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Characteristic Picture Fuzzy Sets and Level Subsets in UP (BCC)-Algebras

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Abstract. The eight new concepts of picture fuzzy sets in UP (BCC)-algebras are introduced by Kankaew et al. in 2022. This idea is extended to the lower and upper level subsets of picture fuzzy sets in UP (BCC)-algebras. Moreover, we define a picture fuzzy set in the same way as a characteristic function and study its characterizations from the related subset.

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [16], BCI-algebras [17], BE-algebras [23], UP-algebras [11], fully UP-semigroups [12], topological UP-algebras [28], UP-hyperalgebras [14], extension of KU/UP-algebras [27] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [17] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [16, 17] in 1966 and have been extensively investigated by many researchers.

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The concept of fuzzy sets was first considered by Zadeh [38] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [38], Atanassov [3, 4] defined a new concept called an intuitionistic fuzzy set which is a generalization of fuzzy set. The concept of picture fuzzy sets was first considered by Cuong and Kreinovich [6] in 2013, which is direct extensions of the fuzzy sets and the intuitionistic fuzzy sets. The picture fuzzy set is characterized by three functions expressing the degree of membership, the degree of neutral membership, and the degree of non-membership. The only constraint is that the sum of the three degrees must not exceed 1. Cuong [5] presented the concept of picture fuzzy sets in the Journal of Computer Science and Cybernetics in 2014. Some operations on picture fuzzy sets with some properties are considered. The Zadeh Extension Principle, picture fuzzy relations, and picture fuzzy soft sets are studied. Several researches were conducted on the generalizations of the concept of picture fuzzy sets in a variety of different fields and its application to a decision-making problem. In 2015, Singh [33] presented a geometrical interpretation of picture fuzzy sets. The author proposed correlation coefficients for picture fuzzy sets which considers the degree of positive membership, degree of neutral membership, degree of negative membership and the degree of refusal membership. In 2017, Wei [35] presented another form of eight similarity measures between picture fuzzy sets based on the cosine function between picture fuzzy sets by considering the degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership in picture fuzzy sets. The author applied these weighted cosine function similarity measures between picture fuzzy sets to strategic decision making. In 2018, Wei and Gao [37] presented some novel Dice similarity measures of picture fuzzy sets and the generalized Dice similarity measures of picture fuzzy sets and indicate that the Dice similarity measures and asymmetric measures (projection measures) are the special cases of the generalized Dice similarity measures in some parameter values. Wei [36] presented some novel process to measure the similarity between picture fuzzy sets. The author applied these similarity measures between picture fuzzy sets to building material recognition and minerals field recognition. In 2020, Ganie et al. [8] introduced two correlation coefficients of picture fuzzy sets. These correlation coefficients of picture fuzzy sets are better than existing ones and effective in expressing the nature of correlation (positive or negative correlation). In 2022, Jun et al. [18] have shown that the concept of UP-algebras (see [11]) and the concept of BCC-algebras (see [24]) are the same concept. Therefore, in this article and future research, our research team will use the name BCC instead of UP in honor of Komori, who first defined it in 1984.

In this paper, we applied the concept of picture fuzzy sets in BCC-algebras to introduce the eight new concepts of picture fuzzy sets: picture fuzzy BCC-subalgebras, picture fuzzy near BCC-filters, picture fuzzy BCC-filters, picture fuzzy implicative BCC-filters, picture fuzzy comparative BCC-filters, picture fuzzy shift BCC-filters, picture fuzzy BCC-ideals, and picture fuzzy strong BCC-ideals. Also, we discuss the relationship between the eight new concepts of picture fuzzy sets in BCC-algebras. This idea is extended to the lower and upper level subsets of picture fuzzy sets in BCC-algebras. Moreover, we define a picture fuzzy set in the same way as a characteristic function and study its characterizations from the related subset.

2. Basic results on BCC-algebras

The concept of BCC-algebras (see [24]) can be redefined without the condition (2.6) as follows:

Definition 2.1. *[10] An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a BCC-algebra, where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms:*

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0), \tag{2.1}$$

$$(\forall x \in X)(0 \cdot x = x), \tag{2.2}$$

$$(\forall x \in X)(x \cdot 0 = 0), \tag{2.3}$$

$$(\forall x, y \in X)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y). \tag{2.4}$$

From [11], we know that the concept of BCC-algebras is a generalization of KU-algebras (see [26]).

The binary relation $\leq$ on a BCC-algebra $X = (X, \cdot, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{2.5}$$

and the following assertions are valid (see [11, 12]).

$$(\forall x \in X)(x \leq x), \tag{2.6}$$

$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z), \tag{2.7}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \cdot x \leq z \cdot y), \tag{2.8}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \cdot z \leq x \cdot z), \tag{2.9}$$

$$(\forall x, y, z \in X)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)), \tag{2.10}$$

$$(\forall x, y \in X)(y \cdot x \leq x \Leftrightarrow x = y \cdot x), \tag{2.11}$$

$$(\forall x, y \in X)(x \leq y \cdot y), \tag{2.12}$$

$$(\forall a, x, y, z \in X)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))), \tag{2.13}$$

$$(\forall a, x, y, z \in X)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z), \tag{2.14}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot z), \tag{2.15}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \cdot y), \tag{2.16}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)), \tag{2.17}$$

$$(\forall a, x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)). \tag{2.18}$$

Example 2.1. *[30] Let U be a nonempty set and let $X \in \mathcal{P}(U)$ where $\mathcal{P}(U)$ means the power set of U. Let $\mathcal{P}_X(U) = \{A \in \mathcal{P}(U) \mid X \subseteq A\}$. Define a binary operation $\triangle$ on $\mathcal{P}_X(U)$ by putting $A \triangle B = B \cap (A^C \cup X)$ for all $A, B \in \mathcal{P}_X(U)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_X(U), \triangle, X)$ is a BCC-algebra. Let $\mathcal{P}^X(U) = \{A \in \mathcal{P}(U) \mid A \subseteq X\}$. Define a binary operation $\blacktriangle$ on $\mathcal{P}^X(U)$ by putting $A \blacktriangle B = B \cup (A^C \cap X)$ for all $A, B \in \mathcal{P}^X(U)$. Then $(\mathcal{P}^X(U), \blacktriangle, X)$ is a BCC-algebra.*

Example 2.2. *[7] Let $\mathbb{Z}^*$ be the set of all nonnegative integers. Define two binary operations $\circ$ and $\star$ on $\mathbb{Z}^*$ by:*

$$(\forall m, n \in \mathbb{Z}^*) \left(m \circ n = \begin{cases} n & \text{if } m < n, \\ 0 & \text{otherwise} \end{cases} \right)$$

and

$$(\forall m, n \in \mathbb{Z}^*) \left(m \star n = \begin{cases} n & \text{if } m > n \text{ or } m = 0, \\ 0 & \text{otherwise} \end{cases} \right).$$

Then $(\mathbb{Z}^*, \circ, 0)$ and $(\mathbb{Z}^*, \star, 0)$ are BCC-algebras.

For more examples of BCC-algebras, see [1, 2, 12, 15, 25, 29–32].

For a nonempty subset S of a BCC-algebra $X = (X, \cdot, 0)$ which satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow x \cdot y \in S). \tag{2.19}$$

Then the constant 0 of X is in S. Indeed, let $x \in S$. By (2.6) and (2.19), we have $0 = x \cdot x \in S$.

Definition 2.2. *[9, 11, 13, 19–21, 34] A nonempty subset S of a BCC-algebra $X = (X, \cdot, 0)$ is called*

(1) *a BCC-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x \cdot y \in S), \tag{2.20}$$

(2) *a near BCC-filter of X if it satisfies the condition (2.19),*

(3) *a BCC-filter of X if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{2.21}$$

$$(\forall x, y \in X)(x \cdot y \in S, x \in S \Rightarrow y \in S), \tag{2.22}$$

(4) *an implicative BCC-filter of X if it satisfies the condition (2.21) and the following condition:*

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, x \cdot y \in S \Rightarrow x \cdot z \in S), \tag{2.23}$$

(5) *a comparative BCC-filter of X if it satisfies the condition (2.21) and the following condition:*

$$(\forall x, y, z \in X)(x \cdot ((y \cdot z) \cdot y) \in S, x \in S \Rightarrow y \in S), \tag{2.24}$$

(6) *a shift BCC-filter of X if it satisfies the condition (2.21) and the following condition:*

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, x \in S \Rightarrow ((z \cdot y) \cdot y) \cdot z \in S), \tag{2.25}$$

(7) *a BCC-ideal of X if it satisfies the condition (2.21) and the following condition:*

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S), \tag{2.26}$$

(8) *a strong BCC-ideal of X if it satisfies the condition (2.21) and the following condition:*

$$(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S). \tag{2.27}$$

Guntasow et al. [9] proved that the only strong BCC-ideal of a BCC-algebra X is X.

The following theorem is easy to verify.

Theorem 2.1. *Let $\mathscr{F}$ be a nonempty family of BCC-subalgebras (resp., near BCC-filters, BCC-filters, implicative BCC-filters, comparative BCC-filters, shift BCC-filters, BCC-ideals, strong BCC-ideals) of a BCC-algebra $X = (X, \cdot, 0)$. Then $\bigcap \mathscr{F}$ is a BCC-subalgebra (resp., near BCC-filter, BCC-filter, implicative BCC-filter, comparative BCC-filter, shift BCC-filter, BCC-ideal, strong BCC-ideal) of X.*

3. PFSs in BCC-algebras

In 2013, Cuong and Kreinovich [6] introduced the concept of picture fuzzy sets as the following definition.

A *picture fuzzy set* (briefly, PFS) in a nonempty set X is a structure of the form:

$$P = \{(x, r_P(x), g_P(x), b_P(x)) \mid x \in X\},$$

where $r_P : X \to [0, 1]$ is a *positive membership*, $g_P : X \to [0, 1]$ is a *neutral membership*, and $b_P : X \to [0, 1]$ is a *negative membership* satisfy the following condition:

$$(\forall x \in X)(r_P(x) + g_P(x) + b_P(x) \leq 1).$$

For our convenience, we will denote a PFS as $P = (X, r_P, g_P, b_P)$.

A PFS P in X is said to be *constant* if P is a constant function from X to $[0, 1]^3$. That is, r_P, g_P, and b_P are constant functions from X to $[0, 1]$.

In what follows, let X denote a BCC-algebra $(X, \cdot, 0)$ unless otherwise specified.

Kankaew et al. [22] introduced the eight new concepts of PFSs in BCC-algebras: picture fuzzy BCC-subalgebras, picture fuzzy near BCC-filters, picture fuzzy BCC-filters, picture fuzzy implicative BCC-filters, picture fuzzy comparative BCC-filters, picture fuzzy shift BCC-filters, picture fuzzy BCC-ideals, and picture fuzzy strong BCC-ideals.

