Strong and $\Delta$-Convergence of a New Iteration for Common Fixed Points of Two Asymptotically Nonexpansive Mappings

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Abstract. The purpose of this paper is to study strong and $\Delta$ - convergence of a newly defined iteration to a common fixed point of two asymptotically nonexpansive self mappings in a hyperbolic space framework. We provide an example and a comparison table to support our assertions.

1. Introduction

Gobel and Kirk [1] introduced the concept of asymptotically nonexpansive mappings and proved that every asymptotically nonexpansive self mapping on a non empty closed subset $K$ of a uniformly convex Banach space $X$ posseses a fixed point. Ever since, many authors (see, [2], [3], [4] and [5]) have established strong and weak convergence theorems for asymptotically nonexpansive mappings based on the modified Mann [6] and Ishikawa [7] iterations. Tan and Xu [8] studied the modified Ishikawa iteration scheme:

$$\begin{align*}
x_1 &\in K \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n \\
y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1
\end{align*} \quad (1.1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ bounded away from 0 and 1.

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Aggarwal et al [9] in an attempt to obtain a faster rate of convergence, modified the above iteration process (1.1) as following:

\[
\begin{align*}
    & x_1 \in K \\
    & x_{n+1} = (1 - \alpha_n)T^nx_n + \alpha_nT^ny_n \\
    & y_n = (1 - \beta_n)x_n + \beta_nT^nx_n, \quad n \geq 1 \\
\end{align*}
\]

This iteration is called the modified S-iteration process. For further results on Ishikawa iteration process, (refer, [10], [11], [12] and [13]). Recently, iterative approximations are defined and investigated in the framework of hyperbolic spaces. Several authors (refer, [14], [15] and [16]) have put forward different notions of hyperbolic spaces in order to blend convexity and metric structure. The following definition given by Kohlenbach [17] is widely used.

**Definition 1.1.** [17] A hyperbolic space is a triplet \((X, d, W)\), where \((X, d)\) is a metric space and \(W : X^2 \to [0, 1]\) is a mapping that satisfies the following conditions:

1. \(d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y)\)
2. \(d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)\)
3. \(W(x, y, \alpha) = W(y, x, (1 - \alpha))\)
4. \(d(W(x, z, \alpha), W(y, v, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, v)\)

for all \(x, y, z, u, v \in X\) and \(\alpha, \beta \in [0, 1]\).

2. Preliminaries

We recall some definitions and basic concepts which will be useful for our work.

**Definition 2.1.** [1] Let \((X, d)\) be a metric space and let \(K\) be a closed convex subset of \(X\). A mapping \(T : K \to K\) is said to be asymptotically nonexpansive, if there is a sequence of real numbers \(\{k_n\} \subseteq [1, \infty)\) such that \(\lim_{n \to \infty} k_n = 1\) and \(d(T^nx, T^ny) \leq k_n d(x, y)\) for all \(x, y \in X\) and \(\forall \ n \in \mathbb{N}\).

The concept of an asymptotically nonexpansive mapping is a natural generalization of a nonexpansive mapping \((d(Tx, Ty) \leq d(x, y))\). The set \(F(T) = \{Tx = x : x \in K\}\) shall denote the set of all fixed points of any mapping \(T\).

**Definition 2.2.** [23] A subset \(K\) of a hyperbolic space \((X, d, W)\) is convex if \(W(x, y, \alpha) \in K\) for all \(x, y \in X\) and \(\forall \ \alpha \in [0, 1]\).

**Definition 2.3.** [24] A hyperbolic space \((X, d, W)\) is said to be uniformly convex if for any \(x, y, z \in X\), \(r > 0\) and \(\epsilon \in (0, 2]\), there is a \(\delta \in (0, 1]\) so that \(d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r\) whenever \(d(x, z) \leq r\), \(d(y, z) \leq r\) and \(d(x, y) \geq \epsilon r\).

**Definition 2.4.** [25], [26] Consider a bounded sequence \(\{x_n\}\) in a hyperbolic space \((X, d, W)\). For any \(x \in X\), define, \(r(x, \{x_n\}) = \lim_{n \to \infty} \sup d(x_n, x)\) and \(r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\}\). The
asymptotic center $A\{x_n\}$ of a bounded sequence $\{x_n\}$ is defined as $A\{x_n\} = \{x \in X/ r(x, \{x_n\}) \leq r(y, \{x_n\}), \forall y \in X\}$.

