Composition Operators on $N_K(p, q)$-Type Spaces on the Unit Ball

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Abstract. We describe the boundedness and compactness of the composition operators $C_\varphi$ acting in $N_K(p, q)$ on the open unit ball $B$.

1. Introduction

For the unit ball $B$ of $\mathbb{C}^n$, $\mathcal{HO}(B)$ denotes the class of all holomorphic functions on $B$ while $H^\infty = H^\infty(B)$ denotes the class of all functions that are holomorphic $u \in \mathcal{HO}(B)$ equipped with the norm $\|u\|_\infty = \sup_{\zeta \in B} |u(\zeta)|$. For any $d > 0$, the weighted Banach space $H^\infty_d = H^\infty_d(B)$ consists of all functions $u \in \mathcal{HO}(B)$ such that

$$\|u\|_d^\infty := \sup_{\zeta \in B} (1 - |\zeta|)^d |u(\zeta)| < \infty.$$ 

The space $H^\infty_{d,0} = H^\infty_{d,0}(B)$ indicate the closed subspace of $H^\infty_d$ such that

$$\lim_{|\zeta| \to 1} |u(\zeta)|(1 - |\zeta|)^d = 0.$$ 

For further details about the properties of $H^\infty_d$ spaces see [10]).

For $\zeta \in B$, we let $dV$ be the Lebesgue measure on $B$ with

$$V(B) = \int_B dV(\zeta) = 1.$$
In addition, we let $d\omega$ be the surface measure on $S$, normalized so that $\omega(S) \equiv 1$. If $u$ is a nonnegative Lebesgue measurable function on $B$, then the measures $V$ and $\omega$ are related by

$$\int_B u(\zeta) dV(\zeta) = 2n \int_0^1 t^{2n-1} dt \int_S u(t\zeta) d\omega(\zeta).$$

Moreover, the formulas for integration on $S$ (see, \cite{11}) as:

$$\int_S u(\zeta) d\omega(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta} \zeta) d\theta,$$

for all $0 \leq \theta \leq 2\pi$.

For any $\psi \in Aut(B)$, $u \in L^1(B)$, the Möbius invariant on $B$ (see e.g., \cite{5}) such that

$$\int_B u(\zeta) d\lambda(\zeta) = \int_B u \circ \psi(\zeta) \frac{dV}{(1 - |\zeta|^2)^{n+1}}.$$

The inner product of $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ in $\mathbb{C}^n$, is given by

$$\langle \zeta, \eta \rangle = \sum_{i=1}^n \zeta_i \eta_i.$$

For any $\zeta \in B$, we define the complex gradient and the radial derivative of the function $u \in \mathcal{H}O_1(B)$ respectively as follows:

$$\nabla u(\zeta) = \left( \frac{\partial u}{\partial \zeta_1}(\zeta), \ldots, \frac{\partial u}{\partial \zeta_n}(\zeta) \right),$$

$$Ru(\zeta) = \langle \nabla u(\zeta), \zeta \rangle = \sum_{i=1}^n \zeta_i \frac{\partial u}{\partial \zeta_1}(\zeta).$$

We know the Bloch space $B^d = B^d(B)$ is the Banach space of functions $u \in \mathcal{H}O_1(B)$ such that $Ru \in H^\infty_g$ which has the norm

$$\|u\|_{B^d} := |f(0)| + \|Ru\|_g^\infty.$$

The involution automorphisms $\Psi_b$ (the Möbius transformation of $B$) is define for $\zeta \in B$ and $b \in B - \{0\}$ as

$$\Psi_b(\zeta) = \frac{b - \langle \zeta, b \rangle b}{|b|^2} - \sqrt{1 - |b|^2} \left( \zeta - \frac{\langle \zeta, b \rangle b}{|b|^2} \right),$$

where $\Psi_0(\zeta) = -\zeta$, $\Psi_b(0) = b$, $\Psi_b(b) = 0$ and $\Psi_b = \Psi_b^{-1}$. It is well known that for any $\zeta \in B$

$$1 - |\Psi_b(\zeta)|^2 = \frac{(1 - |b|^2)(1 - |\zeta|^2)}{|1 - \langle b, \zeta \rangle|^2}.$$