Definition 3.1. *A PFS P in X is called*

(1) *a picture fuzzy BCC-subalgebra of X if it satisfies the following conditions:*

$$(\forall x, y \in X)(r_P(x \cdot y) \geq \min\{r_P(x), r_P(y)\}), \tag{3.1}$$

$$(\forall x, y \in X)(g_P(x \cdot y) \geq \min\{g_P(x), g_P(y)\}), \tag{3.2}$$

$$(\forall x, y \in X)(b_P(x \cdot y) \leq \max\{b_P(x), b_P(y)\}), \tag{3.3}$$

(2) *a picture fuzzy near BCC-filter of X if it satisfies the following conditions:*

$$(\forall x, y \in X)(r_P(x \cdot y) \geq r_P(y)), \tag{3.4}$$

$$(\forall x, y \in X)(g_P(x \cdot y) \geq g_P(y)), \tag{3.5}$$

$$(\forall x, y \in X)(b_P(x \cdot y) \leq b_P(y)), \tag{3.6}$$

(3) *a picture fuzzy BCC-filter of X if it satisfies the following conditions:*

$$(\forall x \in X)(r_P(0) \geq r_P(x)), \tag{3.7}$$

$$(\forall x \in X)(g_P(0) \geq g_P(x)), \tag{3.8}$$

$$(\forall x \in X)(b_P(0) \leq b_P(x)), \tag{3.9}$$

$$(\forall x, y \in X)(r_P(y) \geq \min\{r_P(x \cdot y), r_P(x)\}), \tag{3.10}$$

$$(\forall x, y \in X)(g_P(y) \geq \min\{g_P(x \cdot y), g_P(x)\}), \tag{3.11}$$

$$(\forall x, y \in X)(b_P(y) \leq \max\{b_P(x \cdot y), b_P(x)\}), \tag{3.12}$$

(4) *a picture fuzzy implicative BCC-filter of X if it satisfies the following conditions:* (3.7), (3.8), (3.9), *and*

$$(\forall x, y, z \in X)(r_P(x \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(x \cdot y)\}), \tag{3.13}$$

$$(\forall x, y, z \in X)(g_P(x \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(x \cdot y)\}), \tag{3.14}$$

$$(\forall x, y, z \in X)(b_P(x \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(x \cdot y)\}), \tag{3.15}$$

(5) *a picture fuzzy comparative BCC-filter of X if it satisfies the following conditions:* (3.7), (3.8), (3.9), *and*

$$(\forall x, y, z \in X)(r_P(y) \geq \min\{r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x)\}), \tag{3.16}$$

$$(\forall x, y, z \in X)(g_P(y) \geq \min\{g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x)\}), \tag{3.17}$$

$$(\forall x, y, z \in X)(b_P(y) \leq \max\{b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x)\}), \tag{3.18}$$

(6) *a picture fuzzy shift BCC-filter of X if it satisfies the following conditions:* (3.7), (3.8), (3.9), *and*

$$(\forall x, y, z \in X)(r_P(((z \cdot y) \cdot y) \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(x)\}), \tag{3.19}$$

$$(\forall x, y, z \in X(g_P(((z \cdot y) \cdot y) \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(x)\}), \tag{3.20}$$

$$(\forall x, y, z \in X)(b_P(((z \cdot y) \cdot y) \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(x)\}), \tag{3.21}$$

(7) *a picture fuzzy BCC-ideal of X if it satisfies the following conditions:* (3.7), (3.8), (3.9), *and*

$$(\forall x, y, z \in X)(r_P(x \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(y)\}), \tag{3.22}$$

$$(\forall x, y, z \in X)(g_P(x \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(y)\}), \tag{3.23}$$

$$(\forall x, y, z \in X)(b_P(x \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(y)\}), \tag{3.24}$$

(8) *a picture fuzzy strong BCC-ideal of X if it satisfies the following conditions:* (3.7), (3.8), (3.9), *and*

$$(\forall x, y, z \in X)(r_P(x) \geq \min\{r_P((z \cdot y) \cdot (z \cdot x)), r_P(y)\}), \tag{3.25}$$

$$(\forall x, y, z \in X)(g_P(x) \geq \min\{g_P((z \cdot y) \cdot (z \cdot x)), g_P(y)\}), \tag{3.26}$$

$$(\forall x, y, z \in X)(b_P(x) \leq \max\{b_P((z \cdot y) \cdot (z \cdot x)), b_P(y)\}). \tag{3.27}$$

Kankaew et al. [22] proved the generalization that the concept of picture fuzzy BCC-subalgebras is a generalization of picture fuzzy near BCC-filters, picture fuzzy near BCC-filters is a generalization of picture fuzzy BCC-filters, picture fuzzy BCC-filters is a generalization of picture fuzzy comparative BCC-filters, picture fuzzy BCC-filters is a generalization of picture fuzzy shift BCC-filters, picture fuzzy BCC-filters is a generalization of picture fuzzy BCC-ideals, picture fuzzy BCC-ideals is a generalization of picture fuzzy implicative BCC-filters, and picture fuzzy implicative BCC-filters, picture fuzzy comparative BCC-filters, and picture fuzzy shift BCC-filters is a generalization of picture fuzzy strong BCC-ideals. Moreover, they proved that picture fuzzy strong BCC-ideals and constant PFSs coincide.

In this part, we define a PFS in the same way as a characteristic function and study its characterizations from the related subset.

For any fixed numbers $r^+, r^-, g^+, g^-, b^+, b^- \in [0,1]$ such that $r^+ > r^-, g^+ > g^-, b^+ > b^-$ and a nonempty subset G of X, a PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}] = (X, r_P^G[^{r^+}_{r^-}], g_P^G[^{g^+}_{g^-}], b_P^G[^{b^-}_{b^+}])$ in X where $r_P^G[^{r^+}_{r^-}]$, $g_P^G[^{g^+}_{g^-}]$, and $b_P^G[^{b^-}_{b^+}]$ are functions on X which are given as follows:

$$r_P^G[^{r^+}_{r^-}](x) = \begin{cases} r^+ & \text{if } x \in G, \\ r^- & \text{otherwise,} \end{cases}$$

$$g_P^G[^{g^+}_{g^-}](x) = \begin{cases} g^+ & \text{if } x \in G, \\ g^- & \text{otherwise,} \end{cases}$$

$$b_P^G[^{b^-}_{b^+}](x) = \begin{cases} b^- & \text{if } x \in G, \\ b^+ & \text{otherwise.} \end{cases}$$

Lemma 3.1. *If the constant* 0 *of* X *is in a nonempty subset* G *of* X*, then the PFS* $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ *in* X *satisfies the conditions* (3.7), (3.8), *and* (3.9).

Proof. If $0 \in G$, then $r_P^G[^{r^+}_{r^-}](0) = r^+$, $g_P^G[^{g^+}_{g^-}](0) = g^+$, and $b_P^G[^{b^-}_{b^+}](0) = b^-$. Thus

$$(\forall x \in X) \begin{pmatrix} r_P^G[^{r^+}_{r^-}](0) = r^+ \geq r_P^G[^{r^+}_{r^-}](x) \\ g_P^G[^{g^+}_{g^-}](0) = g^+ \geq g_P^G[^{g^+}_{g^-}](x) \\ b_P^G[^{b^-}_{b^+}](0) = b^- \leq b_P^G[^{b^-}_{b^+}](x) \end{pmatrix}.$$

Hence, $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the conditions (3.7), (3.8), and (3.9). □

Lemma 3.2. *If the PFS* $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ *in* X *satisfies the condition* (3.7) *(resp.,* (3.8), (3.9)*), then the constant* 0 *of* X *is in a nonempty subset* G *of* X.

Proof. Assume that the PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ in X satisfies the condition (3.7). Then $r_P^G[^{r^+}_{r^-}](0) \geq r_P^G[^{r^+}_{r^-}](x)$ for all $x \in X$. Since G is nonempty, there exists $g \in G$. Thus $r_P^G[^{r^+}_{r^-}](g) = r^+$ and so $r_P^G[^{r^+}_{r^-}](0) \geq r_P^G[^{r^+}_{r^-}](g) = r^+ \geq r_P^G[^{r^+}_{r^-}](0)$, that is, $r_P^G[^{r^+}_{r^-}](0) = r^+$. Hence, $0 \in G$. □

Theorem 3.1. *The PFS* $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ *in* X *is a picture fuzzy BCC-subalgebra of* X *if and only if a nonempty subset* G *of* X *is a BCC-subalgebra of* X.

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy BCC-subalgebra of X. Let $x, y \in G$. Then $r_P^G[^{r^+}_{r^-}](x) = r^+ = r_P^G[^{r^+}_{r^-}](y)$. By (3.1), we have

$$r_P^G[^{r^+}_{r^-}](x \cdot y) \geq \min\{r_P^G[^{r^+}_{r^-}](x), r_P^G[^{r^+}_{r^-}](y)\} = \min\{r^+, r^+\} = r^+ \geq r_P^G[^{r^+}_{r^-}](x \cdot y)$$

and so $r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+$. Thus $x \cdot y \in G$. Hence, G is a BCC-subalgebra of X.

Conversely, assume that G is a BCC-subalgebra of X. Let $x, y \in X$.

Case 1: $x, y \in G$. Then

$$r_P^G[^{r^+}_{r^-}](x) = r^+ = r_P^G[^{r^+}_{r^-}](y),$$
$$g_P^G[^{g^+}_{g^-}](x) = g^+ = g_P^G[^{g^+}_{g^-}](y),$$
$$b_P^G[^{b^-}_{b^+}](x) = b^- = b_P^G[^{b^-}_{b^+}](y).$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x), r_P^G[^{r^+}_{r^-}](y)\} = \min\{r^+, r^+\} = r^+,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x), g_P^G[^{g^+}_{g^-}](y)\} = \min\{g^+, g^+\} = g^+,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x), b_P^G[^{b^-}_{b^+}](y)\} = \max\{b^-, b^-\} = b^-.$$

Since G is a BCC-subalgebra of X, we have $x \cdot y \in G$ and so $r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+$, $g_P^G[^{g^+}_{g^-}](x \cdot y) = g^+$, and $b_P^G[^{b^-}_{b^+}](x \cdot y) = b^-$. Hence,

$$r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+ \geq r^+ = \min\{r_P^G[^{r^+}_{r^-}](x), r_P^G[^{r^+}_{r^-}](y)\},$$
$$g_P^G[^{g^+}_{g^-}](x \cdot y) = g^+ \geq g^+ = \min\{g_P^G[^{g^+}_{g^-}](x), g_P^G[^{g^+}_{g^-}](y)\},$$
$$b_P^G[^{b^-}_{b^+}](x \cdot y) = b^- \leq b^- = \max\{b_P^G[^{b^-}_{b^+}](x), b_P^G[^{b^-}_{b^+}](y)\}.$$

Case 2: $x \notin G$ or $y \notin G$. Then

$$r_P^G[^{r^+}_{r^-}](x) = r^- \text{ or } r_P^G[^{r^+}_{r^-}](y) = r^-,$$
$$g_P^G[^{g^+}_{g^-}](x) = g^- \text{ or } g_P^G[^{g^+}_{g^-}](y) = g^-,$$
$$b_P^G[^{b^-}_{b^+}](x) = b^+ \text{ or } b_P^G[^{b^-}_{b^+}](y) = b^+.$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x), r_P^G[^{r^+}_{r^-}](y)\} = r^-,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x), g_P^G[^{g^+}_{g^-}](y)\} = g^-,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x), b_P^G[^{b^-}_{b^+}](y)\} = b^+.$$

Therefore,

$$r_P^G[^{r^+}_{r^-}](x \cdot y) \geq r^- = \min\{r_P^G[^{r^+}_{r^-}](x), r_P^G[^{r^+}_{r^-}](y)\},$$
$$g_P^G[^{g^+}_{g^-}](x \cdot y) \geq g^- = \min\{g_P^G[^{g^+}_{g^-}](x), g_P^G[^{g^+}_{g^-}](y)\},$$
$$b_P^G[^{b^-}_{b^+}](x \cdot y) \leq b^+ = \max\{b_P^G[^{b^-}_{b^+}](x), b_P^G[^{b^-}_{b^+}](y)\}.$$

Hence, $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy BCC-subalgebra of X. □

Theorem 3.2. *The PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ in X is a picture fuzzy near BCC-filter of X if and only if a nonempty subset G of X is a near BCC-filter of X.*

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is picture fuzzy near BCC-filter of X. Let $x \in X$ and $y \in G$. Then $r_P^G[^{r^+}_{r^-}](y) = r^+$. By (3.4), we have

$$r_P^G[^{r^+}_{r^-}](x \cdot y) \geq r_P^G[^{r^+}_{r^-}](y) = r^+ \geq r_P^G[^{r^+}_{r^-}](x \cdot y)$$

and so $r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+$. Thus $x \cdot y \in G$. Hence, G is a near BCC-filter of X.

Conversely, assume that G is a near BCC-filter of X. Let $x, y \in X$.

