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Geometry of Admissible Curves of Constant-Ratio in Pseudo-Galilean Space

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Abstract. An admissible curve of a pseudo-Galilean space is said to be of constant-ratio if the ratio of the length of the tangent and normal components of its position vector function is a constant. In this paper, we investigate and characterize a spacelike admissible curve of constant-ratio in terms of its curvature functions in the pseudo-Galilean space G_3^1. Also, we study some special curves of constant-ratio such as T-constant and N-constant types of these curves. Finally, we give some computational examples for constructing the meant curves to demonstrate our theoretical results.

1. Introduction

According to the space curve theory, it is well known that, a curve $\alpha(s)$ in E^3 lies on a sphere if its position vector lies on its normal plane at each point. If the position vector α lies on its rectifying plane then $\alpha(s)$ is called a rectifying curve [1]. Rectifying curves are characterized by the simple equation:

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s), \tag{1.1}$$

where $\lambda(s)$ and $\mu(s)$ are smooth functions and $T(s)$ and $B(s)$ are tangent and binormal vector fields of α, respectively. In [2] the author provided that a twisted curve is congruent to a non constant linear function of s. On the other hand, in the Minkowski 3-space E_1^3, the rectifying curves were investigated in [3, 4]. Besides, in [4] a characterization of the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space in terms of centrodes were given. The characterization of rectifying curves in three dimensional compact Lee groups as well as in dual spaces were given in [5], [6], respectively. For the study of constant-ratio curves, the authors gave the necessary and sufficient conditions for

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curves in Euclidean and Minkowski spaces to become T-constant or N-constant [7–10]. In analogy with the Euclidean 3-dimensional case, our main goal in this work is to define the spacelike admissible curves of constant-ratio in the pseudo Galilean 3-space as a curve whose position vector always lies in the orthogonal complement $N^{\perp}$ of its principal normal vector field N. Consequently, $N^{\perp}$ is given by

$$N^{\perp} = \{V \in G_3^1 :< V, N >= 0\},$$

where $<\cdot,\cdot>$ denotes the inner product in G_3^1. Hence $N^{\perp}$ is a 2-dimensional plane of G_3^1, spanned by the tangent and binormal vector fields T and B, respectively. Therefore, the position vector with respect to some chosen origin of a considered curve α in G_3^1, satisfies the parametric equation:

$$\alpha(s) = m_o(s)T(s) + m_1(s)N(s) + m_2(s)B(s), \tag{1.2}$$

for some differential functions $m_i(s)$, $0 \leq i \leq 2$, where s is arc-length parameter. Then, we give the necessary and sufficient conditions for the curve α in G_3^1 to be a constant-ratio curve.

2. Pseudo-Galilean geometry

In this section, we introduce the basic concepts, familiar definitions and notations on pseudo-Galilean space which are needed throughout this study. The pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature (0,0,+,-). The absolute of the pseudo-Galilean geometry is an ordered triple $\{w,\ f,\ I\}$ where w is the ideal (absolute) plane, f is a line in w and I is the fixed hyperbolic involution of points of f, for more details, we refer to [11, 12]. The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space which was presented in [11]. The inner and cross product of two vectors $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ in G_3^1 are, respectively defined as follows:

$$g(\mathbf{x}, \mathbf{y}) = \begin{cases} x_1x_2, & if\ x_1 \neq 0 \vee x_2 \neq 0, \\ y_1y_2 - z_1z_2 & if\ x_1 = 0 \wedge x_2 = 0, \end{cases}$$

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

Also the norm of a vector $\mathbf{x} = (x, y, z)$ is given by

$$\|\mathbf{x}\| = \begin{cases} x & , if\ x \neq 0, \\ \sqrt{|y^2 - z^2|} & , if\ x = 0. \end{cases} \tag{2.1}$$

The group of motions of the pseudo-Galilean G_3^1 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned}\bar{x} &= a + x,\\ \bar{y} &= b + cx + y\cosh\varphi + z\sinh\varphi,\\ \bar{z} &= d + ex + y\sinh\varphi + z\cosh\varphi.\end{aligned}$$

According to the motion group in pseudo-Galilean space, a vector $\mathbf{x}(x, y, z)$ is said to be non isotropic if $x \neq 0$. All unit non-isotropic vectors are of the form $(1, y, z)$. For isotropic vectors, $x = 0$ holds. There are four types of isotropic vectors: spacelike $(y^2 - z^2) > 0$, timelike $(y^2 - z^2) < 0$, and two types of lightlike $(y = \pm z)$ vectors. A non-lightlike isotropic vector is a unit vector if $y^2 - z^2 = \pm 1$.