The Bergman metric and the Bergman metric ball on $B$, for $\zeta, \eta \in B$ and $M > 0$ as follows:

$$\beta(\zeta, \eta) = \frac{1}{2} \log \frac{1 + |\Psi_\zeta(\eta)|}{1 - |\Psi_\zeta(\eta)|},$$

$$D(\zeta, M) = \{ \eta \in B : \beta(\zeta, \eta) < M \}.$$
Let $\mathcal{RC}^{+}$ denote the set of all right-continuous nondecreasing functions $K \neq 0$ and $K : [0, \infty) \to [0, \infty)$. For $K \in \mathcal{RC}^{+}$ and $p, q > 0$, the weighted Banach type spaces $N_{K}(p, q) = N_{K}(p, q)(B)$ consists of functions $u \in H_{0}^{1}(B)$ such that

$$N_{K}(p, q) := \{ u \in H(B) : \| u \|_{N_{K}(p, q)}^{p} < \infty \},$$

where

$$\| u \|_{N_{K}(p, q)}^{p} = \sup_{b \in B} \int_{B} \left| u(\zeta) \right|^p (1 - |\zeta|^{2})^q \frac{K(G(b, \zeta))}{K((1 - |\zeta|^{2})^{n})} d\zeta.$$

This space was introduced first by Bakhit and Aljuaid in [1] who study several fundamental properties of $N_{K}(p, q)$-type spaces and its closed subspaces $N_{K,0}(p, q)$, which are Banach spaces of functions that are analytic and their norms determined by a weighted function $K \in \mathcal{RC}^{+}$, together with a M"{o}bius transformation. Also in [1] the authors show that the norm of $N_{K}(p, q)$-type space is equivalent to the norm

$$\| u \|_{N_{K}(p, q)}^{p} = \sup_{b \in B} \int_{B} \left| u(\zeta) \right|^p (1 - |\zeta|^{2})^q \frac{K(G(b, \zeta))}{K((1 - |\zeta|^{2})^{n})} d\zeta < \infty,$$

where $G(b, \zeta) = \log \frac{1}{|\psi_{b}(\zeta)|}$. We set the integral $J_{K,q}(t)$ with $q > n$ as:

$$J_{K,q}(t) = \int_{0}^{1} \frac{t^{2n-1}}{(1 - t^{2})^{n+1-q}} K((1 - t^{2})^{n}) dt. \quad (1.1)$$

Throughout the paper, we suppose that $J_{K,q}(t) < \infty$, then $N_{K}(p, q)$ contain all the polynomials, otherwise $N_{K}(p, q)$ consists only of zero functions.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two function spaces on $B$ and consider $\varphi$ be a holomorphic self-map of $B$. We define the composition operator $C_{\varphi} : \mathcal{X} \to \mathcal{Y}$ by

$$C_{\varphi}(u)(\zeta) = u \circ \varphi, \quad \forall u \in \mathcal{X}.$$

Recall that, for any two normed linear spaces $X$ and $Y$, the linear operator $T : X \to Y$ is said to be bounded if there exists $C > 0$ such that $\| Tu \|_{Y} \leq C \| u \|_{X}, \forall u \in X$. Furthermore, a linear operator $T : X \to Y$ is said to be compact if it maps every bounded set in $X$ to a relatively compact set in $Y$ (i.e., a set whose closure is compact) (see e.g., [12]).

Studying the composition operators acting in different spaces is a quite classical topic since they arise in different problems; see the excellent monographs [2], [3] and [4]. Some of the earlier study on this topic is reflected in [9] descriptions of bounded and compact composition operators on $F(p, q, s)$ spaces were provided [8].