Case 1: $y \in G$. Then $r_P^G[^{r^+}_{r^-}](y) = r^+$, $g_P^G[^{g^+}_{g^-}](y) = g^+$, and $b_P^G[^{b^-}_{b^+}](y) = b^-$. Since G is a near BCC-filter of X, we have $x \cdot y \in G$ and so $r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+$, $g_P^G[^{g^+}_{g^-}](x \cdot y) = g^+$, and $b_P^G[^{b^-}_{b^+}](x \cdot y) = b^-$. Thus

$$r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+ \geq r^+ = r_P^G[^{r^+}_{r^-}](y),$$
$$g_P^G[^{g^+}_{g^-}](x \cdot y) = g^+ \geq g^+ = g_P^G[^{g^+}_{g^-}](y),$$
$$b_P^G[^{b^-}_{b^+}](x \cdot y) = b^- \leq b^- = b_P^G[^{b^-}_{b^+}](y).$$

Case 2: $y \notin G$. Then $r_P^G[^{r^+}_{r^-}](y) = r^-$, $g_P^G[^{g^+}_{g^-}](y) = g^-$, and $b_P^G[^{b^-}_{b^+}](y) = b^+$. Thus

$$r_P^G[^{r^+}_{r^-}](x \cdot y) \geq r^- = r_P^G[^{r^+}_{r^-}](y),$$
$$g_P^G[^{g^+}_{g^-}](x \cdot y) \geq g^- = g_P^G[^{g^+}_{g^-}](y),$$
$$b_P^G[^{b^-}_{b^+}](x \cdot y) \leq b^+ = b_P^G[^{b^-}_{b^+}](y).$$

Hence, $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy near BCC-filter of X. □

Theorem 3.3. *The PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ in X is a picture fuzzy BCC-filter of X if and only if a nonempty subset G of X is a BCC-filter of X.*

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy BCC-filter of X. Since $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the condition (3.7), it follows from Lemma 3.2 that $0 \in G$. Next, let $x, y \in X$ be such that $x \cdot y \in G$ and $x \in G$. Then $r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+ = r_P^G[^{r^+}_{r^-}](x)$. By (3.10), we have

$$r_P^G[^{r^+}_{r^-}](y) \geq \min\{r_P^G[^{r^+}_{r^-}](x \cdot y), r_P^G[^{r^+}_{r^-}](x)\} = \min\{r^+, r^+\} = r^+ \geq r_P^G[^{r^+}_{r^-}](y)$$

and so $r_P^G[^{r^+}_{r^-}](y) = r^+$. Thus $y \in G$. Hence, G is a BCC-filter of X.

Conversely, assume that G is a BCC-filter of X. Since $0 \in G$, it follows from Lemma 3.1 that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the conditions (3.7), (3.8), and (3.9). Next, let $x, y \in X$.

Case 1: $x \cdot y \in G$ and $x \in G$. Then

$$r_P^G[^{r^+}_{r^-}](x \cdot y) = r^+ = r_P^G[^{r^+}_{r^-}](x),$$
$$g_P^G[^{g^+}_{g^-}](x \cdot y) = g^+ = g_P^G[^{g^+}_{g^-}](x),$$
$$b_P^G[^{b^-}_{b^+}](x \cdot y) = b^- = b_P^G[^{b^-}_{b^+}](x).$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x \cdot y), r_P^G[^{r^+}_{r^-}](x)\} = \min\{r^+, r^+\} = r^+,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x \cdot y), g_P^G[^{g^+}_{g^-}](x)\} = \min\{g^+, g^+\} = g^+,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x \cdot y), b_P^G[^{b^-}_{b^+}](x)\} = \max\{b^-, b^-\} = b^-.$$

Since G is a BCC-filter of X, we have $y \in G$ and so $r_P^G[^{r^+}_{r^-}](y) = r^+$, $g_P^G[^{g^+}_{g^-}](y) = g^+$, and $b_P^G[^{b^-}_{b^+}](y) = b^-$. Thus

$$r_P^G[^{r^+}_{r^-}](y) = r^+ \geq r^+ = \min\{r_P^G[^{r^+}_{r^-}](x \cdot y), r_P^G[^{r^+}_{r^-}](x)\},$$
$$g_P^G[^{g^+}_{g^-}](y) = g^+ \geq g^+ = \min\{g_P^G[^{g^+}_{g^-}](x \cdot y), g_P^G[^{g^+}_{g^-}](x)\},$$
$$b_P^G[^{b^-}_{b^+}](y) = b^- \leq b^- = \max\{b_P^G[^{b^-}_{b^+}](x \cdot y), b_P^G[^{b^-}_{b^+}](x)\}.$$

Case 2: $x \cdot y \notin G$ or $x \notin G$. Then

$$r_P^G[^{r^+}_{r^-}](x \cdot y) = r^- \text{ or } r_P^G[^{r^+}_{r^-}](x) = r^-,$$
$$g_P^G[^{g^+}_{g^-}](x \cdot y) = g^- \text{ or } g_P^G[^{g^+}_{g^-}](x) = g^-,$$
$$b_P^G[^{b^-}_{b^+}](x \cdot y) = b^+ \text{ or } b_P^G[^{b^-}_{b^+}](x) = b^+.$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x \cdot y), r_P^G[^{r^+}_{r^-}](x)\} = r^-,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x \cdot y), g_P^G[^{g^+}_{g^-}](x)\} = g^-,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x \cdot y), b_P^G[^{b^-}_{b^+}](x)\} = b^+.$$

Therefore,

$$r_P^G[^{r^+}_{r^-}](y) \geq r^- = \min\{r_P^G[^{r^+}_{r^-}](x \cdot y), r_P^G[^{r^+}_{r^-}](x)\},$$
$$g_P^G[^{g^+}_{g^-}](y) \geq g^- = \min\{g_P^G[^{g^+}_{g^-}](x \cdot y), g_P^G[^{g^+}_{g^-}](x)\},$$
$$b_P^G[^{b^-}_{b^+}](y) \leq b^+ = \max\{b_P^G[^{b^-}_{b^+}](x \cdot y), b_P^G[^{b^-}_{b^+}](x)\}.$$

Hence, $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy BCC-filter of X. □

Theorem 3.4. *The PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ in X is a picture fuzzy implicative BCC-filter of X if and only if a nonempty subset G of X is an implicative BCC-filter of X.*

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy implicative BCC-filter of X. Since $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the condition (3.7), it follows from Lemma 3.2 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $x \cdot y \in G$. Then $r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^+ = r_P^G[^{r^+}_{r^-}](x \cdot y)$. By (3.13), we have

$$\begin{aligned} r_P^G[^{r^+}_{r^-}](x \cdot z) &\geq \min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x \cdot y)\} \\ &= \min\{r^+, r^+\} = r^+ \geq r_P^G[^{r^+}_{r^-}](x \cdot z) \end{aligned}$$

and so $r_P^G[^{r^+}_{r^-}](x \cdot z) = r^+$. Thus $x \cdot z \in G$. Hence, G is an implicative BCC-filter of X.

Conversely, assume that G is an implicative BCC-filter of X. Since $0 \in G$, it follows from Lemma 3.1 that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the conditions (3.7), (3.8), and (3.9). Next, let $x, y, z \in X$.

Case 1: $x \cdot (y \cdot z) \in G$ and $x \cdot y \in G$. Then

$$r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^+ = r_P^G[^{r^+}_{r^-}](x \cdot y),$$
$$g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)) = g^+ = g_P^G[^{g^+}_{g^-}](x \cdot y),$$
$$b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)) = b^- = b_P^G[^{b^-}_{b^+}](x \cdot y).$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x \cdot y)\} = \min\{r^+, r^+\} = r^+,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x \cdot y)\} = \min\{g^+, g^+\} = g^+,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x \cdot y)\} = \max\{b^-, b^-\} = b^-.$$

Since G is an implicative BCC-filter of X, we have $x \cdot z \in G$ and so $r_P^G[^{r^+}_{r^-}](x \cdot z) = r^+$, $g_P^G[^{g^+}_{g^-}](x \cdot z) = g^+$, and $b_P^G[^{b^-}_{b^+}](x \cdot z) = b^-$. Thus

$$r_P^G[^{r^+}_{r^-}](x \cdot z) = r^+ \geq r^+ = \min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x \cdot y)\},$$
$$g_P^G[^{g^+}_{g^-}](x \cdot z) = g^+ \geq g^+ = \min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x \cdot y)\},$$
$$b_P^G[^{b^-}_{b^+}](x \cdot z) = b^- \leq b^- = \max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x \cdot y)\}.$$

Case 2: $x \cdot (y \cdot z) \notin G$ or $x \cdot y \notin G$. Then

$$r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^- \text{ or } r_P^G[^{r^+}_{r^-}](x \cdot y) = r^-,$$
$$g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)) = g^- \text{ or } g_P^G[^{g^+}_{g^-}](x \cdot y) = g^-,$$
$$b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)) = b^+ \text{ or } b_P^G[^{b^-}_{b^+}](x \cdot y) = b^+.$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x \cdot y)\} = r^-,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x \cdot y)\} = g^-,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x \cdot y)\} = b^+.$$

Therefore,

$$r_P^G[^{r^+}_{r^-}](x \cdot z) \geq r^- = \min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x \cdot y)\},$$
$$g_P^G[^{g^+}_{g^-}](x \cdot z) \geq g^- = \min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x \cdot y)\},$$
$$b_P^G[^{b^-}_{b^+}](x \cdot z) \leq b^+ = \max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x \cdot y)\}.$$

Hence, $P^G[^{r^+, g^+, b^-}_{r^-, g^-, b^+}]$ is a picture fuzzy implicative BCC-filter of X. □

Theorem 3.5. *The PFS $P^G[^{r^+, g^+, b^-}_{r^-, g^-, b^+}]$ in X is a picture fuzzy comparative BCC-filter of X if and only if a nonempty subset G of X is a comparative BCC-filter of X.*

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy comparative BCC-filter of X. Since $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the condition (3.7), it follows from Lemma 3.2 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y) \in G$ and $x \in G$. Then $r^G_P[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)) = r^+ = r^G_P[^{r^+}_{r^-}](x)$. By (3.16), we have

$$r^G_P[^{r^+}_{r^-}](y) \geq \min\{r^G_P[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)), r^G_P[^{r^+}_{r^-}](x)\} = \min\{r^+, r^+\} = r^+ \geq r^G_P[^{r^+}_{r^-}](y)$$

and so $r^G_P[^{r^+}_{r^-}](y) = r^+$. Thus $y \in G$. Hence, G is a comparative BCC-filter of X.

Conversely, assume that G is a comparative BCC-filter of X. Since $0 \in G$, it follows from Lemma 3.1 that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the conditions (3.7), (3.8), and (3.9). Next, let $x, y, z \in X$.

Case 1: $x \cdot ((y \cdot z) \cdot y) \in G$ and $x \in G$. Then

$$r^G_P[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)) = r^+ = r^G_P[^{r^+}_{r^-}](x),$$
$$g^G_P[^{g^+}_{g^-}](x \cdot ((y \cdot z) \cdot y)) = g^+ = g^G_P[^{g^+}_{g^-}](x),$$
$$b^G_P[^{b^-}_{b^+}](x \cdot ((y \cdot z) \cdot y)) = b^- = b^G_P[^{b^-}_{b^+}](x).$$

Thus

$$\min\{r^G_P[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)), r^G_P[^{r^+}_{r^-}](x)\} = \min\{r^+, r^+\} = r^+,$$
$$\min\{g^G_P[^{g^+}_{g^-}](x \cdot ((y \cdot z) \cdot y)), g^G_P[^{g^+}_{g^-}](x)\} = \min\{g^+, g^+\} = g^+,$$
$$\max\{b^G_P[^{b^-}_{b^+}](x \cdot ((y \cdot z) \cdot y)), b^G_P[^{b^-}_{b^+}](x)\} = \max\{b^-, b^-\} = b^-.$$

Since G is a comparative BCC-filter of X, we have $y \in G$ and so $r^G_P[^{r^+}_{r^-}](y) = r^+, g^G_P[^{g^+}_{g^-}](y) = g^+$, and $b^G_P[^{b^-}_{b^+}](y) = b^-$. Thus

$$r^G_P[^{r^+}_{r^-}](y) = r^+ \geq r^+ = \min\{r^G_P[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)), r^G_P[^{r^+}_{r^-}](x)\},$$
$$g^G_P[^{g^+}_{g^-}](y) = g^+ \geq g^+ = \min\{g^G_P[^{g^+}_{g^-}](x \cdot ((y \cdot z) \cdot y)), g^G_P[^{g^+}_{g^-}](x)\},$$
$$b^G_P[^{b^-}_{b^+}](y) = b^- \leq b^- = \max\{b^G_P[^{b^-}_{b^+}](x \cdot ((y \cdot z) \cdot y)), b^G_P[^{b^-}_{b^+}](x)\}.$$