A trihedron $(T_o; e_1, e_2, e_3)$ with a proper origin $T_o(x_o, y_o, z_o)$ which is orthonormal in pseudo-Galilean sense if the vectors e_1, e_2, e_3 are of the following form: $e_1 = (1, y_1, z_1)$, $e_2 = (0, y_2, z_2)$ and $e_3 = (0, \varepsilon z_2, \varepsilon y_2)$ with $y^2 - z^2 = \delta$, where ε, δ is $+1$ or -1. Such trihedron $(T_o; e_1, e_2, e_3)$ is called positively oriented if for its vectors, $det(e_1, e_2, e_3) = 1$ holds; that is if $y^2 - z^2 = \varepsilon$.

Let $\alpha(t) : I \subset R \to G_3^1$ be a curve parameterized by $\alpha(t) = (x(t), y(t), z(t))$, where $x(t), y(t), z(t) \in C^3$ (the set of three-times continuously differentiable functions) and t run through a real interval [12].

Definition 2.1. *A curve α given by $\alpha(t) = (x(t), y(t), z(t))$ is admissible if $\dot{x}(t) \neq 0$.*

Also, If α is taken as follows:

$$\alpha(x) = (x, y(x), z(x)), \tag{2.2}$$

with the condition

$$y''^2(x) - z''^2(x) \neq 0, \tag{2.3}$$

then the arc-length parameter s is defined by

$$ds = |\dot{x}(t)dt| = dx. \tag{2.4}$$

Here, we assume that $ds = dx$ and $s = x$ as the arc-length of the curve α [12]. The vector

$$T(s) = \alpha'(s),$$

is called the tangent unit vector of α. Also, the unit vector field is given by

$$N(s) = \frac{\alpha''(s)}{\sqrt{|y''^2(s) - z''^2(s)|}}, \tag{2.5}$$

and the binormal vector is expressed as

$$B(s) = \frac{(0, \varepsilon z''(s), \varepsilon y''(s))}{\sqrt{|y''^2(s) - z''^2(s)|}}, \tag{2.6}$$

and it is orthogonal in pseudo-Galilean sense to the osculating plane of α spanned by the vectors $\alpha'(s)$ and $\alpha''(s)$. The curve α given by Eq. (2.2) is a spacelike (resp. timelike) if $N(s)$ is a timelike (resp.

spacelike) vector. The principal normal vector or simply normal is spacelike if $\varepsilon = +1$ and timelike if $\varepsilon = -1$. Here $\varepsilon = +1$ or -1 is chosen by the criterion $det(T, N, B) = 1$. That means

$$\left|y''^2(s) - z''^2(s)\right| = \varepsilon(y''^2(s) - z''^2(s)). \tag{2.7}$$

Definition 2.2. *In each point of an admissible curve in G_3^1, the associated orthonormal (in pseudo-Galilean sense) trihedron $\{T(s), N(s), B(s)\}$ can be defined. This trihedron is called pseudo-Galilean Frenet trihedron.*

For the pseudo-Galilean Frenet trihedron of an admissible curve α, the Frenet equations are defined as:

$$\begin{aligned} T' &= \kappa N, \\ N' &= \tau B, \\ B' &= \tau N. \end{aligned} \tag{2.8}$$

where κ and τ are the pseudo-Galilean curvatures of α defined as follows:

$$\kappa(s) = \sqrt{|y''^2(s) - z''^2(s)|}, \tag{2.9}$$

$$\tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}, \tag{2.10}$$

and the pseudo-Galilean torsion can be written in the form

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}. \tag{2.11}$$

The Serret-Frenet equations (2.8) can be written in matrix form as

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

The Pseudo-Galilean sphere with radius r is defined by

$$S_{\pm}^2 = \{u \in G_3^1 : g(u, u) = \pm r^2\},$$

3. Spacelike curves of constant-ratio in G_3^1

Let $\alpha : I \subset R \to G_3^1$ be an arbitrary spacelike admissible curve. In the light of which introduced in [13–15], we consider the following theorem.