This paper is organized as follows: in Section 2 we shortly give the preliminaries and background information. In Section 3 we establish proving our main results respectively.

We use the notation $a \lesssim b$ in what follows to mean that there is a constant $C > 0$ with $a \leq Cb$. and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$. 
2. Preliminaries

For $0 < t < \infty$, we use the auxiliary function $\phi_K(t) = \sup_{s \in (0,1]} \frac{K(st)}{K(s)}$ (see e.g., [6], [7]). The following constraints on $\phi_K(t)$ play a significant role in the study of any class of $N_K(p,q)$ spaces:

$$J_K(t) = \int_0^1 \phi_K(t) \frac{dt}{t} < \infty,$$

(2.1)

and

$$\int_1^\infty \phi_K(t) \frac{dt}{t^{1+q}} < \infty,$$

(2.2)

and more generally,

$$\int_1^\infty \phi_K(t) \frac{dt}{t^{1+\varrho}} < \infty, \quad \varrho > 0.$$

(2.3)

In the case that $K$ satisfies condition (2.1), then $K(2t) \lesssim K(t) \forall 0 \leq 2t \leq 1$. If we started with the property that $K(t) = K(1)$ for $t \geq 1$, then $K(2t) \approx K(t)$ for $t > 0$ (see, [6]).

The following results will have an important role in the subsequent. The following lemma was proven in [1].

Lemma 2.1. Let $K \in \mathcal{RC}^+, \ p \geq 1$ and $q > 0$ then

- $N_K(p,q) \subseteq H_{q/p}^\infty(\mathbb{B})$.
- $N_K(p,q) = H_{q/p}^\infty(\mathbb{B})$ if

$$I_K(t) = \int_0^1 \frac{t^{2n-1}}{(1-t^2)^{n+1}} K((1-t^2)^n) \ dt < \infty.$$  

(2.4)

We can find the subsequent result in [11].

Lemma 2.2. Let $\delta \in (0,1]$ then there is a sequence $\{n_i\} \in \mathbb{B}$ such that

- $\lim_{i \to \infty} |n_i| = 1$.
- $\mathbb{B} = \bigcup_{i=1}^\infty D(m_i, \delta)$.
- Let $N > 0$ be an integer, then $\zeta \in \bigcap_{k=1}^{N+1} D(m_k, 4\delta)$ and $m_k \in D(\zeta, 4\delta)$ for each $\zeta \in \mathbb{B}, 1 \leq k \leq N + 1$.

Lemma 2.3. For any $K \in \mathcal{RC}^+, \delta > 0$, let $p, q > 0$ and $\zeta, b \in \mathbb{B}$. Then there is a positive constant $C$, such that

$$|u(\zeta)|^p \leq \frac{(1-|z|^2-q-n-1)}{K((1-|\psi_b(z)|^2)^n)} \int_{D(z,2\delta)} |u(w)|^p (1-|w|^2)^q K(1-|\psi_b(w)|^2) \ dV(w),$$

for all $\zeta \in D(z, \delta)$ and $u \in \mathcal{HOL}(\mathbb{B})$.

Proof. By the result in Lemma 2.24 in [5], we obtain

$$|u(\zeta)|^p \leq \frac{1}{(1-|\zeta|^2)^{n+1}} \int_{D(\zeta, \delta)} |u(w)|^p \ dV(w),$$

for all $\zeta \in \mathbb{B}$ and $u \in \mathcal{HOL}(\mathbb{B})$. 