It is well known that in uniformly convex Banach spaces, bounded sequences have unique asymptotic centers. The following Lemma proved by Leustean [27] guarantees that complete uniformly convex hyperbolic spaces also enjoy this property.

**Lemma 2.1.** [27] Let $(X,d,W)$ be a complete uniformly convex hyperbolic space. Then every bounded sequence $\{x_n\}$ in $X$ has a unique asymptotic center.

**Definition 2.5.** [28] A sequence $\{x_n\}$ in a hyperbolic space $(X,d,W)$ is said to $\Delta$-converge to a point $x \in X$, if every subsequence $\{z_{n_k}\}$ of $\{x_n\}$ has $x$ as its unique asymptotic center.

**Lemma 2.2.** [29] Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $(X,d,W)$ and let $\{x_n\}$ be a bounded sequence in $K$ such that $A\{x_n\} = \{z\}$ and $r(\{x_n\}) = \omega$. If $\{z_m\}$ is a sequence in $K$ such that $\lim_{{m \to \infty}} r(z_m,\{x_n\}) = \omega$, then $\lim_{{m \to \infty}} z_m = z$.

**Lemma 2.3.** [29] Let $(X,d,W)$ be a uniformly convex hyperbolic space. Let $x \in X$ and let $\{t_n\}$ be a sequence in $(0,1)$ such that $\delta \leq t_n \leq 1 - \delta$ for all $n \in \mathbb{N}$ and for some $\delta > 0$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_{{n \to \infty}} \sup_{{n \in \mathbb{N}}} d(x_n, x) \leq c$, $\lim_{{n \to \infty}} \sup_{{n \in \mathbb{N}}} d(y_n, x) \leq c$ and $\lim_{{n \to \infty}} d(W(x_n, y_n, t_n), x) = c$ for some $c \geq 0$, then $\lim_{{n \to \infty}} d(x_n, y_n) = 0$.

**Lemma 2.4.** [3] Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences of nonnegative numbers such that

$$\delta_{n+1} \leq \alpha_n \delta_n + \beta_n \quad \forall \quad n \in \mathbb{N}.$$ 

If $\alpha_n \geq 1 \quad \forall \quad n \in \mathbb{N}$ and $\sum_{{n=1}}^{{\infty}} (\alpha_n - 1) < \infty$ and $\beta_n < \infty$, then $\lim_{{n \to \infty}} \delta_n$ exists.

Uniformly convex Banach spaces and CAT(0) spaces are some of the known examples of hyperbolic spaces. Sahin and Basarir [18] studied the following iterative process in a hyperbolic space setting and established some convergence results under suitable conditions:

$$\begin{cases}
    x_1 & \in K \\
    x_{n+1} = W(T^n x_n, T^n y_n, \alpha_n) \\
    y_n = W(x_n, T^n x_n, \beta_n), \quad n \geq 1
\end{cases}$$  \hspace{1cm} (2.1)

Ishikawa type iteration is also employed to study the convergence of common fixed points of two asymptotically nonexpansive mappings. In a Banach space framework, Das and Debata [19] initiated the study of two mapping iterative procedure. The authors in [20] and [21] have studied the following iteration for the convergence of common fixed points:

$$\begin{cases}
    x_1 & \in K \\
    x_{n+1} = W(x_n, S^n y_n, \alpha_n) \\
    y_n = W(x_n, T^n x_n, \beta_n), \quad n \geq 1
\end{cases}$$  \hspace{1cm} (2.2)
where $S$ and $T$ are asymptotically nonexpansive mappings with at least one common fixed point and \{\alpha_n\} and \{\beta_n\} are sequences in $(0, 1)$.

Recently, Saluja [22] modified the iterative procedure introduced by Khan et al [13] in hyperbolic spaces to obtain a faster iterative procedure:

$$
\begin{align*}
&x_1 \in K \\
n &x_{n+1} = W(T^n x, S^n y, \alpha_n) \\
y_n = W(x, T^n x, \beta_n), \quad n \geq 1
\end{align*}
$$

where $S$ and $T$ are asymptotically nonexpansive mappings with at least one common fixed point and \{\alpha_n\} and \{\beta_n\} are sequences in $(0, 1)$.