Theorem 3.1. *The position vector of α with curvatures $\kappa(s)$ and $\tau(s) \neq 0$, and with respect to the Frenet frame in the pseudo-Galilean space G_3^1, it can be written as*

$$\begin{aligned}\alpha &= (s+c_o)T + e^{-\int \tau(s)ds}\left(c_1\ e^{2\int \tau(s)ds} + e^{2\int \tau(s)ds}\int \frac{\kappa(s)(s+c_o)}{2}e^{-\int \tau(s)ds}ds\right.\\ &\quad \left.-\int \frac{\kappa(s)(s+c_o)}{2}e^{\int \tau(s)ds}ds + c_2\right)N + e^{-\int \tau(s)ds}\left(c_1\ e^{2\int \tau(s)ds}\right.\\ &\quad \left.+e^{2\int \tau(s)ds}\int \frac{\kappa(s)(s+c_o)}{2}e^{-\int \tau(s)ds}ds + \int \frac{\kappa(s)(s+c_o)}{2}e^{\int \tau(s)ds}ds - c_2\right)B.\end{aligned} \tag{3.1}$$

where c_o, c_1 and c_2 are arbitrary constants.

Proof. Let α be an arbitrary spacelike curve in the pseudo-Galilean space G_3^1, then we may express its position vector as

$$\alpha(s) = m_o(s)T(s) + m_1(s)N(s) + m_2(s)B(s).$$

Differentiating this equation with respect to the arc-length parameter s and using the Serret-Frenet equations (2.8), we obtain

$$\begin{aligned}\alpha'(s) &= m_o'(s)T(s) + (m_1'(s) + \kappa(s)m_o(s) + \tau(s)m_2(s))N(s)\\ &\quad +(m_2'(s) + \tau(s)m_1(s))B(s),\end{aligned}$$

it follows that

$$\begin{aligned} m_o'(s) &= 1,\\ m_1'(s) + \kappa(s)m_o(s) + \tau(s)m_2(s) &= 0,\\ m_2'(s) + \tau(s)m_1(s) &= 0.\end{aligned} \tag{3.2}$$

From Eqs. (3.2), we have

$$m_o(s) = s + c_o. \tag{3.3}$$

It is useful to change the variable s to the variable $t = \int \tau(s)ds$. Therefore all functions of s will transform to the functions of t. Here, we will use dot to denote the derivative with respect to t (where the prime denotes the derivative with respect to s). Also, From Eq. (3.2), we get

$$m_1(t) = -\dot{m}_2(t), \quad where\ \dot{m}_2 = \frac{dm_2}{dt}, \tag{3.4}$$

it leads to

$$\ddot{m}_2(t) - m_2(t) = \frac{y(t)\kappa(t)}{\tau(t)}, \qquad y(t) = m_o(s) = s + c_o. \tag{3.5}$$

The general solution of this equation is given by

$$m_2(t) = e^{-t}\left[c_1\ e^{2t} + e^{2t}\int \frac{\kappa(t)y(t)}{2\tau(t)}e^{-t}dt + \int \frac{\kappa(t)y(t)}{2\tau(t)}e^{t}dt - c_2\right], \tag{3.6}$$

where c_1 and c_2 are arbitrary constants. From Eqs. (3.4) and (3.6), we obtain the function $m_1(t)$ as

$$m_1(t) = e^{-t}\left[c_1\ e^{2t} + e^{2t}\int \frac{\kappa(t)y(t)}{2\tau(t)}e^{-t}dt - \int \frac{\kappa(t)y(t)}{2\tau(t)}e^{t}dt + c_2\right]. \tag{3.7}$$