Now let $\zeta \in D(z, \delta)$ and $w \in D(\zeta, \delta)$, then obtain $\beta(w, z) \leq \beta(w, \zeta) < 2\delta$. Thus, $D(\zeta, \delta) \subset D(z, 2\delta)$. From some results in [5], we obtain

$$1 - |\zeta|^2 \asymp 1 - |z|^2 \asymp 1 - |w|^2,$$
$$|1 - \langle b, w \rangle| \asymp |1 - \langle b, z \rangle|.$$

Thus,

$$|u(\zeta)|^p \leq \frac{(1 - |z|^2)^{-q-n-1}}{K((1 - |\Psi_b(\zeta)|^2)^n)} \int_{D(z,2\delta)} |u(w)|^p(1 - |w|^2)^qK((1 - |\Psi_b(w)|^2)^n) dV(w).$$

\[\square\]

**Lemma 2.4.** Let $\phi$ be a holomorphic self-map of $\mathbb{B}$ and $b \in \mathbb{B}$. If $u$ is a nonnegative Lebesgue measurable function on $\mathbb{B}$, then

$$\int_{\mathbb{B}} u(\zeta) d\lambda_{K,q,\phi}(\zeta) = \int_{\mathbb{B}} u(\phi(\zeta)) (1 - |\zeta|^2)^qK((1 - |\Psi_b(\zeta)|^2)^n) dV(\zeta),$$

where

$$\lambda_{K,q,\phi} = \int_{\phi^{-1}(E)} (1 - |\zeta|^2)^qK((1 - |\Psi_b(\zeta)|^2)^n) dV(\zeta),$$

for any Borel measurable set $E \subseteq \mathbb{B}$.

**Proof.** Let $u$ be a nonnegative simple Lebesgue measurable function. Assume that

$$u(\zeta) = \sum_{i=1}^{n} b_i \chi_{E_i},$$

where $E_i$ is the measurable set on $\mathbb{B}$. Then,

$$\int_{\mathbb{B}} u(\zeta) d\lambda_{K,q,\phi}(\zeta) = \sum_{i=1}^{n} b_i \lambda_{K,q,\phi}(E_i) = \sum_{i=1}^{n} b_i \int_{E_i} d\lambda_{K,q,\phi}(\zeta)$$

$$= \sum_{i=1}^{n} b_i \int_{E_i} \phi^{-1}(E) (1 - |\zeta|^2)^qK((1 - |\Psi_b(\zeta)|^2)^n) dV(\zeta)$$

$$= \int_{\mathbb{B}} \left( \sum_{i=1}^{n} b_i \chi_{\phi^{-1}(E_i) \cap B} \right) (1 - |\zeta|^2)^qK((1 - |\Psi_b(\zeta)|^2)^n) dV(\zeta)$$

$$= \int_{\mathbb{B}} u(\phi(\zeta))(1 - |\zeta|^2)^qK((1 - |\Psi_b(\zeta)|^2)^n) dV(\zeta).$$

If $u$ is a nonnegative Lebesgue measurable function, for $\zeta \in \mathbb{B}$ then there is a monotone increasing simple measurable function sequence $\{u_j\}$ such that

$$\lim_{j \to \infty} u_j(\zeta) = u(\zeta).$$

Thus,

$$\lim_{j \to \infty} \int_{\mathbb{B}} u_j(\zeta) d\lambda_{K,q,\phi}(\zeta) = \int_{\mathbb{B}} u(\zeta) d\lambda_{K,q,\phi}(\zeta).$$
Now let the function sequence \( \{U_j(K, q, \phi)\} = \{u_j(\varphi(\xi))(1 - |\xi|^2)^q K((1 - |\varphi(\xi)|^2)^n)\} \), then \( \{U_j(K, q, \phi)\} \) is a monotone increasing measurable function. Moreover,

\[
\lim_{j \to \infty} U_j(K, q, \phi) = u(\varphi(\xi))(1 - |\xi|^2)^q K((1 - |\varphi(\xi)|^2)^n),
\]

which implies that

\[
\int_B u(\zeta) \, d\lambda_{K, a, \varphi}(\zeta) = \lim_{j \to \infty} \int_B u_j(\zeta) \, d\lambda_{K, a, \varphi}(\zeta)
= \lim_{j \to \infty} \int_B u_j(\varphi(\zeta))(1 - |\zeta|^2)^q K((1 - |\varphi(\zeta)|^2)^n) \, dV(\zeta)
= \lim_{j \to \infty} \int_B u(\varphi(\zeta))(1 - |\zeta|^2)^q K((1 - |\varphi(\zeta)|^2)^n) \, dV(\zeta).
\]

