sup d(x_n, p) = c$, we have,

$$d(W(x_n, S^n(T^n x_n), \beta_n), p) \to c$$

$$\text{Further, } \lim_{n\to\infty} \sup d(S^n(T^n x_n), p) \leq c. \tag{3.10}$$

So, using Lemma 2.3, we conclude that,

$$\lim_{n\to\infty} d(x_n, S^n(T^n x_n)) = 0. \tag{3.11}$$

$$\begin{aligned} \text{Now, } d(y_n, x_n) &= d(W(x_n, S^n(T^n x_n), \beta_n), x_n) \\ &\leq (1-\beta_n) d(x_n, x_n) + \beta_n d(S^n(T^n x_n), x_n). \end{aligned}$$

$$\text{Using (3.11), } \lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.12}$$

$$\text{From } d(y_n, S^n(T^n x_n)) \leq d(y_n, x_n) + d(x_n, S^n(T^n x_n)),$$

$$\text{we have, } \lim_{n\to\infty} d(y_n, S^n(T^n x_n)) = 0. \tag{3.13}$$

Now,

$$\begin{aligned} d(x_{n+1}, S^n x_n) &= d(W(S^n x_n, T^n y_n, \alpha_n), S^n x_n) \\ &\leq (1-\alpha_n) d(S^n x_n, S^n x_n) + \alpha_n d(T^n y_n, S^n x_n) \\ &\leq (1-\alpha_n) k_n d(x_n, x_n) + \alpha_n d(T^n y_n, S^n x_n). \end{aligned}$$

$$\text{So, } \lim_{n\to\infty} d(x_{n+1}, S^n x_n) = 0. \tag{3.14}$$

Further,

$$\begin{aligned} d(x_{n+1}, T^n y_n) &= d(W(S^n x_n, T^n y_n, \alpha_n), T^n y_n) \\ &\leq (1-\alpha_n) d(S^n x_n, T^n y_n) + \alpha_n d(T^n y_n, T^n y_n) \\ &\leq (1-\alpha_n) d(S^n x_n, T^n y_n) + \alpha_n k_n d(y_n, y_n) \end{aligned}$$

$$\text{yields, } \lim_{n\to\infty} d(x_{n+1}, T^n y_n) = 0. \tag{3.15}$$

$$\text{Now, } d(x_n, S^n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, S^n x_n)$$

$$\text{So, } \lim_{n\to\infty} d(x_n, S^n x_n) = 0. \tag{3.16}$$

Now consider,

$$\begin{aligned} d(x_n, Sx_n) \leq\ & d(x_n, x_{n+1}) + d(x_{n+1}, S^{n+1}x_{n+1}) + d(S^{n+1}x_{n+1}, S^{n+1}x_n) \\ & + d(S^{n+1}x_n, Sx_n) \\ \leq\ & d(x_n, x_{n+1}) + d(x_{n+1}, S^{n+1}x_{n+1}) + k_{n+1}d(x_{n+1}, x_n) + k_1 d(S^n x_n, x_n). \end{aligned}$$

Thus, we conclude that, $\lim_{n\to\infty} d(x_n, Sx_n) = 0.$ (3.17)

And from,

$$d(x_n, T^n y_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^n y_n)$$

we obtain, $\lim_{n\to\infty} d(x_n, T^n y_n) = 0$ (3.18)

and therefore, $\lim_{n\to\infty} d(y_n, T^n y_n) = 0.$ (3.19)

Also, $d(y_n, y_{n+1}) \leq d(y_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}).$

Thus, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$ (3.20)

Now consider,

$$\begin{aligned} d(y_n, Ty_n) \leq\ & d(y_n, y_{n+1}) + d(y_{n+1}, T^{n+1}y_{n+1}) \\ & + d(T^{n+1}y_{n+1}, T^{n+1}y_n) + d(T^{n+1}y_n, Ty_n) \\ \leq\ & d(y_n, y_{n+1}) + d(y_{n+1}, T^{n+1}y_{n+1}) + k_{n+1}d(y_{n+1}, y_n) + k_1 d(T^n y_n, y_n). \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(y_n, Ty_n) = 0.$ (3.21)

By the asymptotic nonexpansive property of T, $d(Tx_n, Ty_n) \leq k_1 d(x_n, y_n)$.