The purpose of this paper is to introduce and study a new iterative procedure (3.1) even in Banach spaces to approximate the common fixed points of two asymptotically nonexpansive mappings. We prove strong and $\Delta$ - convergence of such an iteration in the general nonlinear framework of hyperbolic spaces.

3. Main Results

In this section, we introduce and study a new iterative scheme to approximate common fixed points of two asymptotically nonexpansive mappings in a hyperbolic space.

Let $(X, d, W)$ be a uniformly convex hyperbolic space. Let $K$ be a non-empty subset of $X$. Let $S$ and $T$ be two asymptotically nonexpansive self mappings on $K$. Let \{\alpha_n\} and \{\beta_n\} be sequences in $(0, 1)$ such that, $\delta \leq \alpha_n, \beta_n \leq 1 - \delta$, for all $n \in \mathbb{N}$ and for some $\delta > 0$.

We define the following iteration:

$$
\begin{align*}
&x_1 \in K \\
n &x_{n+1} = W(S^n x, T^n y, \alpha_n) \\
y_n = W(x, S^n (T^n x), \beta_n), \quad n \geq 1
\end{align*}
$$

**Lemma 3.1.** Let $K$ be a non-empty subset of a uniformly convex hyperbolic space $X$. Let $S$ and $T$ be asymptotically nonexpansive self mappings on $K$ with a common sequence of real numbers $k_n \geq 1$ satisfying $\sum (k_n^2 - 1) < \infty$. Let $F$ denote the set of all common fixed points of $S$ and $T$. i.e., $F = F(S) \cap F(T)$. Let $p \in F$. If \{\alpha_n\} and \{\beta_n\} are sequences as defined in (3.1), then $\lim_{n \to \infty} d(x_n, p)$ and $\lim_{n \to \infty} d(y_n, p)$ exist and

$$
\lim_{n \to \infty} d(x_n, p) = \lim_{n \to \infty} d(y_n, p).
$$

**Proof.** Since $p \in F(S) \cap F(T)$,

$$
\begin{align*}
d(x_{n+1}, p) &= d(W(S^n x, T^n y, \alpha_n), p) \\
&\leq (1 - \alpha_n)d(S^n x, p) + \alpha_n d(T^n y, p)
\end{align*}
$$

for all $n \geq 1$. Therefore, $\lim_{n \to \infty} d(x_n, p)$ and $\lim_{n \to \infty} d(y_n, p)$ exist.
By substituting (3.3) in (3.2), we get,

\[ d(y_n, p) = d(W(x_n, S^n(T^n x_n), \beta_n), p) \]

\[ \leq (1 - \beta_n)d(x_n, p) + \beta_n d(S^n(T^n x_n), p) \]

\[ = (1 - \beta_n)d(x_n, p) + \beta_n d(S^n(T^n x_n), S^n(T^n p)) \]

\[ \leq (1 - \beta_n)d(x_n, p) + \beta_n k_n d(T^n x_n, T^n p) \]

\[ \leq (1 - \beta_n)d(x_n, p) + \beta_n k_n^2 d(x_n, p) \]

\[ = d(x_n, p)[(1 - \beta_n) + \beta_n k_n^2] \] (3.3)

By substituting (3.3) in (3.2), we get,

\[ d(x_{n+1}, p) \leq (1 - \alpha_n)k_n d(x_n, p) + \alpha_n k_n\left[(1 - \beta_n) + \beta_n k_n^2\right]d(x_n, p) \]

\[ = [(1 - \alpha_n)k_n + \alpha_n k_n(1 - \beta_n) + \beta_n k_n^2]d(x_n, p) \]

\[ = [k_n - \alpha_n k_n \beta_n + \alpha_n \beta_n k_n^3]d(x_n, p) \]

\[ = [1 + (k_n - 1) - \alpha_n k_n \beta_n + \alpha_n \beta_n k_n^3]d(x_n, p) \]

\[ = [1 + (k_n - 1) + (k_n^2 - 1)\alpha_n k_n \beta_n]d(x_n, p) \] (3.4)

Hence,

\[ d(x_{n+1}, p) \leq [1 + (k_n - 1) + (k_n^2 - 1)\alpha_n k_n \beta_n]d(x_n, p) \]

By Lemma 2.4, \( \lim_{n \to \infty} d(x_n, p) \) exists.