This completes the proof. \( \square \)

**Lemma 2.5.** For \( K \in \mathcal{RC}^+ \) and \( p > 0, q + n + 1 > 0 \) if (2.4) holds, then \( u_w(\zeta) \in \mathcal{N}_K(p, q) \), where

\[
u_w(\zeta) = \frac{(1 - |w|^2)^{\frac{p}{n+1} + 1}}{(1 - \langle \zeta, w \rangle)^{n+1+q+p}}.
\]

**Proof.** Firstly, suppose that (2.4) holds, to show that \( u_w(\zeta) \in \mathcal{N}_K(p, q) \), it suffices to show that there is \( \varepsilon > 0 \), such that

\[
\sup_{b \in \mathcal{B}} \int_B \frac{(1 - |z|^2)^p(1 - |\zeta|^2)^q}{|1 - \langle \zeta, z \rangle_n^{n+1+q+p}} K((1 - |\varphi(\zeta)|^2)^n) \, dV(\zeta) \leq \varepsilon, \quad \forall z \in \mathcal{B}.
\]

Now we let \( \frac{1}{\sqrt{2}} < |\varphi(\zeta)| < 1 \), by the fact that \( (1 - |\zeta|) \leq |1 - \langle \zeta, b \rangle| \) and Theorem 1.4.10 in [5], therefore

\[
\int_{\frac{1}{\sqrt{2}} < |\varphi(\zeta)| < 1} \frac{(1 - |z|^2)^p(1 - |\zeta|^2)^q}{|1 - \langle \zeta, z \rangle_n^{n+1+q+p}} K((1 - |\varphi(\zeta)|^2)^n) \, dV(\zeta)
\leq \varepsilon \int_B (1 - |\zeta|^2)^{-n-1} K((1 - |\varphi(\zeta)|^2)^n) \, dV(\zeta)
\leq \varepsilon \int_0^1 \frac{t^{2n-1}}{(1 - t^2)^{n+1+q+p}} K((1 - t^2)^n) \, dt < \varepsilon.
\]

At the same time,

\[
\int_{|\varphi(\zeta)| \leq \frac{1}{\sqrt{2}}} \frac{(1 - |z|^2)^p(1 - |\zeta|^2)^q}{|1 - \langle \zeta, z \rangle_n^{n+1+q+p}} K((1 - |\varphi(\zeta)|^2)^n) \, dV(\zeta)
\leq \int_{|w| \leq \frac{1}{\sqrt{2}}} \frac{(1 - |z|^2)^p(1 - |\varphi(\zeta)|^2)^q}{|1 - \langle \varphi(\zeta), w \rangle|^{n+1+q+p}} \frac{1}{1 - \langle w, b \rangle^{2n+1+q+p}} K((1 - |w|^2)^n) \, dV(w)
\]

This completes the proof. \( \square \)
Let
\[
\begin{align*}
\leq \varepsilon \int_{|w| \leq \frac{1}{2}} \frac{(1 - |b|^2)^{n+1}}{(1 - |\psi_b(w)|^2)^{n+1}|1 - \langle w, b \rangle|^2} K((1 - |w|^2)^n) dV(w) \\
\leq \varepsilon \int_{|w| \leq \frac{1}{2}} K((1 - |w|^2)^n) \frac{dV(w)}{|1 - \langle w, b \rangle|^n} \\
\leq \varepsilon \int_{|w| \leq \frac{1}{2}} K((1 - |w|^2)^n) \frac{dV(w)}{(1 - |w|^2)^n} \\
\leq \varepsilon \int_{B} K((1 - |w|^2)^n) dV(w) < \varepsilon.
\end{align*}
\] (2.6)