Hence, Eqs. (3.6) and (3.7) take the following forms:

$$m_1 = e^{-\int \tau(s)ds}\left[c_1\ e^{2\int \tau(s)ds} + e^{2\int \tau(s)ds}\int \frac{(s+c_o)\kappa}{2}e^{-\int \tau(s)ds}ds - \int \frac{(s+c_o)\kappa}{2}e^{\int \tau(s)ds}ds + c_2\right], \quad (3.8)$$

$$m_2 = e^{-\int \tau(s)ds}\left[c_1\ e^{2\int \tau(s)ds} + e^{2\int \tau(s)ds}\int \frac{(s+c_o)\kappa}{2}e^{-\int \tau(s)ds}ds + \int \frac{(s+c_o)\kappa}{2}e^{\int \tau(s)ds}ds - c_2\right]. \quad (3.9)$$

Substituting from Eqs. (3.3), (3.8) and (3.9) in Eq. (1.2), the result (3.1) is obtained and thus, the proof is completed. □

Theorem 3.2. *Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve with $\kappa \neq 0$ and $\tau \neq 0$ in G_3^1. Then the position vector and curvatures of α satisfy a vector differential equation of third order.*

Proof. Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve with curvatures $\kappa \neq 0$ and $\tau \neq 0$ in G_3^1. From Frenet equations (2.8), one can write

$$N = \frac{T'}{\kappa}, \quad (3.10)$$

$$B = \frac{N'}{\tau}. \quad (3.11)$$

Substituting Eq. (3.10) in Eq. (2.8), we get

$$B' = \frac{\tau}{\kappa}T'. \quad (3.12)$$

Differentiating Eq. (3.10) with respect to s and substituting in Eq. (3.10), we find

$$B = \frac{1}{\tau}\left[\left(\frac{1}{\kappa}\right)' T' + \left(\frac{1}{\kappa}\right) T''\right]. \quad (3.13)$$

Similarly, taking the differentiation of Eq. (3.13) and equalize with Eq. (2.8), we obtain

$$\frac{1}{\tau\kappa}T''' + \left[2\frac{1}{\tau}\left(\frac{1}{\kappa}\right)' - \left(\frac{1}{\tau}\right)'\frac{1}{\kappa}\right]T'' + \left[\frac{1}{\tau}\left(\left(\frac{1}{\kappa}\right)'' - \frac{\tau^2}{\kappa}\right) - \left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)'\right]T' = 0. \quad (3.14)$$

Hence, it completes the proof. □

Theorem 3.3. *The position vector $\alpha(s)$ of a spacelike admissible curve with curvature $\kappa(s)$ and torsion $\tau(s)$ in the pseudo-Galilean space G_3^1 is computed from the intrinsic representation form*

$$\alpha(s) = \left(s, -\int\left[\int \kappa(s)\sinh[\int \tau(s)ds]ds\right]ds, \int\left[\int \kappa(s)\cosh[\int \tau(s)ds]ds\right]ds\right),$$

with tangent, principal normal and binormal vectors respectively, are given by

$$T(s) = \left(1, -\int \kappa(s)\sinh[\int \tau(s)ds]ds, \int \kappa(s)\cosh[\int \tau(s)ds]ds\right),$$

$$N(s) = \left(0, -\sinh[\int \tau(s)ds], \cosh[\int \tau(s)ds]\right),$$

$$B(s) = \left(0, -\cosh[\int \tau(s)ds], \sinh[\int \tau(s)ds]\right).$$

Now, for each given $\alpha : I \subset R \to G_3^1$, there is a natural orthogonal decomposition of the position vector α at each point on α; namely,

$$\alpha = \alpha^T + \alpha^N, \tag{3.15}$$

where α^T and α^N denote the tangential and normal components of α at the point, respectively. Let $\|\alpha^T\|$ and $\|\alpha^N\|$ denote the length of α^T and α^N, respectively. In what follows we introduce the notion of constant-ratio curves. So, similar to the Euclidean case [16], we consider the following definitions [17].

Definition 3.1. *A curve α of the pseudo-Galilean space G_3^1 is said to be of constant-ratio curve if the ratio $\|\alpha^T\| : \|\alpha^N\|$ is constant on $\alpha(I)$.*

Clearly, for a constant-ratio curve in G_3^1, we have

$$\frac{m_o^2}{m_2^2 - m_1^2} = c_3, \tag{3.16}$$

for some constant c_3.