Case 2: $x \cdot ((y \cdot z) \cdot y) \notin G$ or $x \notin G$. Then

$$r^G_P[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)) = r^- \text{ or } r^G_P[^{r^+}_{r^-}](x) = r^-,$$
$$g^G_P[^{g^+}_{g^-}](x \cdot ((y \cdot z) \cdot y)) = g^- \text{ or } g^G_P[^{g^+}_{g^-}](x) = g^-,$$
$$b^G_P[^{b^-}_{b^+}](x \cdot ((y \cdot z) \cdot y)) = b^+ \text{ or } b^G_P[^{b^-}_{b^+}](x) = b^+.$$

Thus

$$\min\{r^G_P[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)), r^G_P[^{r^+}_{r^-}](x)\} = r^-,$$
$$\min\{g^G_P[^{g^+}_{g^-}](x \cdot ((y \cdot z) \cdot y)), g^G_P[^{g^+}_{g^-}](x)\} = g^-,$$
$$\max\{b^G_P[^{b^-}_{b^+}](x \cdot ((y \cdot z) \cdot y)), b^G_P[^{b^-}_{b^+}](x)\} = b^+.$$

Therefore,

$$r_P^G[^{r^+}_{r^-}](y) \geq r^- = \min\{r_P^G[^{r^+}_{r^-}](x \cdot ((y \cdot z) \cdot y)), r_P^G[^{r^+}_{r^-}](x)\},$$
$$g_P^G[^{g^+}_{g^-}](y) \geq g^- = \min\{g_P^G[^{g^+}_{g^-}](x \cdot ((y \cdot z) \cdot y)), g_P^G[^{g^+}_{g^-}](x)\},$$
$$b_P^G[^{b^-}_{b^+}](y) \leq b^+ = \max\{b_P^G[^{b^-}_{b^+}](x \cdot ((y \cdot z) \cdot y)), b_P^G[^{b^-}_{b^+}](x)\}.$$

Hence, $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy comparative BCC-filter of X. □

Theorem 3.6. *The PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ in X is a picture fuzzy shift BCC-filter of X if and only if a nonempty subset G of X is a shift BCC-filter of X.*

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy shift BCC-filter of X. Since $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the condition (3.7), it follows from Lemma 3.2 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $x \in G$. Then $r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^+ = r_P^G[^{r^+}_{r^-}](x)$. By (3.19), we have

$$\begin{aligned} r_P^G[^{r^+}_{r^-}](((z \cdot y) \cdot y) \cdot z) &\geq \min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x)\} \\ &= \min\{r^+, r^+\} = r^+ \geq r_P^G[^{r^+}_{r^-}](((z \cdot y) \cdot y) \cdot z) \end{aligned}$$

and so $r_P^G[^{r^+}_{r^-}](((z \cdot y) \cdot y) \cdot z) = r^+$. Thus $((z \cdot y) \cdot y) \cdot z \in G$. Hence, G is a shift BCC-filter of X.

Conversely, assume that G is a shift BCC-filter of X. Since $0 \in G$, it follows from Lemma 3.1 that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the conditions (3.7), (3.8), and (3.9). Next, let $x, y, z \in X$.

Case 1: $x \cdot (y \cdot z) \in G$ and $x \in G$. Then

$$r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^+ = r_P^G[^{r^+}_{r^-}](x),$$
$$g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)) = g^+ = g_P^G[^{g^+}_{g^-}](x),$$
$$b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)) = b^- = b_P^G[^{b^-}_{b^+}](x).$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x)\} = \min\{r^+, r^+\} = r^+,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x)\} = \min\{g^+, g^+\} = g^+,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x)\} = \max\{b^-, b^-\} = b^-.$$

Since G is a shift BCC-filter of X, we have $((z \cdot y) \cdot y) \cdot z \in G$ and so $r_P^G[^{r^+}_{r^-}](((z \cdot y) \cdot y) \cdot z) = r^+$, $g_P^G[^{g^+}_{g^-}](((z \cdot y) \cdot y) \cdot z) = g^+$, and $b_P^G[^{b^-}_{b^+}](((z \cdot y) \cdot y) \cdot z) = b^-$. Thus

$$r_P^G[^{r^+}_{r^-}](((z \cdot y) \cdot y) \cdot z) = r^+ \geq r^+ = \min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x)\},$$
$$g_P^G[^{g^+}_{g^-}](((z \cdot y) \cdot y) \cdot z) = g^+ \geq g^+ = \min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x)\},$$
$$b_P^G[^{b^-}_{b^+}](((z \cdot y) \cdot y) \cdot z) = b^- \leq b^- = \max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x)\}.$$

Case 2: $x \cdot (y \cdot z) \notin G$ or $x \notin G$. Then

$$r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^- \text{ or } r_P^G[^{r^+}_{r^-}](x) = r^-,$$
$$g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)) = g^- \text{ or } g_P^G[^{g^+}_{g^-}](x) = g^-,$$
$$b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)) = b^+ \text{ or } b_P^G[^{b^-}_{b^+}](x) = b^+.$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x)\} = r^-,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x)\} = g^-,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x)\} = b^+.$$

Therefore,

$$r_P^G[^{r^+}_{r^-}](((z \cdot y) \cdot y) \cdot z) \geq r^- = \min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](x)\},$$
$$g_P^G[^{g^+}_{g^-}](((z \cdot y) \cdot y) \cdot z) \geq g^- = \min\{g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)), g_P^G[^{g^+}_{g^-}](x)\},$$
$$b_P^G[^{b^-}_{b^+}](((z \cdot y) \cdot y) \cdot z) \leq b^+ = \max\{b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)), b_P^G[^{b^-}_{b^+}](x)\}.$$

Hence, $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy shift BCC-filter of X. □

Theorem 3.7. *The PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ in X is a picture fuzzy BCC-ideal of X if and only if a nonempty subset G of X is a BCC-ideal of X.*

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy BCC-ideal of X. Since $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the condition (3.7), it follows from Lemma 3.2 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $y \in G$. Then $r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^+ = r_P^G[^{r^+}_{r^-}](y)$. By (3.22), we have

$$r_P^G[^{r^+}_{r^-}](x \cdot z) \geq \min\{r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)), r_P^G[^{r^+}_{r^-}](y)\} = \min\{r^+, r^+\} = r^+ \geq r_P^G[^{r^+}_{r^-}](x \cdot z)$$

and so $r_P^G[^{r^+}_{r^-}](x \cdot z) = r^+$. Thus $x \cdot z \in G$. Hence, G is a BCC-ideal of X.

Conversely, assume that G is a BCC-ideal of X. Since $0 \in G$, it follows from Lemma 3.1 that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ satisfies the conditions (3.7), (3.8), and (3.9). Next, let $x, y, z \in X$.

Case 1: $x \cdot (y \cdot z) \in G$ and $y \in G$. Then

$$r_P^G[^{r^+}_{r^-}](x \cdot (y \cdot z)) = r^+ = r_P^G[^{r^+}_{r^-}](y),$$
$$g_P^G[^{g^+}_{g^-}](x \cdot (y \cdot z)) = g^+ = g_P^G[^{g^+}_{g^-}](y),$$
$$b_P^G[^{b^-}_{b^+}](x \cdot (y \cdot z)) = b^- = b_P^G[^{b^-}_{b^+}](y).$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x\cdot(y\cdot z)), r_P^G[^{r^+}_{r^-}](y)\} = \min\{r^+, r^+\} = r^+,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x\cdot(y\cdot z)), g_P^G[^{g^+}_{g^-}](y)\} = \min\{g^+, g^+\} = g^+,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x\cdot(y\cdot z)), b_P^G[^{b^-}_{b^+}](y)\} = \max\{b^-, b^-\} = b^-.$$

Since G is a BCC-ideal of X, we have $x\cdot z \in G$ and so $r_P^G[^{r^+}_{r^-}](x\cdot z) = r^+$, $g_P^G[^{g^+}_{g^-}](x\cdot z) = g^+$, and $b_P^G[^{b^-}_{b^+}](x\cdot z) = b^-$. Thus

$$r_P^G[^{r^+}_{r^-}](x\cdot z) = r^+ \geq r^+ = \min\{r_P^G[^{r^+}_{r^-}](x\cdot(y\cdot z)), r_P^G[^{r^+}_{r^-}](y)\},$$
$$g_P^G[^{g^+}_{g^-}](x\cdot z) = g^+ \geq g^+ = \min\{g_P^G[^{g^+}_{g^-}](x\cdot(y\cdot z)), g_P^G[^{g^+}_{g^-}](y)\},$$
$$b_P^G[^{b^-}_{b^+}](x\cdot z) = b^- \leq b^- = \max\{b_P^G[^{b^-}_{b^+}](x\cdot(y\cdot z)), b_P^G[^{b^-}_{b^+}](y)\}.$$

Case 2: $x\cdot(y\cdot z) \notin G$ or $y \notin G$. Then

$$r_P^G[^{r^+}_{r^-}](x\cdot(y\cdot z)) = r^- \text{ or } r_P^G[^{r^+}_{r^-}](y) = r^-,$$
$$g_P^G[^{g^+}_{g^-}](x\cdot(y\cdot z)) = g^- \text{ or } g_P^G[^{g^+}_{g^-}](y) = g^-,$$
$$b_P^G[^{b^-}_{b^+}](x\cdot(y\cdot z)) = b^+ \text{ or } b_P^G[^{b^-}_{b^+}](y) = b^+.$$

Thus

$$\min\{r_P^G[^{r^+}_{r^-}](x\cdot(y\cdot z)), r_P^G[^{r^+}_{r^-}](y)\} = r^-,$$
$$\min\{g_P^G[^{g^+}_{g^-}](x\cdot(y\cdot z)), g_P^G[^{g^+}_{g^-}](y)\} = g^-,$$
$$\max\{b_P^G[^{b^-}_{b^+}](x\cdot(y\cdot z)), b_P^G[^{b^-}_{b^+}](y)\} = b^+.$$

Therefore,

$$r_P^G[^{r^+}_{r^-}](x\cdot z) \geq r^- = \min\{r_P^G[^{r^+}_{r^-}](x\cdot(y\cdot z)), r_P^G[^{r^+}_{r^-}](y)\},$$
$$g_P^G[^{g^+}_{g^-}](x\cdot z) \geq g^- = \min\{g_P^G[^{g^+}_{g^-}](x\cdot(y\cdot z)), g_P^G[^{g^+}_{g^-}](y)\},$$
$$b_P^G[^{b^-}_{b^+}](x\cdot z) \leq b^+ = \max\{b_P^G[^{b^-}_{b^+}](x\cdot(y\cdot z)), b_P^G[^{b^-}_{b^+}](y)\}.$$

Hence, $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy BCC-ideal of X. □

Theorem 3.8. *The PFS $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ in X is a picture fuzzy strong BCC-ideal of X if and only if a nonempty subset G of X is a strong BCC-ideal of X.*

Proof. Assume that $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy strong BCC-ideal of X. Then $P^G[^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is constant, that is, $r_P^G[^{r^+}_{r^-}]$ is constant. Since G is nonempty, we have $r_P^G[^{r^+}_{r^-}](x) = r^+$ for all $x \in X$. Thus $G = X$. Hence, G is a strong BCC-ideal of X.

Conversely, assume that G is a strong BCC-ideal of X. Then $G = X$, so

$$(\forall x \in X)\begin{pmatrix} r_P^G[{}^{r^+}_{r^-}](x) = r^+ \\ g_P^G[{}^{g^+}_{g^-}](x) = g^+ \\ b_P^G[{}^{b^-}_{b^+}](x) = b^- \end{pmatrix}.$$

Thus $r_P^G[{}^{r^+}_{r^-}]$, $g_P^G[{}^{g^+}_{g^-}]$, and $b_P^G[{}^{b^-}_{b^+}]$ are constant, that is, $P^G[{}^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is constant. Hence, $P^G[{}^{r^+,g^+,b^-}_{r^-,g^-,b^+}]$ is a picture fuzzy strong BCC-ideal of X. □

4. Level subsets of a PFS

In this section, we discuss the relationships between picture fuzzy BCC-subalgebras (resp., picture fuzzy near BCC-filters, picture fuzzy BCC-filters, picture fuzzy implicative BCC-filters, picture fuzzy comparative BCC-filters, picture fuzzy shift BCC-filters, picture fuzzy BCC-ideals, and picture fuzzy strong BCC-ideals) of BCC-algebras and their level subsets.