Let \( \lim_{n \to \infty} d(x_n, p) = c. \) (3.5)

From (3.2), we have,

\[ d(y_n, p) \leq [(1 - \beta_n) + \beta_n k_n^2]d(x_n, p) \]

Hence, \( \lim_{n \to \infty} \sup d(y_n, p) \leq \lim_{n \to \infty} \sup d(x_n, p) \)

i.e., \( \lim_{n \to \infty} \sup d(y_n, p) \leq c. \) (3.6)

Now consider,

\[ d(x_{n+1}, p) = d(W(S^n x_n, T^n y_n, \alpha_n), p) \]

\[ \leq (1 - \alpha_n)k_n d(x_n, p) + \alpha_n k_n d(y_n, p) \]

\[ = [1 + (k_n - 1) + (k_n^2 - 1)\alpha_n \beta_n k_n]d(x_n, p) \]

By (3.5), we have, \( \lim_{n \to \infty} \sup d(x_{n+1}, p) = c \) and \( \lim_{n \to \infty} \sup d(x_n, p) = c. \) Hence, from (3.2) and (3.4),

\[ \lim_{n \to \infty} \sup [(1 - \alpha_n)k_n d(x_n, p) + \alpha_n k_n d(y_n, p)] = c. \]
i.e,
\[
\lim_{n \to \infty} \sup \left[ k_n d(x_n, p) - k_n \alpha_n d(x_n, p) + \alpha_n k_n d(y_n, p) \right] = c.
\]
Since, \( \lim_{n \to \infty} k_n = 1 \), we have,
\[
c + \lim_{n \to \infty} \sup \alpha_n k_n \left[ d(y_n, p) - d(x_n, p) \right] = c \implies \lim_{n \to \infty} \sup \alpha_n k_n \left[ d(y_n, p) - d(x_n, p) \right] = 0.
\]
Since, \( \lim_{n \to \infty} \sup \alpha_n k_n > 0 \), this will imply that,
\[
\lim_{n \to \infty} \sup \left[ d(y_n, p) - d(x_n, p) \right] = 0.
\]
Therefore,
\[
\lim_{n \to \infty} \sup d(y_n, p) = c.
\]
Similarly, we can show that, \( \lim_{n \to \infty} \inf d(y_n, p) = c \).

Hence,
\[
\lim_{n \to \infty} d(y_n, p) = c \tag{3.7}
\]

\[\square\]

**Lemma 3.2.** Let \( K \) be a non-empty subset of a uniformly convex hyperbolic space \( X \). Let \( S \) and \( T \) be asymptotically nonexpansive self mappings on \( K \) with a common sequence of real numbers \( k_n \geq 1 \) satisfying \( \sum (k_n^2 - 1) < \infty \). If \( \{x_n\} \) is a sequence as defined in (3.1) and \( d(x_n, x_{n+1}) \to 0 \) as \( n \to \infty \), then \( \lim_{n \to \infty} d(x_n, Sx_n) = 0 \) and \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \).

**Proof.** Let \( F \) denote the set of all common fixed points of \( S \) and \( T \).
i.e., \( F = F(S) \cap F(T) \). Let \( p \in F \). Now since, \( \lim_{n \to \infty} k_n = 1 \), from (3.5), we have,
\[
\lim_{n \to \infty} \sup d(T^n y_n, p) \leq \lim_{n \to \infty} \sup d(x_n, p) = c.
\]
Similarly,
\[
\lim_{n \to \infty} \sup d(S^n x_n, p) \leq c \tag{3.8}
\]
Now, \( d(x_{n+1}, p) = d(W(S^n x_n, T^n y_n, \alpha_n), p) \)
\[
\leq [1 + (k_n - 1) + (k_n^2 - 1)\alpha_n \beta_n k_n] d(x_n, p).
\]
From (3.5), we have, \( d(W(S^n x_n, T^n y_n, \alpha_n), p) = c \).
By Lemma 2.3, we have,
\[
\lim_{n \to \infty} d(S^n x_n, T^n y_n) = 0 \tag{3.9}
\]
Now consider,

\[ d(y_n, p) = d(W(x_n, S^n(T^n x_n), \beta_n), p) \]
\[ \leq (1 - \beta_n)d(x_n, p) + \beta_n d(S^n(T^n x_n), p) \]
\[ = d(x_n, p)[(1 - \beta_n) + \beta_n k_n^2]. \]