Definition 3.2. *Let $\alpha : I \subset R \to G_3^1$ be an admissible curve in G_3^1. If $\|\alpha^T\|$ is constant, then α is called T-constant curve. Further, T-constant curve α is called of first kind if $\|\alpha^T\| = 0$, otherwise is called of second kind.*

Definition 3.3. *Let $\alpha : I \subset R \to G_3^1$ be an admissible curve in G_3^1. If $\|\alpha^N\|$ is constant, then α is called a N-constant curve. For a N-constant curve α, either $\|\alpha^N\| = 0$ or $\|\alpha^N\| = \mu$ for some non-zero smooth function μ. Further, a N-constant curve α is called of first kind if $\|\alpha^N\| = 0$, otherwise it is of second kind.*

For N-constant curve α in G_3^1, we can write

$$\left\|\alpha^N(s)\right\|^2 = m_2^2(s) - m_1^2(s) = c_4, \tag{3.17}$$

where c_4 is constant.
In what follows, we characterize the admissible curves in terms of their curvature functions $m_i(s)$ and give the necessary and sufficient conditions for these curves to be T-constant or N-constant curves.

Theorem 3.4. *Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve in G_3^1. Then α is of constant-ratio if and only if*

$$\left(\frac{\kappa' - \kappa^3 c_3(s + c_o)}{c_3 \kappa^2 \tau}\right)' = \frac{-\tau}{c_3 \kappa}.$$

Proof. Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve given with the invariant parameter s. Then, we have

$$m_o(s) = s + c_o,$$

where c_o is an arbitrary constant. Also, from Eq. (3.16), the curvature functions $m_i(s)$, $0 \leq i \leq 2$ satisfy

$$m_2(s)m_2'(s) - m_1(s)m_1'(s) = \frac{s + c_o}{c_3}. \tag{3.18}$$

By using Eqs. (3.2) with Eq. (3.18), we obtain

$$m_1 = \frac{1}{c_3\kappa},$$

it follows that

$$m_2 = \frac{\kappa' - \kappa^3 c_3(s + c_o)}{c_3\kappa^2\tau},$$

thus, the result is clear. □

3.1. **T-constant spacelike curves in G_3^1.**

Proposition 3.1. *There are no T-constant spacelike curves in pseudo-Galilean space G_3^1.*

Proof. Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve in G_3^1. Then $\left\|\alpha^T\right\| = m_o$, where m_o is equal to zero or a nonzero constant. Since $m_o = x + c_o$, this contradicts the fact of value of m_o. □

3.2. **N-constant spacelike curves in G_3^1.**

Lemma 3.1. *Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve in G_3^1. Then α is N-constant curve if and only if the following condition:*

$$m_2(s)m_2'(s) - m_1(s)m_1'(s) = 0,$$

holds together Eqs. (3.2), where $m_i(s)$, $0 \leq i \leq 2$ are differentiable functions.

Proposition 3.2. *Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve in G_3^1. Then α is a N-constant curve of first kind if α is a straight line in G_3^1.*

Proof. Suppose that α is N-constant curve of first kind in G_3^1, then

$$m_2^2(s) - m_1^2(s) = 0.$$

So, we have two cases to be discussed:

Case 1.

$$m_2(s) = m_1(s).$$

Using Eqs. (3.2), we get

$$\kappa = 0.$$

Case 2.

$$m_2(s) = -m_1(s).$$

Also, from Eqs. (3.2), we obtain

$$\kappa = 0.$$

It means that the curve α is a straight line in G_3^1. □

Theorem 3.5. *Let $\alpha : I \subset R \to G_3^1$ be a spacelike curve in G_3^1. If α is N-constant curve of second kind, then the position vector α has the parametrization:*

$$\begin{aligned}\alpha(s) &= (s+c_o)T(s) + \left[\frac{1}{4}e^{-u(s)}\left(-4c_4 + e^{2u(s)}\right) - \frac{1}{2}e^{u(s)}\right]N(s) \\ &\quad + \left[\frac{1}{4}e^{-u(s)}\left(-4c_4 + e^{2u(s)}\right)\right]B(s), \end{aligned} \tag{3.19}$$

where $u(s) = \int \tau(s)ds + c_5$, c_5 is integral constant.