Hence, $\lim_{n\to\infty} d(Tx_n, Ty_n) = 0.$ (3.22)

From,

$$d(x_n, Tx_n) \leq d(x_n, y_n) + d(y_n, Ty_n) + d(Ty_n, Tx_n),$$

we conclude that, $\lim_{n\to\infty)} d(x_n, Tx_n) = 0.$ This completes the proof. (3.23)

□

Theorem 3.1. *Let K be a non-empty closed convex subset of a uniformly convex hyperbolic space (X, d, W). Let $T : K \to K$ and $S : K \to K$ be asymptotically nonexpansive mappings with $F(T) \neq \phi$ and $F(S) \neq \phi$ and $k_n \geq 1$ satisfying $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. For any initial point $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (3.1). Suppose $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, then, $\{x_n\}$ Δ-converges to an element of $F(T) \cap F(S)$.*

Proof. From Lemma 3.2, $d(x_n, Tx_n) \to 0$ and $d(x_n, Sx_n) \to 0$ as $n \to \infty$.

Lemma 2.1 ensures that any bounded sequence has a unique asymptotic center.

Let $\{z_n\}$ be a subsequence of $\{x_n\}$. Since $\{x_n\}$ is bounded, $\{z_n\}$ is also bounded and suppose that $A(\{x_n\}) = x$ and $A(\{z_n\}) = z$.

Using the asymptotic nonexpansive property of T, we have, $\lim_{n\to\infty} d(T^k z_n, T^{k+1} z_n) = 0$, where $k = 1, 2, 3, ...$

Our purpose is to show that, $z = x$ and $z \in F(T) \cap F(S)$.

Let m and n be positive integers.

$$\text{Now, } d(T^m z, z_n) \le d(T^m z, T^m z_n) + d(T^m z_n, T^{m-1} z_n) + ... + d(Tz_n, z_n)$$
$$\le k_m d(z, z_n) + \sum_{k=0}^{m-1} d(T^k z_n, T^{k+1} z_n).$$

Taking lim sup as $n \to \infty$, for any fixed m, we have,

$$\begin{aligned} r(T^m z, \{z_n\}) &= \lim_{n\to\infty} \sup d(T^m z, \{z_n\}) \\ &\le k_m \lim_{n\to\infty} \sup d(z, \{z_n\}) \\ &= k_m r(z, \{z_n\}). \end{aligned}$$

Now, taking lim sup as $m \to \infty$, we obtain, $\lim_{m\to\infty} \sup r(T^m z, \{z_n\}) \le r(z, \{z_n\})$.

Since $A(\{z_n\}) = z$, we have, $r(z, \{z_n\}) \le r(T^m z, \{z_n\})$, for any fixed $m \in \mathbb{N}$, which implies that, $\lim_{m\to\infty} r(T^m z, \{z_n\}) = r(z, \{z_n\})$. Using Lemma 2.2, we conclude that, $T^m z \to z$ and $z \in F(T)$. By a similar argument, we can show that $z \in F(S)$.

We now claim that, z is the unique asymptotic center for each subsequence $\{z_n\}$ of $\{x_n\}$.

Suppose $x \neq z$. Since $z \in F(T) \cap F(S)$, by Lemma 3.1, $\lim_{n\to\infty} d(x_n, z)$ exists and therefore by the uniqueness of asymptotic centers, we have,

$$\begin{aligned} \lim_{n\to\infty} \sup d(z_n, z) &< \lim_{n\to\infty} \sup d(z_n, x) \\ &\le \lim_{n\to\infty} \sup d(x_n, x) \\ &< \lim_{n\to\infty} \sup d(x_n, z) \\ &= \lim_{n\to\infty} \sup d(z_n, z). \end{aligned}$$