Since, \( \lim_{n \to \infty} \sup d(y_n, p) = c \) and \( \lim_{n \to \infty} \sup d(x_n, p) = c \), we have,

\[ d(W(x_n, S^n(T^n x_n), \beta_n), p) \to c. \]

Further, \( \lim_{n \to \infty} \sup d(S^n(T^n x_n), p) \leq c. \) (3.10)

So, using Lemma 2.3, we conclude that,

\[ \lim_{n \to \infty} d(x_n, S^n(T^n x_n)) = 0. \] (3.11)

Now, \( d(y_n, x_n) = d(W(x_n, S^n(T^n x_n), \beta_n), x_n) \)
\[ \leq (1 - \beta_n)d(x_n, x_n) + \beta_n d(S^n(T^n x_n), x_n). \]

Using (3.11), \( \lim_{n \to \infty} d(x_n, y_n) = 0. \) (3.12)

From \( d(y_n, S^n(T^n x_n)) \leq d(y_n, x_n) + d(x_n, S^n(T^n x_n)) \),
we have, \( \lim_{n \to \infty} d(y_n, S^n(T^n x_n)) = 0. \) (3.13)

Now,

\[ d(x_{n+1}, S^n x_n) = d(W(S^n x_n, T^n y_n, \alpha_n), S^n x_n) \]
\[ \leq (1 - \alpha_n)d(S^n x_n, S^n x_n) + \alpha_n d(T^n y_n, S^n x_n) \]
\[ \leq (1 - \alpha_n)k_n d(x_n, x_n) + \alpha_n d(T^n y_n, S^n x_n). \]

So, \( \lim_{n \to \infty} d(x_{n+1}, S^n x_n) = 0. \) (3.14)

Further,

\[ d(x_{n+1}, T^n y_n) = d(W(S^n x_n, T^n y_n, \alpha_n), T^n y_n) \]
\[ \leq (1 - \alpha_n)d(S^n x_n, T^n y_n) + \alpha_n d(T^n y_n, T^n y_n) \]
\[ \leq (1 - \alpha_n)d(S^n x_n, T^n y_n) + \alpha_n k_n d(y_n, y_n) \]
yields, \( \lim_{n \to \infty} d(x_{n+1}, T^n y_n) = 0. \) (3.15)

Now, \( d(x_n, S^n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, S^n x_n) \)
So, \( \lim_{n \to \infty} d(x_n, S^n x_n) = 0. \) (3.16)
Now consider,
\[
d(x_n, Sx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, S^{n+1}x_{n+1}) + d(S^{n+1}x_{n+1}, S^{n+1}x_n) \\
+ d(S^{n+1}x_n, Sx_n) \\
\leq d(x_n, x_{n+1}) + d(x_{n+1}, S^{n+1}x_{n+1}) + k_{n+1}d(x_{n+1}, x_n) + k_1d(S^nx_n, x_n).
\]
Thus, we conclude that, \(\lim_{n \to \infty} d(x_n, Sx_n) = 0.\) (3.17)

And from,
\[
d(x_n, T^ny_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^n y_n)
\]
we obtain, \(\lim_{n \to \infty} d(x_n, T^n y_n) = 0\) (3.18)
and therefore, \(\lim_{n \to \infty} d(y_n, T^n y_n) = 0.\) (3.19)

Also, \(d(y_n, y_{n+1}) \leq d(y_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}).\)
Thus, \(\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.\) (3.20)

Now consider,
\[
d(y_n, Ty_n) \leq d(y_n, y_{n+1}) + d(y_{n+1}, T^{n+1}y_{n+1}) \\
+ d(T^{n+1}y_{n+1}, T^{n+1}y_n) + d(T^{n+1}y_n, Ty_n) \\
\leq d(y_n, y_{n+1}) + d(y_{n+1}, T^{n+1}y_{n+1}) + k_{n+1}d(y_{n+1}, y_n) + k_1d(T^ny_n, y_n).
\]
Therefore, \(\lim_{n \to \infty} d(y_n, Ty_n) = 0.\) (3.21)

By the asymptotic nonexpansive property of \(T, d(Tx_n, Ty_n) \leq k_1d(x_n, y_n).\)
Hence, \(\lim_{n \to \infty} d(Tx_n, Ty_n) = 0.\) (3.22)