Proof. From Eq. (3.3), we have

$$m_o(s) = (s + c_o).$$

Besides, from of Eq. (3.2) and Eq. (3.17), we obtain

$$m_2'^2(s) - \tau^2(s)m_2^2(s) - c_4\tau^2(s) = 0,$$

where $c_4 \neq 0$ is a real constant. The solution of this equation is given by

$$m_2(s) = \frac{1}{4}e^{-u(s)}\left(-4c_4 + e^{2u(s)}\right). \tag{3.20}$$

If we substitute Eq. (3.3) in Eq. (3.2), we can get

$$m_1(s) = \frac{1}{4}e^{-u(s)}\left(-4c_4 + e^{2u(s)}\right) - \frac{1}{2}e^{u(s)}, \tag{3.21}$$

hence, in light of Eqs. (3.3), (3.20) and (3.21), we obtain the required result. □

Theorem 3.6. *Let α be a spacelike curve in G_3^1 with its pseudo-Galilean trihedron $\{T(s), N(s), B(s)\}$. If the curve α lies on a pseudo-Galilean sphere $S_\pm^2$, then it is N-constant curve of second kind and the center of a pseudo-Galilean sphere of α at the point $c(s)$ is given by*

$$c(s) = \alpha(s) + m_1(s)N(s) + m_2(s)B(s).$$

Proof. Let $S_\pm^2$ be a sphere in G_3^1, then $S_\pm^2$ is given by

$$S_\pm^2 = \{u \in G_3^1 : g(u,u) = \pm r^2\},$$

where r is the radius of the pseudo-Galilean sphere and it is a constant. Let c be the center of the pseudo-Galilean sphere, then we have

$$g(c(s) - \alpha(s), c(s) - \alpha(s)) = \pm r^2.$$

Differentiating this equation with respect to s, we get

$$g(-T(s), c(s) - \alpha(s)) = 0, \tag{3.22}$$

more differentiation yields

$$g(-T'(s), c(s) - \alpha(s)) + g(-T(s), -T(s)) = 0.$$

From Eq. (2.8), we find

$$-\kappa(s)g(N(s), c(s) - \alpha(s)) + 1 = 0, \tag{3.23}$$

and since $c(s) - \alpha(s) \in Sp\{T(s), N(s), B(s)\}$, then we can write

$$c(s) - \alpha(s) = m_o(s)T(s) + m_1(s)N(s) + m_2(s)B(s). \tag{3.24}$$

Now, from Eq. (3.23) and (3.24), we find

$$\kappa(s)m_1(s) + 1 = 0,$$

it follows that

$$m_1(s) = -\frac{1}{\kappa(s)}.$$

Also, from Eq. (3.22) and (3.24), one can write

$$g(T(s), c(s) - \alpha(s)) = m_o(s),$$

which gives

$$m_o(s) = 0,$$

and then Eq. (3.24) becomes

$$c(s) - \alpha(s) = m_1(s)N(s) + m_2(s)B(s).$$

Besides, the derivation of Eq. (3.23) leads to

$$m_2(s) = \frac{-m_1'(s)}{\tau(s)}.$$

Now, from aforementioned information, we obtain

$$m_2^2(s) - m_1^2(s) = \pm r^2 = const.$$

which completes the proof. □

Theorem 3.7. *Let α be N-constant curve of second kind which lies on a pseudo-Galilean sphere $S_\pm^2$ with constant radius r in G_3^1. Then*

$$m_2'(s) - \tau(s)m_1(s) = 0,$$

where $m_2(s) \neq 0$, $\tau(s) \neq 0$.

Proof. Let α be a N-constant curve in G_3^1, then we have

$$m_2^2(s) - m_1^2(s) = \pm r^2,$$

since r is constant, then

$$m_2(s)m_2'(s) - m_1(s)m_1'(s) = 0.$$

Substituting $m_2(s) = \frac{m_1'(s)}{\tau(s)}$ in this equation, we get

$$m_2'(s) - \tau(s)m_1(s) = 0.$$

Thus, the proof is completed. □

Theorem 3.8. *Let $\alpha(s)$ be a spacelike curve in G_3^1 with $\kappa(s) \neq 0$, $\tau(s) \neq 0$. The image of the N-constant curve α lies on a pseudo-Galilean sphere $S_{\pm}^2$ if and only if for each $s \in I \subset R$, its curvatures satisfy the following equalities:*