This contradiction proves that z must be equal to x. Since the choice of the subsequence $\{z_n\}$ is arbitrary, we have, $A(\{z_n\}) = \{x\}$, for all subsequences $\{z_n\}$ of $\{x_n\}$. Thus, we conclude that, $\{x_n\}$ Δ-converges to a common fixed point of T and S. □

Theorem 3.2. *Let K be a non-empty subset of a uniformly convex hyperbolic space X. Let S and T be asymptotically nonexpansive self mappings on K. Let $\{x_n\}$ and $\{y_n\}$ be sequences as defined in*

(3.1) and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If either of the mappings T or S is demi-compact, then $\{x_n\}$ and $\{y_n\}$ converge strongly to an element of $F(T) \cap F(S)$.

Proof. Assume T is demi-compact. By Theorem 3.1, we have, $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Then, there exists a subsequence $\{x_{np}\}$ of $\{x_n\}$ such that $Tx_{np} \to z^*$.
Now, $d(x_{np}, z^*) \leq d(x_{np}, Tx_{np}) + d(Tx_{np}, z^*) \to 0$ as $p \to \infty$. Since, $\lim_{n\to\infty} d(x_n, Tx_n) \to 0$, we have $z^* \in F(T)$. Also, $\lim_{n\to\infty} d(x_n, z^*)$ exists. Hence, $x_n \to z^*$ and $d(x_n, y_n) \to 0$ implies that $\lim_{n\to\infty} d(y_n, z^*)$ exists. Further, $d(x_n, Sx_n) \to 0$ implies that $z^* \in F(S)$.
Hence, $\{x_n\}$ and $\{y_n\}$ converges strongly to $z^* \in F(T) \cap F(S)$. □

As an illusration, we consider the following example in a Banach space setting.

Example 3.1. *Consider $K = B(0; 0.9)$, the ball centred at 0 and radius 0.9 in $\mathbb{R}^2$. Let S and T be self mappings on K defined by $S(x_1, x_2) = (x_1^2, x_2^2)$ and $T(x_1, x_2) = (\sin x_1, \sin x_2)$. Let $x, y \in K$, so that $x = (x_1, x_2)$ and $y = (y_1, y_2)$.*

Assume that $y_1 < x_1$ and $y_2 < x_2$.

$$\begin{aligned}
\text{Now, } d(S^n x, S^n y) &= \|S^n x - S^n y\| \\
&= \left\| \left(x_1^{2n}, x_2^{2n}\right) - \left(y_1^{2n}, y_2^{2n}\right) \right\| \\
&= \left[\left(x_1^{2n} - y_1^{2n}\right)^2 + \left(x_2^{2n} - y_2^{2n}\right)^2 \right]^{\frac{1}{2}} \\
&= \Big[|x_1 - y_1|^2 \left\{ x_1^{2n-1} + y_1 x_1^{2n-2} + \ldots + y_1^{2n-1} \right\}^2 \\
&\quad + |x_2 - y_2|^2 \left\{ x_2^{2n-1} + y_2 x_2^{2n-2} + \ldots + y_2^{2n-1} \right\}^2 \Big]^{\frac{1}{2}} \\
&\leq \left[|x_1 - y_1|^2 \left\{ 2^n x_1^{2n-1} \right\}^2 + |x_2 - y_2|^2 \left\{ 2^n x_2^{2n-1} \right\}^2 \right]^{\frac{1}{2}}
\end{aligned}$$

Take $l_n = \max\left\{1, 2^n x_1^{2n-1}\right\}$ and $m_n = \max\left\{1, 2^n x_2^{2n-1}\right\}$. Let $k_n = \max\{l_n, m_n\}$. Then clearly $k_n \to 1$ as $n \to \infty$.

$$\begin{aligned}
\text{So, } d(S^n x, S^n y) &\leq k_n \left[|x_1 - y_1|^2 + |x_2 - y_2|^2 \right]^{\frac{1}{2}} \\
&= k_n \|x - y\|.
\end{aligned}$$

Hence S is an asymptotically nonexpansive mapping on K. Also T is a nonexpansive mapping on K and $(0, 0)$ is a common fixed point of T and S.