From,
\[
d(x_n, Tx_n) \leq d(x_n, y_n) + d(y_n, Ty_n) + d(Ty_n, Tx_n).
\]
we conclude that, \(\lim_{n \to \infty} d(x_n, Tx_n) = 0.\) This completes the proof. (3.23)

**Theorem 3.1.** Let \(K\) be a non-empty closed convex subset of a uniformly convex hyperbolic space \((X, d, W)\). Let \(T : K \to K\) and \(S : K \to K\) be asymptotically nonexpansive mappings with \(F(T) \neq \emptyset\) and \(F(S) \neq \emptyset\) and \(k_n \geq 1\) satisfying \(\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty.\) For any initial point \(x_1 \in K, define the sequence \(\{x_n\}\) iteratively by (3.1). Suppose \(d(x_n, x_{n+1}) \to 0\) as \(n \to \infty, then, \{x_n\} \Delta\)-converges to an element of \(F(T) \cap F(S)\). □
Proof. From Lemma 3.2, \(d(x_n, Tx_n) \to 0\) and \(d(x_n, Sx_n) \to 0\) as \(n \to \infty\).

Lemma 2.1 ensures that any bounded sequence has a unique asymptotic center.

Let \(\{z_n\}\) be a subsequence of \(\{x_n\}\). Since \(\{x_n\}\) is bounded, \(\{z_n\}\) is also bounded and suppose that \(A(\{x_n\}) = x\) and \(A(\{z_n\}) = z\).

Using the asymptotic nonexpansive property of \(T\), we have, \(\lim_{n \to \infty} d(T^kz_n, T^{k+1}z_n) = 0\), where \(k = 1, 2, 3, \ldots\)

Our purpose is to show that, \(z = x\) and \(z \in F(T) \cap F(S)\).

Let \(m\) and \(n\) be positive integers.

Now, \(d(T^mz, z_n) \leq d(T^mz, T^mz_n) + d(T^mz_n, T^{m-1}z_n) + \ldots + d(Tz_n, z_n)\)

\[\leq k_md(z, z_n) + \sum_{k=0}^{m-1} d(T^kz_n, T^{k+1}z_n).\]

Taking \(\limsup\) as \(n \to \infty\), for any fixed \(m\), we have,

\[r(T^mz, \{z_n\}) = \lim_{n \to \infty} \sup d(T^mz, \{z_n\})\]

\[\leq k_m \lim_{n \to \infty} \sup d(z, \{z_n\})\]

\[= k_mr(z, \{z_n\}).\]

Now, taking \(\limsup\) as \(m \to \infty\), we obtain, \(\lim_{m \to \infty} r(T^mz, \{z_n\}) \leq r(z, \{z_n\}).\)

Since \(A(\{z_n\}) = z\), we have, \(r(z, \{z_n\}) \leq r(T^mz, \{z_n\})\), for any fixed \(m \in \mathbb{N}\), which implies that, \(\lim_{m \to \infty} r(T^mz, \{z_n\}) = r(z, \{z_n\})\). Using Lemma 2.2, we conclude that, \(T^mz \to z\) and \(z \in F(T)\). By a similar argument, we can show that \(z \in F(S)\).

We now claim that, \(z\) is the unique asymptotic center for each subsequence \(\{z_n\}\) of \(\{x_n\}\).

Suppose \(x \neq z\). Since \(z \in F(T) \cap F(S)\), by Lemma 3.1, \(\lim_{n \to \infty} d(x_n, z)\) exists and therefore by the uniqueness of asymptotic centers, we have,

\[\lim_{n \to \infty} \sup d(z_n, z) < \lim_{n \to \infty} \sup d(z_n, x)\]

\[\leq \lim_{n \to \infty} \sup d(x_n, x)\]

\[< \lim_{n \to \infty} \sup d(x_n, z)\]

\[= \lim_{n \to \infty} \sup d(z_n, z).\]

This contradiction proves that \(z\) must be equal to \(x\). Since the choice of the subsequence \(\{z_n\}\) is arbitrary, we have, \(A(\{z_n\}) = \{x\}\), for all subsequences \(\{z_n\}\) of \(\{x_n\}\). Thus, we conclude that, \(\{x_n\}\) \(\Delta\)-converges to a common fixed point of \(T\) and \(S\). \(\Box\)

**Theorem 3.2.** Let \(K\) be a non-empty subset of a uniformly convex hyperbolic space \(X\). Let \(S\) and \(T\) be asymptotically nonexpansive self mappings on \(K\). Let \(\{x_n\}\) and \(\{y_n\}\) be sequences as defined in
(3.1) and \( d(x_n, x_{n+1}) \to 0 \) as \( n \to \infty \). If either of the mappings \( T \) or \( S \) is demi-compact, then \( \{x_n\} \)
and \( \{y_n\} \) converge strongly to an element of \( F(T) \cap F(S) \).