$$\begin{aligned} s + c_o &= 0, \\ \frac{1}{4}e^{-u(s)}\left(-4c_4 + e^{2u(s)}\right) - \frac{1}{2}e^{u(s)} &= \frac{1}{\kappa(s)}, \\ \frac{1}{4}e^{-u(s)}\left(-4c_4 + e^{2u(s)}\right) &= \frac{\kappa'(s)}{\kappa^2(s)\tau(s)}, \end{aligned} \tag{3.25}$$

where $u(s) = \int \tau(s)ds + c_5$ and c_o, c_4 and $c_5 \in R$.

Proof. By assumption, we have

$$g(\alpha(s), \alpha(s)) = r^2,$$

for every $s \in I \subset R$ and r is the radius of the pseudo-Galilean sphere. Differentiating this equation with respect to s gives

$$g(T(s), \alpha(s)) = 0. \tag{3.26}$$

Again, differentiation leads to

$$g(N(s), \alpha(s)) = -\frac{1}{\kappa(s)}, \tag{3.27}$$

and also

$$g(B(s), \alpha(s)) = \frac{\kappa'(s)}{\kappa^2(s)\tau(s)}. \tag{3.28}$$

Using Eqs. (3.26)-(3.28) in Eq. (3.19), we obtain the required result: Eq. (3.25).
Conversely, we assume that Eq. (3.25) holds, for each $s \in I \subset R$, then from Eq. (3.19), the position vector of α can be expressed as

$$\alpha(s) = -\frac{1}{\kappa(s)}N(s) + \frac{\kappa'(s)}{\kappa^2(s)\tau(s)}B(s),$$

which satisfies the equation: $g(\alpha(s), \alpha(s)) = r^2$. It means that the curve α lies on the pseudo-Galilean sphere $S_{\pm}^2$. Hence, the proof is completed. □

Theorem 3.9. *Let α be a spacelike curve in G_3^1. If α is a circle then α is N-constant curve of second kind.*

Proof. If α is a circle, then we have

$$\kappa(s) = const \quad \text{and} \quad \tau(s) = 0.$$

Also, from Theorem 3.4, one can write

$$m_1 = \frac{1}{c_3\kappa} = const.,$$

$$m_2 = \int \left(\frac{-\tau}{c_3\kappa}\right) ds = const.,$$

which leads to

$$m_2^2(s) - m_1^2(s) = const.$$

thus, it completes the proof. □

4. Examples

In this section, we give some examples to illustrate our main results.

Example 4.1. *Consider the following spacelike curve $\alpha : I \subset R \to G_3^1$, given by*

$$\alpha(s) = \left(s, \frac{s}{6}\left[2\sinh(2\ln s) - \cosh(2\ln s)\right], \frac{s}{6}\left[2\cosh(2\ln s) - \sinh(2\ln s)\right]\right). \tag{4.1}$$

Differentiating Eq. (4.1), we get

$$\alpha'(s) = \left(1, \frac{1}{2}\cosh(2\ln s), \frac{1}{2}\sinh(2\ln s)\right). \tag{4.2}$$

Pseudo-Galilean inner product follows that $\langle \alpha', \alpha' \rangle = 1$. So the curve is parameterized by the arc-length. The tangent vector is

$$T' = \left(0, \frac{1}{s}\sinh(2\ln s), \frac{1}{s}\cosh(2\ln s)\right),$$

by taking the norm of both sides, we have $\kappa(s) = \frac{1}{s}$. Thereafter, we have

$$N = (0, \sinh(2\ln s), \cosh(2\ln s)),$$

and the binormal vector is

$$B = (0, -\cosh(2\ln s), -\sinh(2\ln s)).$$

From Serret-Frenet equations, one can obtain $\tau(s) = \frac{-2}{s}$. Moreover, the curvature functions $m_i(s)$ are

$$m_o = s, \quad m_1 = \frac{s}{c_3}, \quad m_2 = -\Omega\ s,\ \Omega = \left(\frac{1+c_3}{2c_3}\right) = const.$$

So, from Eq. (3.16), we get

$$\frac{m_o^2}{m_2^2 - m_1^2} = F, \quad F = \frac{4(c_3)^2}{(c_3+1)^2 - 4} = const.$$

Under the above considerations, α is of constant-ratio and the ratio is equal F. Also, since

$$\left\|\alpha^N(s)\right\|^2 = m_2^2(s) - m_1^2(s) = \left(\frac{(c_3+1)^2 - 4}{4(c_3)^2}\right) s^2 \neq const.,$$

then the curve α is a constant-ratio curve but not N-constant curve, see Fig(1a).