Definition 4.1. *[34] Let f be a fuzzy set in X. For any $t \in [0, 1]$, the sets*

$$U(f; t) = \{x \in X \mid f(x) \geq t\},$$
$$U^+(f; t) = \{x \in X \mid f(x) > t\},$$
$$L(f; t) = \{x \in X \mid f(x) \leq t\},$$
$$L^-(f; t) = \{x \in X \mid f(x) < t\},$$
$$E(f; t) = \{x \in X \mid f(x) = t\}$$

are called an upper t-level subset, an upper t-strong level subset, a lower t-level subset, a lower t-strong level subset and an equal t-level subset of f, respectively.

Theorem 4.1. *A PFS P in X is a picture fuzzy BCC-subalgebra of X if and only if for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-subalgebras of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy BCC-subalgebra of X. Let $t \in [0, 1]$ be such that $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x, y \in U(r_P; t)$. Then $r_P(x) \geq t$ and $r_P(y) \geq t$, so t is a lower bound of $\{r_P(x), r_P(y)\}$. By (3.1), we have $r_P(x \cdot y) \geq \min\{r_P(x), r_P(y)\} \geq t$. Thus $x \cdot y \in U(r_P; t)$.

Let $x, y \in U(g_P; t)$. Then $g_P(x) \geq t$ and $g_P(y) \geq t$, so t is a lower bound of $\{g_P(x), g_P(y)\}$. By (3.2), we have $g_P(x \cdot y) \geq \min\{g_P(x), g_P(y)\} \geq t$. Thus $x \cdot y \in U(g_P; t)$.

Let $x, y \in L(b_P; t)$. Then $b_P(x) \leq t$ and $b_P(y) \leq t$, so t is an upper bound of $\{b_P(x), b_P(y)\}$. By (3.3), we have $b_P(x \cdot y) \leq \max\{b_P(x), b_P(y)\} \leq t$. Thus $x \cdot y \in L(b_P; t)$.

Hence, $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-subalgebras of X.

Conversely, assume that for all $t \in [0,1]$, the sets $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are BCC-subalgebras of X if $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are nonempty.

Let $x, y \in X$. Then $r_P(x), r_P(y) \in [0,1]$. Choose $t = \min\{r_P(x), r_P(y)\}$. Thus $r_P(x) \geq t$ and $r_P(y) \geq t$, so $x, y \in U(r_P;t) \neq \emptyset$. By assumption, we have $U(r_P;t)$ is a BCC-subalgebra of X and so $x \cdot y \in U(r_P;t)$. Thus $r_P(x \cdot y) \geq t = \min\{r_P(x), r_P(y)\}$.

Let $x, y \in X$. Then $g_P(x), g_P(y) \in [0,1]$. Choose $t = \min\{g_P(x), g_P(y)\}$. Thus $g_P(x) \geq t$ and $g_P(y) \geq t$, so $x, y \in U(g_P;t) \neq \emptyset$. By assumption, we have $U(g_P;t)$ is a BCC-subalgebra of X and so $x \cdot y \in U(g_P;t)$. Thus $g_P(x \cdot y) \geq t = \min\{g_P(x), g_P(y)\}$.

Let $x, y \in X$. Then $b_P(x), b_P(y) \in [0,1]$. Choose $t = \max\{b_P(x), b_P(y)\}$. Thus $b_P(x) \leq t$ and $b_P(y) \leq t$, so $x, y \in L(b_P;t) \neq \emptyset$. By assumption, we have $L(b_P;t)$ is a BCC-subalgebra of X and so $x \cdot y \in L(b_P;t)$. Thus $b_P(x \cdot y) \leq t = \max\{b_P(x), b_P(y)\}$.

Therefore, P is a picture fuzzy BCC-subalgebra of X. □

Theorem 4.2. *If P is a picture fuzzy BCC-subalgebra of X, then for all $t \in [0,1]$, the sets $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are BCC-subalgebras of X if $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy BCC-subalgebra of X. Let $t \in [0,1]$ be such that $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are nonempty.

Let $x, y \in U^+(r_P;t)$. Then $r_P(x) > t$ and $r_P(y) > t$, so t is a lower bound of $\{r_P(x), r_P(y)\}$. By (3.1), we have $r_P(x \cdot y) \geq \min\{r_P(x), r_P(y)\} > t$. Thus $x \cdot y \in U^+(r_P;t)$.

Let $x, y \in U^+(g_P;t)$. Then $g_P(x) > t$ and $g_P(y) > t$, so t is a lower bound of $\{g_P(x), g_P(y)\}$. By (3.2), we have $g_P(x \cdot y) \geq \min\{g_P(x), g_P(y)\} > t$. Thus $x \cdot y \in U^+(g_P;t)$.

Let $x, y \in L^-(b_P;t)$. Then $b_P(x) < t$ and $b_P(y) < t$, so t is an upper bound of $\{b_P(x), b_P(y)\}$. By (3.3), we have $b_P(x \cdot y) \leq \max\{b_P(x), b_P(y)\} < t$. Thus $x \cdot y \in L^-(b_P;t)$.

Hence, $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are BCC-subalgebras of X. □

Theorem 4.3. *A PFS P in X is a picture fuzzy near BCC-filter of X if and only if for all $t \in [0,1]$, the sets $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are near BCC-filters of X if $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy near BCC-filter of X. Let $t \in [0,1]$ be such that $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are nonempty.

Let $x \in X$ and $y \in U(r_P;t)$. Then $r_P(y) \geq t$. By (3.4), we have $r_P(x \cdot y) \geq r_P(y) \geq t$. Thus $x \cdot y \in U(r_P;t)$.

Let $x \in X$ and $y \in U(g_P;t)$. Then $g_P(y) \geq t$. By (3.5), we have $g_P(x \cdot y) \geq g_P(y) \geq t$. Thus $x \cdot y \in U(g_P;t)$.

Let $x \in X$ and $y \in L(b_P;t)$. Then $b_P(y) \leq t$. By (3.6), we have $b_P(x \cdot y) \leq b_P(y) \leq t$. Thus $x \cdot y \in L(b_P;t)$.

Hence, $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are near BCC-filters of X.

Conversely, assume that for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are near BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x, y \in X$. Then $r_P(y) \in [0, 1]$. Choose $t = r_P(y)$. Thus $r_P(y) \geq t$, so $y \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a near BCC-filter of X and so $x \cdot y \in U(r_P; t)$. Thus $r_P(x \cdot y) \geq t = r_P(y)$.

Let $x, y \in X$. Then $g_P(y) \in [0, 1]$. Choose $t = g_P(y)$. Thus $g_P(y) \geq t$, so $y \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a near BCC-filter of X and so $x \cdot y \in U(g_P; t)$. Thus $g_P(x \cdot y) \geq t = g_P(y)$.

Let $x, y \in X$. Then $b_P(y) \in [0, 1]$. Choose $t = b_P(y)$. Thus $b_P(y) \leq t$, so $y \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a near BCC-filter of X and so $x \cdot y \in L(b_P; t)$. Thus $b_P(x \cdot y) \leq t = b_P(y)$.

Therefore, P is a picture fuzzy near BCC-filter of X. □

Theorem 4.4. *If P is a picture fuzzy near BCC-filter of X, then for all $t \in [0, 1]$, the sets $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are near BCC-filters of X if $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy near BCC-filter of X. Let $t \in [0, 1]$ be such that $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.

Let $x \in X$ and $y \in U^+(r_P; t)$. Then $r_P(y) > t$. By (3.4), we have $r_P(x \cdot y) \geq r_P(y) > t$. Thus $x \cdot y \in U^+(r_P; t)$.

Let $x \in X$ and $y \in U^+(g_P; t)$. Then $g_P(y) > t$. By (3.5), we have $g_P(x \cdot y) \geq g_P(y) > t$. Thus $x \cdot y \in U^+(g_P; t)$.

Let $x \in X$ and $y \in L^-(b_P; t)$. Then $b_P(y) < t$. By (3.6), we have $b_P(x \cdot y) \leq b_P(y) < t$. Thus $x \cdot y \in L^-(b_P; t)$.

Hence, $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are near BCC-filters of X. □

Theorem 4.5. *A PFS P in X is a picture fuzzy BCC-filter of X if and only if for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy BCC-filter of X. Let $t \in [0, 1]$ be such that $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in U(r_P; t)$. Then $r_P(x) \geq t$. By (3.7), we have $r_P(0) \geq r_P(x) \geq t$. Thus $0 \in U(r_P; t)$. Next, let $x, y \in X$ be such that $x \cdot y \in U(r_P; t)$ and $x \in U(r_P; t)$. Then $r_P(x \cdot y) \geq t$ and $r_P(x) \geq t$, so t is a lower bound of $\{r_P(x \cdot y), r_P(x)\}$. By (3.10), we have $r_P(y) \geq \min\{r_P(x \cdot y), r_P(x)\} \geq t$. Thus $y \in U(r_P; t)$.

Let $x \in U(g_P; t)$. Then $g_P(x) \geq t$. By (3.8), we have $g_P(0) \geq g_P(x) \geq t$. Thus $0 \in U(g_P; t)$. Next, let $x, y \in X$ be such that $x \cdot y \in U(g_P; t)$ and $x \in U(g_P; t)$. Then $g_P(x \cdot y) \geq t$ and $g_P(x) \geq t$, so t is a lower bound of $\{g_P(x \cdot y), g_P(x)\}$. By (3.11), we have $g_P(y) \geq \min\{g_P(x \cdot y), g_P(x)\} \geq t$. Thus $y \in U(g_P; t)$.

Let $x \in L(b_P; t)$. Then $b_P(x) \leq t$. By (3.9), we have $b_P(0) \leq b_P(x) \leq t$. Thus $0 \in L(b_P; t)$. Next, let $x, y \in X$ be such that $x \cdot y \in L(b_P; t)$ and $x \in L(b_P; t)$. Then $b_P(x \cdot y) \leq t$ and $b_P(x) \leq t$, so t is an upper bound of $\{b_P(x \cdot y), b_P(x)\}$. By (3.12), we have $b_P(y) \leq \max\{b_P(x \cdot y), b_P(x)\} \leq t$. Thus $y \in L(b_P; t)$.

Hence, $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-filters of X.

Conversely, assume that for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in X$. Then $r_P(x) \in [0, 1]$. Choose $t = r_P(x)$. Thus $r_P(x) \geq t$, so $x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a BCC-filter of X and so $0 \in U(r_P; t)$. Thus $r_P(0) \geq t = r_P(x)$. Next, let $x, y \in X$. Then $r_P(x \cdot y), r_P(x) \in [0, 1]$. Choose $t = \min\{r_P(x \cdot y), r_P(x)\}$. Thus $r_P(x \cdot y) \geq t$ and $r_P(x) \geq t$, so $x \cdot y, x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a BCC-filter of X and so $y \in U(r_P; t)$. Thus $r_P(y) \geq t = \min\{r_P(x \cdot y), r_P(x)\}$.

Let $x \in X$. Then $g_P(x) \in [0, 1]$. Choose $t = g_P(x)$. Thus $g_P(x) \geq t$, so $x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a BCC-filter of X and so $0 \in U(g_P; t)$. Thus $g_P(0) \geq t = g_P(x)$. Next, let $x, y \in X$. Then $g_P(x \cdot y), g_P(x) \in [0, 1]$. Choose $t = \min\{g_P(x \cdot y), g_P(x)\}$. Thus $g_P(x \cdot y) \geq t$ and $g_P(x) \geq t$, so $x \cdot y, x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a BCC-filter of X and so $y \in U(g_P; t)$. Thus $g_P(y) \geq t = \min\{g_P(x \cdot y), g_P(x)\}$.