Proof. Assume \( T \) is demi-compact. By Theorem 3.1, we have, \( d(x_n, Tx_n) \to 0 \) as \( n \to \infty \). Then, there exists a subsequence \( \{x_{np}\} \) of \( \{x_n\} \) such that \( Tx_{np} \to z^* \).

Now, \( d(x_{np}, z^*) \leq d(x_{np}, Tx_{np}) + d(Tx_{np}, z^*) \to 0 \) as \( p \to \infty \). Since, \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \), we have \( z^* \in F(T) \). Also, \( \lim_{n \to \infty} d(x_n, z^*) \) exists. Hence, \( x_n \to z^* \) and \( d(x_n, y_n) \to 0 \) implies that \( \lim_{n \to \infty} d(y_n, z^*) \) exists. Further, \( d(x_n, Sx_n) \to 0 \) implies that \( z^* \in F(S) \).

Hence, \( \{x_n\} \) and \( \{y_n\} \) converges strongly to \( z^* \in F(T) \cap F(S) \).

As an illustration, we consider the following example in a Banach space setting.

Example 3.1. Consider \( K = B(0; 0.9) \), the ball centred at 0 and radius 0.9 in \( \mathbb{R}^2 \). Let \( S \) and \( T \) be self mappings on \( K \) defined by \( S(x_1, x_2) = (x_1^2, x_2^2) \) and \( T(x_1, x_2) = (\sin x_1, \sin x_2) \). Let \( x, y \in K \), so that \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \).

Assume that \( y_1 < x_1 \) and \( y_2 < x_2 \).

Now, \( d(S^n x, S^n y) = \|S^n x - S^n y\| \)
\[= \|(x_1^{2n}, x_2^{2n}) - (y_1^{2n}, y_2^{2n})\| \]
\[= \left[ (x_1^{2n} - y_1^{2n})^2 + (x_2^{2n} - y_2^{2n})^2 \right]^\frac{1}{2} \]
\[= \left[ |x_1 - y_1|^2 \left\{ x_1^{2n-1} + y_1 x_1^{2n-2} + ... + y_1^{2n-1} \right\}^2 \right]^\frac{1}{2} \]
\[+ |x_2 - y_2|^2 \left\{ x_2^{2n-1} + y_2 x_2^{2n-2} + ... + y_2^{2n-1} \right\}^2 \]
\[\leq \left[ |x_1 - y_1|^2 \left\{ 2^n x_1^{2n-1} \right\}^2 + |x_2 - y_2|^2 \left\{ 2^n x_2^{2n-1} \right\}^2 \right]^\frac{1}{2} \]

Take \( l_n = \max \{ 1, 2^n x_1^{2n-1} \} \) and \( m_n = \max \{ 1, 2^n x_2^{2n-1} \} \). Let \( k_n = \max \{ l_n, m_n \} \). Then clearly \( k_n \to 1 \) as \( n \to \infty \).

So, \( d(S^n x, S^n y) \leq k_n \left[ |x_1 - y_1|^2 + |x_2 - y_2|^2 \right]^\frac{1}{2} \]
\[= k_n \|x - y\|. \]

Hence \( S \) is an asymptotically nonexpansive mapping on \( K \). Also \( T \) is a nonexpansive mapping on \( K \) and \( (0, 0) \) is a common fixed point of \( T \) and \( S \).

The following table shows that our new iterative scheme has a comparatively better rate of convergence than some of the existing iterative schemes. Here, we take \( x_1 = \left( \frac{3}{4}, \frac{3}{4} \right) \) and \( \alpha_n = \beta_n = \frac{1}{2}, \forall n \in \mathbb{N} \).
<table>
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<tr>
<th>Iterations</th>
<th>new iteration defined as in (3.1)</th>
<th>iteration defined as in (2.3)</th>
<th>iteration defined as in (2.2)</th>
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<td>$y_3 = (0.092728, 0.092728)$</td>
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Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