The following table shows that our new iterative scheme has a comparitively better rate of convergence than some of the existing iterative schemes. Here, we take $x_1 = \left(\frac{3}{4}, \frac{3}{4}\right)$ and $\alpha_n = \beta_n = \frac{1}{2}, \forall n \in \mathbb{N}$.

$Iterations$	new iteration defined as in (3.1)	iteration defined as in (2.3)	iteration defined as in (2.2)
I	$y_1 = (0.607316, 0.607316)$ $x_2 = (0.566583, 0.566583)$	$y_1 = (0.715819, 0.715819)$ $x_2 = (0.597018, 0.597018)$	$y_1 = (0.715819, 0.715819)$ $x_2 = (0.631199, 0.631199)$
II	$y_2 = (0.286736, 0.286736)$ $x_3 = (0.091520, 0.091520)$	$y_2 = (0.456532, 0.456532)$ $x_3 = (0.179742, 0.179742)$	$y_2 = (0.489716, 0.489716)$ $x_3 = (0.344357, 0.344357)$
III	$y_3 = (0.045760, 0.045760)$ $x_4 = (0.000048, 0.000048)$	$y_3 = (0.092728, 0.092728)$ $x_4 = (0.002857, 0.002857)$	$y_3 = (0.191416, 0.191416)$ $x_4 = (0.172179, 0.172179)$
IV	$y_4 = (0.000024, 0.000024)$ $x_5 = (0.000000, 0.000000)$	$y_4 = (0.001428, 0.001428)$ $x_5 = (0.000000, 0.000000)$	$y_4 = (0.086520, 0.086520)$ $x_5 = (0.086090, 0.086090)$
V	$y_5 = (0.000000, 0.000000)$ $x_6 = (0.000000, 0.000000)$	$y_5 = (0.000000, 0.000000)$ $x_6 = (0.000000, 0.000000)$	$y_5 = (0.043047, 0.043047)$ $x_6 = (0.043045, 0.043045)$
VI			$y_6 = (0.021522, 0.021522)$ $x_7 = (0.021522, 0.021522)$
VII			$y_7 = (0.010761, 0.010761)$ $x_8 = (0.010761, 0.010761)$
$VIII$			$y_8 = (0.005381, 0.005381)$ $x_9 = (0.005381, 0.005381)$
IX			$y_9 = (0.002690, 0.002690)$ $x_{10} = (0.002690, 0.002690)$
X			$y_{10} = (0.001345, 0.001345)$ $x_{11} = (0.001345, 0.001345)$
XI			$y_{11} = (0.000673, 0.000673)$ $x_{12} = (0.000673, 0.000673)$
XII			$y_{12} = (0.000336, 0.000336)$ $x_{13} = (0.000336, 0.000336)$
$XIII$			$y_{13} = (0.000168, 0.000168)$ $x_{14} = (0.000168, 0.000168)$
XIV			$y_{14} = (0.000084, 0.000084)$ $x_{15} = (0.000084, 0.000084)$
XV			$y_{15} = (0.000042, 0.000042)$ $x_{16} = (0.000042, 0.000042)$
XVI			$y_{16} = (0.000021, 0.000021)$ $x_{17} = (0.000021, 0.000021)$
$XVII$			$y_{17} = (0.000011, 0.000011)$ $x_{18} = (0.000011, 0.000011)$
$XVIII$			$y_{18} = (0.000005, 0.000005)$ $x_{19} = (0.000005, 0.000005)$
XIX			$y_{19} = (0.000003, 0.000003)$ $x_{20} = (0.000003, 0.000003)$
XX			$y_{20} = (0.000001, 0.000001)$ $x_{21} = (0.000001, 0.000001)$
XXI			$y_{21} = (0.000001, 0.000001)$ $x_{22} = (0.000001, 0.000001)$
$XXII$			$y_{22} = (0.000000, 0.000000)$ $x_{23} = (0.000000, 0.000000)$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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