Let $x \in X$. Then $b_P(x) \in [0, 1]$. Choose $t = b_P(x)$. Thus $b_P(x) \leq t$, so $x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a BCC-filter of X and so $0 \in L(b_P; t)$. Thus $b_P(0) \leq t = b_P(x)$. Next, let $x, y \in X$. Then $b_P(x \cdot y), b_P(x) \in [0, 1]$. Choose $t = \max\{b_P(x \cdot y), b_P(x)\}$. Thus $b_P(x \cdot y) \leq t$ and $b_P(x) \leq t$, so $x \cdot y, x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a BCC-filter of X and so $y \in L(b_P; t)$. Thus $b_P(y) \leq t = \max\{b_P(x \cdot y), b_P(x)\}$.

Therefore, P is a picture fuzzy BCC-filter of X. □

Theorem 4.6. *If P is a picture fuzzy BCC-filter of X, then for all $t \in [0, 1]$, the sets $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are BCC-filters of X if $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy BCC-filter of X. Let $t \in [0, 1]$ be such that $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.

Let $x \in U^+(r_P; t)$. Then $r_P(x) > t$. By (3.7), we have $r_P(0) \geq r_P(x) > t$. Thus $0 \in U^+(r_P; t)$. Next, let $x, y \in X$ be such that $x \cdot y \in U^+(r_P; t)$ and $x \in U^+(r_P; t)$. Then $r_P(x \cdot y) > t$ and $r_P(x) > t$, so t is a lower bound of $\{r_P(x \cdot y), r_P(x)\}$. By (3.10), we have $r_P(y) \geq \min\{r_P(x \cdot y), r_P(x)\} > t$. Thus $y \in U^+(r_P; t)$.

Let $x \in U^+(g_P; t)$. Then $g_P(x) > t$. By (3.8), we have $g_P(0) \geq g_P(x) > t$. Thus $0 \in U^+(g_P; t)$. Next, let $x, y \in X$ be such that $x \cdot y \in U^+(g_P; t)$ and $x \in U^+(g_P; t)$. Then $g_P(x \cdot y) > t$ and $g_P(x) > t$, so t is a lower bound of $\{g_P(x \cdot y), g_P(x)\}$. By (3.11), we have $g_P(y) \geq \min\{g_P(x \cdot y), g_P(x)\} > t$. Thus $y \in U^+(g_P; t)$.

Let $x \in L^-(b_P; t)$. Then $b_P(x) < t$. By (3.9), we have $b_P(0) \leq b_P(x) < t$. Thus $0 \in L^-(b_P; t)$. Next, let $x, y \in X$ be such that $x \cdot y \in L^-(b_P; t)$ and $x \in L^-(b_P; t)$. Then $b_P(x \cdot y) < t$ and $b_P(x) < t$, so t is an upper bound of $\{b_P(x \cdot y), b_P(x)\}$. By (3.12), we have $b_P(y) \leq \max\{b_P(x \cdot y), b_P(x)\} < t$. Thus $y \in L^-(b_P; t)$.

Hence, $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are BCC-filters of X. □

Theorem 4.7. *A PFS P in X is a picture fuzzy implicative BCC-filter of X if and only if for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are implicative BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy implicative BCC-filter of X. Let $t \in [0, 1]$ be such that $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in U(r_P; t)$. Then $r_P(x) \geq t$. By (3.7), we have $r_P(0) \geq r_P(x) \geq t$. Thus $0 \in U(r_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(r_P; t)$ and $x \cdot y \in U(r_P; t)$. Then $r_P(x \cdot (y \cdot z)) \geq t$ and $r_P(x \cdot y) \geq t$, so t is a lower bound of $\{r_P(x \cdot (y \cdot z)), r_P(x \cdot y)\}$. By (3.13), we have $r_P(x \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(x \cdot y)\} \geq t$. Thus $x \cdot z \in U(r_P; t)$.

Let $x \in U(g_P; t)$. Then $g_P(x) \geq t$. By (3.8), we have $g_P(0) \geq g_P(x) \geq t$. Thus $0 \in U(g_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(g_P; t)$ and $x \cdot y \in U(g_P; t)$. Then $g_P(x \cdot (y \cdot z)) \geq t$ and $g_P(x \cdot y) \geq t$, so t is a lower bound of $\{g_P(x \cdot (y \cdot z)), g_P(x \cdot y)\}$. By (3.14), we have $g_P(x \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(x \cdot y)\} \geq t$. Thus $x \cdot z \in U(g_P; t)$.

Let $x \in L(b_P; t)$. Then $b_P(x) \leq t$. By (3.9), we have $b_P(0) \leq b_P(x) \leq t$. Thus $0 \in L(b_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L(b_P; t)$ and $x \cdot y \in L(b_P; t)$. Then $b_P(x \cdot (y \cdot z)) \leq t$ and $b_P(x \cdot y) \leq t$, so t is an upper bound of $\{b_P(x \cdot (y \cdot z)), b_P(x \cdot y)\}$. By (3.15), we have $b_P(x \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(x \cdot y)\} \leq t$. Thus $x \cdot z \in L(b_P; t)$.

Hence, $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are implicative BCC-filters of X.

Conversely, assume that for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are implicative BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in X$. Then $r_P(x) \in [0, 1]$. Choose $t = r_P(x)$. Thus $r_P(x) \geq t$, so $x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is an implicative BCC-filter of X and so $0 \in U(r_P; t)$. Thus $r_P(0) \geq t = r_P(x)$. Next, let $x, y, z \in X$. Then $r_P(x \cdot (y \cdot z)), r_P(x \cdot y) \in [0, 1]$. Choose $t = \min\{r_P(x \cdot (y \cdot z)), r_P(x \cdot y)\}$. Thus $r_P(x \cdot (y \cdot z)) \geq t$ and $r_P(x \cdot y) \geq t$, so $x \cdot (y \cdot z), x \cdot y \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is an implicative BCC-filter of X and so $x \cdot z \in U(r_P; t)$. Thus $r_P(x \cdot z) \geq t = \min\{r_P(x \cdot (y \cdot z)), r_P(x \cdot y)\}$.

Let $x \in X$. Then $g_P(x) \in [0, 1]$. Choose $t = g_P(x)$. Thus $g_P(x) \geq t$, so $x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is an implicative BCC-filter of X and so $0 \in U(g_P; t)$. Thus

$g_P(0) \geq t = g_P(x)$. Next, let $x, y, z \in X$. Then $g_P(x \cdot (y \cdot z)), g_P(x \cdot y) \in [0, 1]$. Choose $t = \min\{g_P(x\cdot(y\cdot z)), g_P(x\cdot y)\}$. Thus $g_P(x\cdot(y\cdot z)) \geq t$ and $g_P(x\cdot y) \geq t$, so $x\cdot(y\cdot z), x\cdot y \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is an implicative BCC-filter of X and so $x \cdot z \in U(g_P; t)$. Thus $g_P(x \cdot z) \geq t = \min\{g_P(x \cdot (y \cdot z)), g_P(x \cdot y)\}$.

Let $x \in X$. Then $b_P(x) \in [0, 1]$. Choose $t = b_P(x)$. Thus $b_P(x) \leq t$, so $x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is an implicative BCC-filter of X and so $0 \in L(b_P; t)$. Thus $b_P(0) \leq t = b_P(x)$. Next, let $x, y, z \in X$. Then $b_P(x \cdot (y \cdot z)), b_P(x \cdot y) \in [0, 1]$. Choose $t = \max\{b_P(x \cdot (y \cdot z)), b_P(x \cdot y)\}$. Thus $b_P(x \cdot (y \cdot z)) \leq t$ and $b_P(x \cdot y) \leq t$, so $x \cdot (y \cdot z), x \cdot y \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is an implicative BCC-filter of X and so $x\cdot z \in L(b_P; t)$. Thus $b_P(x \cdot z) \leq t = \max\{b_P(x \cdot (y \cdot z)), b_P(x \cdot y)\}$.

Therefore, P is a picture fuzzy implicative BCC-filter of X. □

Theorem 4.8. *If P is a picture fuzzy implicative BCC-filter of X, then for all $t \in [0, 1]$, the sets $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are implicative BCC-filters of X if $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy implicative BCC-filter of X. Let $t \in [0, 1]$ be such that $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.

Let $x \in U^+(r_P; t)$. Then $r_P(x) > t$. By (3.7), we have $r_P(0) \geq r_P(x) > t$. Thus $0 \in U^+(r_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U^+(r_P; t)$ and $x \cdot y \in U^+(r_P; t)$. Then $r_P(x \cdot (y \cdot z)) > t$ and $r_P(x \cdot y) > t$, so t is a lower bound of $\{r_P(x \cdot (y \cdot z)), r_P(x \cdot y)\}$. By (3.13), we have $r_P(x \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(x \cdot y)\} > t$. Thus $x \cdot z \in U^+(r_P; t)$.

Let $x \in U^+(g_P; t)$. Then $g_P(x) > t$. By (3.8), we have $g_P(0) \geq g_P(x) > t$. Thus $0 \in U^+(g_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U^+(g_P; t)$ and $x \cdot y \in U^+(g_P; t)$. Then $g_P(x \cdot (y \cdot z)) > t$ and $g_P(x \cdot y) > t$, so t is a lower bound of $\{g_P(x \cdot (y \cdot z)), g_P(x \cdot y)\}$. By (3.14), we have $g_P(x \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(x \cdot y)\} > t$. Thus $x \cdot z \in U^+(g_P; t)$.

Let $x \in L^-(b_P; t)$. Then $b_P(x) < t$. By (3.9), we have $b_P(0) \leq b_P(x) < t$. Thus $0 \in L^-(b_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L^-(b_P; t)$ and $x \cdot y \in L^-(b_P; t)$. Then $b_P(x \cdot (y \cdot z)) < t$ and $b_P(x \cdot y) < t$, so t is an upper bound of $\{b_P(x \cdot (y \cdot z)), b_P(x \cdot y)\}$. By (3.15), we have $b_P(x \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(x \cdot y)\} < t$. Thus $x \cdot z \in L^-(b_P; t)$.

Hence, $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are implicative BCC-filters of X. □

Theorem 4.9. *A PFS P in X is a picture fuzzy comparative BCC-filter of X if and only if for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are comparative BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy comparative BCC-filter of X. Let $t \in [0, 1]$ be such that $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in U(r_P; t)$. Then $r_P(x) \geq t$. By (3.7), we have $r_P(0) \geq r_P(x) \geq t$. Thus $0 \in U(r_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y) \in U(r_P; t)$ and $x \in U(r_P; t)$. Then $r_P(x \cdot ((y \cdot z) \cdot y)) \geq t$ and $r_P(x) \geq t$, so t is a lower bound of $\{r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x)\}$. By (3.16), we have $r_P(y) \geq \min\{r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x)\} \geq t$. Thus $y \in U(r_P; t)$.

Let $x \in U(g_P; t)$. Then $g_P(x) \geq t$. By (3.8), we have $g_P(0) \geq g_P(x) \geq t$. Thus $0 \in U(g_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y) \in U(g_P; t)$ and $x \in U(g_P; t)$. Then $g_P(x \cdot ((y \cdot z) \cdot y)) \geq t$ and $g_P(x) \geq t$, so t is a lower bound of $\{g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x)\}$. By (3.17), we have $g_P(y) \geq \min\{g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x)\} \geq t$. Thus $y \in U(g_P; t)$.

Let $x \in L(b_P; t)$. Then $b_P(x) \leq t$. By (3.9), we have $b_P(0) \leq b_P(x) \leq t$. Thus $0 \in L(b_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y) \in L(b_P; t)$ and $x \in L(b_P; t)$. Then $b_P(x \cdot ((y \cdot z) \cdot y)) \leq t$ and $b_P(x) \leq t$, so t is an upper bound of $\{b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x)\}$. By (3.18), we have $b_P(y) \leq \max\{b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x)\} \leq t$. Thus $y \in L(b_P; t)$.

Hence, $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are comparative BCC-filters of X.

Conversely, assume that for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are comparative BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in X$. Then $r_P(x) \in [0, 1]$. Choose $t = r_P(x)$. Thus $r_P(x) \geq t$, so $x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a comparative BCC-filter of X and so $0 \in U(r_P; t)$. Thus $r_P(0) \geq t = r_P(x)$. Next, let $x, y, z \in X$. Then $r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x) \in [0, 1]$. Choose $t = \min\{r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x)\}$. Thus $r_P(x \cdot ((y \cdot z) \cdot y)) \geq t$ and $r_P(x) \geq t$, so $x \cdot ((y \cdot z) \cdot y), x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a comparative BCC-filter of X and so $y \in U(r_P; t)$. Thus $r_P(y) \geq t = \min\{r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x)\}$.