Example 4.2. *Consider a spacelike curve $\gamma(s)$ in G_3^1 parameterized by*

$$\alpha(s) = \left(s, -a\int\left(\int \sinh(\frac{s^2}{2})ds\right)ds, a\int\left(\int \cosh(\frac{s^2}{2})ds\right)ds\right),$$

where $a \in R$.

Then we have

$$\gamma'(s) = T(s) = \left(1, -a\int \sinh(\frac{s^2}{2})ds, a\int \cosh(\frac{s^2}{2})ds\right),$$
$$T'(s) = \left(0, -a\sinh(\frac{s^2}{2}), a\cosh(\frac{s^2}{2})\right).$$

By a straightforward calculations, we obtain

$$N(s) = \left(0, -\sinh(\frac{s^2}{2}), \cosh(\frac{s^2}{2})\right),$$

$$B(s) = \left(0, -\cosh(\frac{s^2}{2}), \sinh(\frac{s^2}{2})\right),$$

where $\kappa(s) = a = const$ and $\tau(s) = s$.

Since the curve has a constant curvature and non-constant torsion, so it is a Salkowski curve.

From Theorem 3.4, we have the curvature functions:

$$m_1 = \frac{1}{c_3\kappa} = \frac{1}{ac_3},$$
$$m_2 = \frac{\kappa' - \kappa^3 c_3 s}{c_3\kappa^2\tau} = -a, \quad a \text{ is constant},$$

which leads to

$$m_2^2(s) - m_1^2(s) = (-a)^2 - \left(\frac{1}{ac_3}\right)^2 = const.$$

It follows that γ is N-constant curve but not constant-ratio curve, see Fig(1b).

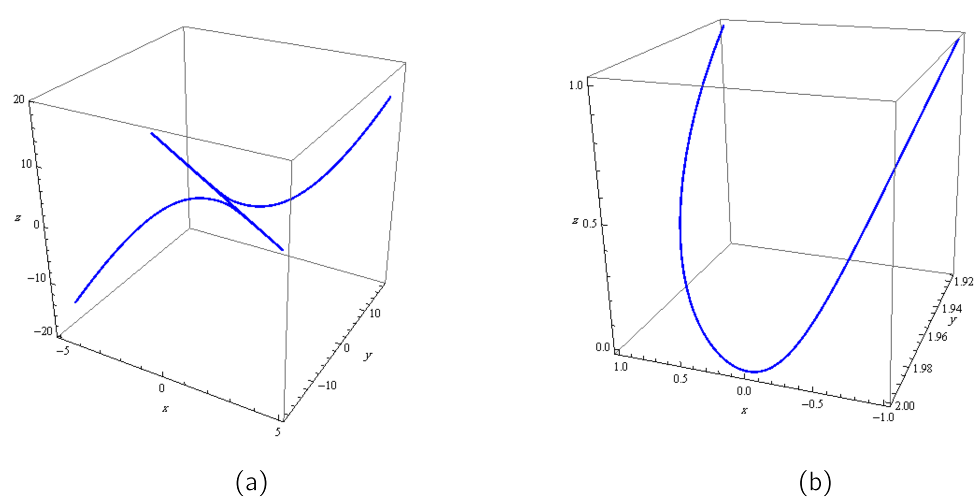

Figure 1. (A) The constant-ratio curve α, (B) the N-constant Salkowski curve γ; $a = 2$.

5. Conclusion

In the three-dimensional pseudo-Galilean space, spacelike admissible curves of constant-ratio and some special curves such as T-constant and N-constant curves have been studied. Furthermore, the spherical images of these curves have been studied. Some interesting results of $N-constant$ curves have been obtained. Finally, as an application for this work, two examples are given and plotted to confirm our main results.

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