Let $x \in X$. Then $g_P(x) \in [0, 1]$. Choose $t = g_P(x)$. Thus $g_P(x) \geq t$, so $x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a comparative BCC-filter of X and so $0 \in U(g_P; t)$. Thus $g_P(0) \geq t = g_P(x)$. Next, let $x, y, z \in X$. Then $g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x) \in [0, 1]$. Choose $t = \min\{g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x)\}$. Thus $g_P(x \cdot ((y \cdot z) \cdot y)) \geq t$ and $g_P(x) \geq t$, so $x \cdot ((y \cdot z) \cdot y), x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a comparative BCC-filter of X and so $y \in U(g_P; t)$. Thus $g_P(y) \geq t = \min\{g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x)\}$.

Let $x \in X$. Then $b_P(x) \in [0, 1]$. Choose $t = b_P(x)$. Thus $b_P(x) \leq t$, so $x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a comparative BCC-filter of X and so $0 \in L(b_P; t)$. Thus $b_P(0) \leq t = b_P(x)$. Next, let $x, y, z \in X$. Then $b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x) \in [0, 1]$. Choose $t = \max\{b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x)\}$. Thus $b_P(x \cdot ((y \cdot z) \cdot y)) \leq t$ and $b_P(x) \leq t$, so $x \cdot ((y \cdot z) \cdot y), x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a comparative BCC-filter of X and so $y \in L(b_P; t)$. Thus $b_P(y) \leq t = \max\{b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x)\}$.

Therefore, P is a picture fuzzy comparative BCC-filter of X. □

Theorem 4.10. *If P is a picture fuzzy comparative BCC-filter of X, then for all $t \in [0,1]$, the sets $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are comparative BCC-filters of X if $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy comparative BCC-filter of X. Let $t \in [0,1]$ be such that $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are nonempty.

Let $x \in U^+(r_P;t)$. Then $r_P(x) > t$. By (3.7), we have $r_P(0) \geq r_P(x) > t$. Thus $0 \in U^+(r_P;t)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y) \in U^+(r_P;t)$ and $x \in U^+(r_P;t)$. Then $r_P(x \cdot ((y \cdot z) \cdot y)) > t$ and $r_P(x) > t$, so t is a lower bound of $\{r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x)\}$. By (3.16), we have $r_P(y) \geq \min\{r_P(x \cdot ((y \cdot z) \cdot y)), r_P(x)\} > t$. Thus $y \in U^+(r_P;t)$.

Let $x \in U^+(g_P;t)$. Then $g_P(x) > t$. By (3.8), we have $g_P(0) \geq g_P(x) > t$. Thus $0 \in U^+(g_P;t)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y) \in U^+(g_P;t)$ and $x \in U^+(g_P;t)$. Then $g_P(x \cdot ((y \cdot z) \cdot y)) > t$ and $g_P(x) > t$, so t is a lower bound of $\{g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x)\}$. By (3.17), we have $g_P(y) \geq \min\{g_P(x \cdot ((y \cdot z) \cdot y)), g_P(x)\} > t$. Thus $y \in U^+(g_P;t)$.

Let $x \in L^-(b_P;t)$. Then $b_P(x) < t$. By (3.9), we have $b_P(0) \leq b_P(x) < t$. Thus $0 \in L^-(b_P;t)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y) \in L^-(b_P;t)$ and $x \in L^-(b_P;t)$. Then $b_P(x \cdot ((y \cdot z) \cdot y)) < t$ and $b_P(x) < t$, so t is an upper bound of $\{b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x)\}$. By (3.18), we have $b_P(y) \leq \max\{b_P(x \cdot ((y \cdot z) \cdot y)), b_P(x)\} < t$. Thus $y \in L^-(b_P;t)$.

Hence, $U^+(r_P;t), U^+(g_P;t)$, and $L^-(b_P;t)$ are comparative BCC-filters of X. □

Theorem 4.11. *A PFS P in X is a picture fuzzy shift BCC-filter of X if and only if for all $t \in [0,1]$, the sets $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are shift BCC-filters of X if $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy shift BCC-filter of X. Let $t \in [0,1]$ be such that $U(r_P;t), U(g_P;t)$, and $L(b_P;t)$ are nonempty.

Let $x \in U(r_P;t)$. Then $r_P(x) \geq t$. By (3.7), we have $r_P(0) \geq r_P(x) \geq t$. Thus $0 \in U(r_P;t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(r_P;t)$ and $x \in U(r_P;t)$. Then $r_P(x \cdot (y \cdot z)) \geq t$ and $r_P(x) \geq t$, so t is a lower bound of $\{r_P(x \cdot (y \cdot z)), r_P(x)\}$. By (3.19), we have $r_P(((z \cdot y) \cdot y) \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(x)\} \geq t$. Thus $((z \cdot y) \cdot y) \cdot z \in U(r_P;t)$.

Let $x \in U(g_P;t)$. Then $g_P(x) \geq t$. By (3.8), we have $g_P(0) \geq g_P(x) \geq t$. Thus $0 \in U(g_P;t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(g_P;t)$ and $x \in U(g_P;t)$. Then $g_P(x \cdot (y \cdot z)) \geq t$ and $g_P(x) \geq t$, so t is a lower bound of $\{g_P(x \cdot (y \cdot z)), g_P(x)\}$. By (3.20), we have $g_P(((z \cdot y) \cdot y) \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(x)\} \geq t$. Thus $((z \cdot y) \cdot y) \cdot z \in U(g_P;t)$.

Let $x \in L(b_P;t)$. Then $b_P(x) \leq t$. By (3.9), we have $b_P(0) \leq b_P(x) \leq t$. Thus $0 \in L(b_P;t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L(b_P;t)$ and $x \in L(b_P;t)$. Then $b_P(x \cdot (y \cdot z)) \leq t$ and $b_P(x) \leq t$, so t is an upper bound of $\{b_P(x \cdot (y \cdot z)), b_P(x)\}$. By (3.21), we have $b_P(((z \cdot y) \cdot y) \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(x)\} \leq t$. Thus $((z \cdot y) \cdot y) \cdot z \in L(b_P;t)$.

Hence, $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are shift BCC-filters of X.

Conversely, assume that for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are shift BCC-filters of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in X$. Then $r_P(x) \in [0, 1]$. Choose $t = r_P(x)$. Thus $r_P(x) \geq t$, so $x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a shift BCC-filter of X and so $0 \in U(r_P; t)$. Thus $r_P(0) \geq t = r_P(x)$. Next, let $x, y, z \in X$. Then $r_P(x \cdot (y \cdot z)), r_P(x) \in [0, 1]$. Choose $t = \min\{r_P(x \cdot (y \cdot z)), r_P(x)\}$. Thus $r_P(x \cdot (y \cdot z)) \geq t$ and $r_P(x) \geq t$, so $x \cdot (y \cdot z), x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a shift BCC-filter of X and so $((z \cdot y) \cdot y) \cdot z \in U(r_P; t)$. Thus $r_P(((z \cdot y) \cdot y) \cdot z) \geq t = \min\{r_P(x \cdot (y \cdot z)), r_P(x)\}$.

Let $x \in X$. Then $g_P(x) \in [0, 1]$. Choose $t = g_P(x)$. Thus $g_P(x) \geq t$, so $x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a shift BCC-filter of X and so $0 \in U(g_P; t)$. Thus $g_P(0) \geq t = g_P(x)$. Next, let $x, y, z \in X$. Then $g_P(x \cdot (y \cdot z)), g_P(x) \in [0, 1]$. Choose $t = \min\{g_P(x \cdot (y \cdot z)), g_P(x)\}$. Thus $g_P(x \cdot (y \cdot z)) \geq t$ and $g_P(x) \geq t$, so $x \cdot (y \cdot z), x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a shift BCC-filter of X and so $((z \cdot y) \cdot y) \cdot z \in U(g_P; t)$. Thus $g_P(((z \cdot y) \cdot y) \cdot z) \geq t = \min\{g_P(x \cdot (y \cdot z)), g_P(x)\}$.

Let $x \in X$. Then $b_P(x) \in [0, 1]$. Choose $t = b_P(x)$. Thus $b_P(x) \leq t$, so $x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a shift BCC-filter of X and so $0 \in L(b_P; t)$. Thus $b_P(0) \leq t = b_P(x)$. Next, let $x, y, z \in X$. Then $b_P(x \cdot (y \cdot z)), b_P(x) \in [0, 1]$. Choose $t = \max\{b_P(x \cdot (y \cdot z)), b_P(x)\}$. Thus $b_P(x \cdot (y \cdot z)) \leq t$ and $b_P(x) \leq t$, so $x \cdot (y \cdot z), x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a shift BCC-filter of X and so $((z \cdot y) \cdot y) \cdot z \in L(b_P; t)$. Thus $b_P(((z \cdot y) \cdot y) \cdot z) \leq t = \max\{b_P(x \cdot (y \cdot z)), b_P(x)\}$.

Therefore, P is a picture fuzzy shift BCC-filter of X. □

Theorem 4.12. *If P is a picture fuzzy shift BCC-filter of X, then for all $t \in [0, 1]$, the sets $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are shift BCC-filters of X if $U^+(r_P; t)$, $U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy shift BCC-filter of X. Let $t \in [0, 1]$ be such that $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.

Let $x \in U^+(r_P; t)$. Then $r_P(x) > t$. By (3.7), we have $r_P(0) \geq r_P(x) > t$. Thus $0 \in U^+(r_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U^+(r_P; t)$ and $x \in U^+(r_P; t)$. Then $r_P(x \cdot (y \cdot z)) > t$ and $r_P(x) > t$, so t is a lower bound of $\{r_P(x \cdot (y \cdot z)), r_P(x)\}$. By (3.19), we have $r_P(((z \cdot y) \cdot y) \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(x)\} > t$. Thus $((z \cdot y) \cdot y) \cdot z \in U^+(r_P; t)$.

Let $x \in U^+(g_P; t)$. Then $g_P(x) > t$. By (3.8), we have $g_P(0) \geq g_P(x) > t$. Thus $0 \in U^+(g_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U^+(g_P; t)$ and $x \in U^+(g_P; t)$. Then $g_P(x \cdot (y \cdot z)) > t$ and $g_P(x) > t$, so t is a lower bound of $\{g_P(x \cdot (y \cdot z)), g_P(x)\}$. By (3.20), we have $g_P(((z \cdot y) \cdot y) \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(x)\} > t$. Thus $((z \cdot y) \cdot y) \cdot z \in U^+(g_P; t)$.

Let $x \in L^-(b_P; t)$. Then $b_P(x) < t$. By (3.9), we have $b_P(0) \leq b_P(x) < t$. Thus $0 \in L^-(b_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L^-(b_P; t)$ and $x \in L^-(b_P; t)$. Then $b_P(x \cdot (y \cdot z)) < t$ and

$b_P(x) < t$, so t is an upper bound of $\{b_P(x \cdot (y \cdot z)), b_P(x)\}$. By (3.21), we have $b_P(((z \cdot y) \cdot y) \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(x)\} < t$. Thus $((z \cdot y) \cdot y) \cdot z \in L^-(b_P; t)$.

Hence, $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are shift BCC-filters of X. □

Theorem 4.13. *A PFS P in X is a picture fuzzy BCC-ideal of X if and only if for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-ideals of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy BCC-ideal of X. Let $t \in [0, 1]$ be such that $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in U(r_P; t)$. Then $r_P(x) \geq t$. By (3.7), we have $r_P(0) \geq r_P(x) \geq t$. Thus $0 \in U(r_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(r_P; t)$ and $y \in U(r_P; t)$. Then $r_P(x \cdot (y \cdot z)) \geq t$ and $r_P(y) \geq t$, so t is a lower bound of $\{r_P(x \cdot (y \cdot z)), r_P(y)\}$. By (3.22), we have $r_P(x \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(y)\} \geq t$. Thus $x \cdot z \in U(r_P; t)$.

Let $x \in U(g_P; t)$. Then $g_P(x) \geq t$. By (3.8), we have $g_P(0) \geq g_P(x) \geq t$. Thus $0 \in U(g_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(g_P; t)$ and $y \in U(g_P; t)$. Then $g_P(x \cdot (y \cdot z)) \geq t$ and $g_P(y) \geq t$, so t is a lower bound of $\{g_P(x \cdot (y \cdot z)), g_P(y)\}$. By (3.23), we have $g_P(x \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(y)\} \geq t$. Thus $x \cdot z \in U(g_P; t)$.

Let $x \in L(b_P; t)$. Then $b_P(x) \leq t$. By (3.9), we have $b_P(0) \leq b_P(x) \leq t$. Thus $0 \in L(b_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L(b_P; t)$ and $y \in L(b_P; t)$. Then $b_P(x \cdot (y \cdot z)) \leq t$ and $b_P(y) \leq t$, so t is an upper bound of $\{b_P(x \cdot (y \cdot z)), b_P(y)\}$. By (3.24), we have $b_P(x \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(y)\} \leq t$. Thus $x \cdot z \in L(b_P; t)$.

Hence, $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-ideals of X.

Conversely, assume that for all $t \in [0, 1]$, the sets $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are BCC-ideals of X if $U(r_P; t), U(g_P; t)$, and $L(b_P; t)$ are nonempty.

Let $x \in X$. Then $r_P(x) \in [0, 1]$. Choose $t = r_P(x)$. Thus $r_P(x) \geq t$, so $x \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a BCC-ideal of X and so $0 \in U(r_P; t)$. Thus $r_P(0) \geq t = r_P(x)$. Next, let $x, y, z \in X$. Then $r_P(x \cdot (y \cdot z)), r_P(y) \in [0, 1]$. Choose $t = \min\{r_P(x \cdot (y \cdot z)), r_P(y)\}$. Thus $r_P(x \cdot (y \cdot z)) \geq t$ and $r_P(y) \geq t$, so $x \cdot (y \cdot z), y \in U(r_P; t) \neq \emptyset$. By assumption, we have $U(r_P; t)$ is a BCC-ideal of X and so $x \cdot z \in U(r_P; t)$. Thus $r_P(x \cdot z) \geq t = \min\{r_P(x \cdot (y \cdot z)), r_P(y)\}$.

Let $x \in X$. Then $g_P(x) \in [0, 1]$. Choose $t = g_P(x)$. Thus $g_P(x) \geq t$, so $x \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a BCC-ideal of X and so $0 \in U(g_P; t)$. Thus $g_P(0) \geq t = g_P(x)$. Next, let $x, y, z \in X$. Then $g_P(x \cdot (y \cdot z)), g_P(y) \in [0, 1]$. Choose $t = \min\{g_P(x \cdot (y \cdot z)), g_P(y)\}$. Thus $g_P(x \cdot (y \cdot z)) \geq t$ and $g_P(y) \geq t$, so $x \cdot (y \cdot z), y \in U(g_P; t) \neq \emptyset$. By assumption, we have $U(g_P; t)$ is a BCC-ideal of X and so $x \cdot z \in U(g_P; t)$. Thus $g_P(x \cdot z) \geq t = \min\{g_P(x \cdot (y \cdot z)), g_P(y)\}$.

Let $x \in X$. Then $b_P(x) \in [0, 1]$. Choose $t = b_P(x)$. Thus $b_P(x) \leq t$, so $x \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a BCC-ideal of X and so $0 \in L(b_P; t)$. Thus $b_P(0) \leq t = b_P(x)$. Next, let $x, y, z \in X$. Then $b_P(x \cdot (y \cdot z)), b_P(y) \in [0, 1]$. Choose $t = \max\{b_P(x \cdot (y \cdot z)), b_P(y)\}$.

Thus $b_P(x \cdot (y \cdot z)) \leq t$ and $b_P(y) \leq t$, so $x \cdot (y \cdot z), y \in L(b_P; t) \neq \emptyset$. By assumption, we have $L(b_P; t)$ is a BCC-ideal of X and so $x \cdot z \in L(b_P; t)$. Thus $b_P(x \cdot z) \leq t = \max\{b_P(x \cdot (y \cdot z)), b_P(y)\}$.

Therefore, P is a picture fuzzy BCC-ideal of X. □

Theorem 4.14. *If P in X is a picture fuzzy BCC-ideal of X, then for all $t \in [0, 1]$, the sets $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are BCC-ideals of X if $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.*

Proof. Assume that P is a picture fuzzy BCC-ideal of X. Let $t \in [0, 1]$ be such that $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are nonempty.

Let $x \in U^+(r_P; t)$. Then $r_P(x) > t$. By (3.7), we have $r_P(0) \geq r_P(x) > t$. Thus $0 \in U^+(r_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U^+(r_P; t)$ and $y \in U^+(r_P; t)$. Then $r_P(x \cdot (y \cdot z)) > t$ and $r_P(y) > t$, so t is a lower bound of $\{r_P(x \cdot (y \cdot z)), r_P(y)\}$. By (3.22), we have $r_P(x \cdot z) \geq \min\{r_P(x \cdot (y \cdot z)), r_P(y)\} > t$. Thus $x \cdot z \in U^+(r_P; t)$.

Let $x \in U^+(g_P; t)$. Then $g_P(x) > t$. By (3.8), we have $g_P(0) \geq g_P(x) > t$. Thus $0 \in U^+(g_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U^+(g_P; t)$ and $y \in U^+(g_P; t)$. Then $g_P(x \cdot (y \cdot z)) > t$ and $g_P(y) > t$, so t is a lower bound of $\{g_P(x \cdot (y \cdot z)), g_P(y)\}$. By (3.23), we have $g_P(x \cdot z) \geq \min\{g_P(x \cdot (y \cdot z)), g_P(y)\} > t$. Thus $x \cdot z \in U^+(g_P; t)$.

Let $x \in L^-(b_P; t)$. Then $b_P(x) < t$. By (3.9), we have $b_P(0) \leq b_P(x) < t$. Thus $0 \in L^-(b_P; t)$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L^-(b_P; t)$ and $y \in L^-(b_P; t)$. Then $b_P(x \cdot (y \cdot z)) < t$ and $b_P(y) < t$, so t is an upper bound of $\{b_P(x \cdot (y \cdot z)), b_P(y)\}$. By (3.24), we have $b_P(x \cdot z) \leq \max\{b_P(x \cdot (y \cdot z)), b_P(y)\} < t$. Thus $x \cdot z \in L^-(b_P; t)$.

Hence, $U^+(r_P; t), U^+(g_P; t)$, and $L^-(b_P; t)$ are BCC-ideals of X. □

Theorem 4.15. *A PFS P in X is a picture fuzzy strong BCC-ideal of X if and only if the sets $E(r_P; r_P(0)), E(g_P; g_P(0))$, and $E(b_P; b_P(0))$ are strong BCC-ideals of X.*

Proof. Assume that P is a picture fuzzy strong BCC-ideal of X. Then P is constant, that is, r_P, g_P, and b_P are constant. Thus

$$(\forall x \in X) \begin{pmatrix} r_P(x) = r_P(0) \\ g_P(x) = g_P(0) \\ b_P(x) = b_P(0) \end{pmatrix}.$$

Hence, $E(r_P; r_P(0)) = X, E(g_P; g_P(0)) = X$, and $E(b_P; b_P(0)) = X$ and so $E(r_P; r_P(0)), E(g_P; g_P(0))$, and $E(b_P; b_P(0))$ are strong BCC-ideals of X.

Conversely, assume that $E(r_P; r_P(0))$, $E(g_P; g_P(0))$, and $E(b_P; b_P(0))$ are strong BCC-ideals of X. Then $E(r_P; r_P(0)) = X$, $E(g_P; g_P(0)) = X$, and $E(b_P; b_P(0)) = X$ and so

$$(\forall x \in X) \begin{pmatrix} r_P(x) = r_P(0) \\ g_P(x) = g_P(0) \\ b_P(x) = b_P(0) \end{pmatrix}.$$

Thus r_P, g_P, and b_P are constant, that is, P is constant. Hence, P is a picture fuzzy strong BCC-ideal of X. □

Definition 4.2. *Let P be a PFS in X. For any $\alpha, \beta, \gamma \in [0, 1]$, the sets*

$$UUL_P(\alpha, \beta, \gamma) = \{x \in X \mid r_P(x) \geq \alpha, g_P(x) \geq \beta, b_P(x) \leq \gamma\},$$

$$LLU_P(\alpha, \beta, \gamma) = \{x \in X \mid r_P(x) \leq \alpha, g_P(x) \leq \beta, b_P(x) \geq \gamma\},$$

$$E_P(\alpha, \beta, \gamma) = \{x \in X \mid r_P(x) = \alpha, g_P(x) = \beta, b_P(x) = \gamma\}$$

are called a UUL-(α, β, γ)-level subset, a LLU-(α, β, γ)-level subset, and an E-(α, β, γ)-level subset of P, respectively. Then we see that

$$UUL_P(\alpha, \beta, \gamma) = U(r_P; \alpha) \cap U(g_P; \beta) \cap L(b_P; \gamma),$$

$$LLU_P(\alpha, \beta, \gamma) = L(r_P; \alpha) \cap L(g_P; \beta) \cap U(b_P; \gamma),$$

$$E_P(\alpha, \beta, \gamma) = E(r_P; \alpha) \cap E(g_P; \beta) \cap E(b_P; \gamma).$$

Corollary 4.1. *A PFS P in X is a picture fuzzy BCC-subalgebra of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, $UUL_P(\alpha, \beta, \gamma)$ is a BCC-subalgebra of X if $UUL_P(\alpha, \beta, \gamma)$ is nonempty.*

Proof. It is straightforward by Theorems 4.1 and 2.1. □

Corollary 4.2. *A PFS P in X is a picture fuzzy near BCC-filter of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, $UUL_P(\alpha, \beta, \gamma)$ is a near BCC-filter of X if $UUL_P(\alpha, \beta, \gamma)$ is nonempty.*

Proof. It is straightforward by Theorems 4.3 and 2.1. □

Corollary 4.3. *A PFS P in X is a picture fuzzy BCC-filter of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, $UUL_P(\alpha, \beta, \gamma)$ is a BCC-filter of X if $UUL_P(\alpha, \beta, \gamma)$ is nonempty.*

Proof. It is straightforward by Theorems 4.5 and 2.1. □

Corollary 4.4. *A PFS P in X is a picture fuzzy implicative BCC-filter of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, $UUL_P(\alpha, \beta, \gamma)$ is a implicative BCC-filter of X if $UUL_P(\alpha, \beta, \gamma)$ is nonempty.*

Proof. It is straightforward by Theorems 4.7 and 2.1. □

Corollary 4.5. *A PFS P in X is a picture fuzzy comparative BCC-filter of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, $UUL_P(\alpha, \beta, \gamma)$ is a comparative BCC-filter of X if $UUL_P(\alpha, \beta, \gamma)$ is nonempty.*

Proof. It is straightforward by Theorems 4.9 and 2.1. □

Corollary 4.6. *A PFS P in X is a picture fuzzy shift BCC-filter of X if and only if for all $\alpha, \beta, \gamma \in [0, 1], UUL_P(\alpha, \beta, \gamma)$ is a shift BCC-filter of X if $UUL_P(\alpha, \beta, \gamma)$ is nonempty.*

Proof. It is straightforward by Theorems 4.11 and 2.1. □

Corollary 4.7. *A PFS P in X is a picture fuzzy BCC-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0, 1], UUL_P(\alpha, \beta, \gamma)$ is a BCC-ideal of X if $UUL_P(\alpha, \beta, \gamma)$ is nonempty.*

Proof. It is straightforward by Theorems 4.13 and 2.1. □

Corollary 4.8. *A PFS P in X is a picture fuzzy strong BCC-ideal of X if and only if $E_P(r_P(0), g_P(0), b_P(0))$ is a strong BCC-ideal of X, that is, $E(r_P, r_P(0)) = X, E(g_P, g_P(0)) = X$, and $E(b_P, b_P(0)) = X$.*

Proof. It is straightforward by Theorems 4.15 and 2.1. □

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