Combining (2.5) and (2.6), it follows that
\[
\sup_{b \in B} \int_{B} \frac{(1 - |z|^2)^p(1 - |\zeta|^2)^q}{|1 - \langle \zeta, z \rangle|^{n+q+\rho}} K((1 - |\psi_b(\zeta)|^2)^n) dV(\zeta) \leq \varepsilon, \quad \forall \ z \in B.
\]
□

3. Main Results

3.1. Boundedness.

**Theorem 3.1.** Let \( K \in \mathcal{RC}^+ \) and \( 0 < p, q < \infty \). Then the operator \( C_\varphi \) is bounded on \( N_K(p, q) \) if and only if
\[
\sup_{w, b \in B} (1 - |w|^2)^p \left( \int_{B} \frac{(1 - |\zeta|^2)^q}{|1 - \langle \varphi(\zeta), w \rangle|^{q+n+1}} K((1 - |\psi_b(\zeta)|^2)^n) dV(\zeta) \right) < \infty. \] (3.1)

**Proof.** Let \( C_\varphi \) be the bounded operator in \( N_K(p, q) \). Consider the function
\[
u_{w}(\zeta) = \frac{(1 - |w|^2)^p}{(1 - \langle \zeta, w \rangle)^{\frac{q+n+1}{p}}}.\]

Then by Lemma 2.5, we obtain
\[
\begin{align*}
\int_{B} |\nu_{w}(\zeta)|^p(1 - |\zeta|^2)^q K((1 - |\psi_b(\zeta)|^2)^n) dV(\zeta) \\
\leq \int_{B} \frac{(1 - |w|^2)^p(1 - |\zeta|^2)^q}{|1 - \langle \zeta, w \rangle|^{p+q+n+1}} K((1 - |\psi_b(\zeta)|^2)^n) dV(\zeta) \leq \varepsilon,
\end{align*}
\]
which exactly
\[
\|C_\varphi(\nu_{w})\|_{N_K(p, q)} \leq \|C_\varphi\| \|\nu_{w}\|_{N_K(p, q)} \leq \varepsilon^\frac{1}{p} \|C_\varphi\|.
\]

That is
\[
\sup_{w, b \in B} (1 - |w|^2)^p \int_{B} \frac{(1 - |\zeta|^2)^q}{|1 - \langle \varphi(\zeta), w \rangle|^{q+n+1}} K((1 - |\psi_b(\zeta)|^2)^n) dV(\zeta) \leq \varepsilon \|C_\varphi\|^p.
\]

Conversely, suppose that (3.1) holds, then by Lemma (2.3), there exists a constant \( \varepsilon \) such that
\[
\frac{(1 - |w|^2)^p}{K((1 - |\psi_b(w)|^2)^n)} \int_{B} \frac{d\lambda_{K, b, \varphi}(\zeta)}{|1 - \langle \zeta, w \rangle|^{q+n+1}} \leq \varepsilon, \quad \forall \ w, b \in B,
\]
where
\[ \lambda_{K,q,\varphi} = \int_{\psi^{-1}(E)} (1 - |\zeta|^2)^q K((1 - |\Psi_b(\zeta)|^2)^p) dV(\zeta), \; \forall \; E \subseteq \mathbb{B}. \]

Fixed \( \delta > 0 \), so that
\[ \frac{(1 - |w|^2)^p}{K(1 - |\Psi_b(w)|^2)} \int_{D(w,\delta)} d\lambda_{K,q,\varphi}(\zeta) \leq \varepsilon, \; \forall \; w, b \in \mathbb{B}. \]

Then, we have
\[ \lambda_{K,q,\varphi}(D(w, \delta)) \leq (1 - |w|^2)^{q+n+1} K(1 - |\Psi_b(w)|^2). \]

If \( u \in \mathcal{N}_K(p, q) \), then
\[ \int_{\mathbb{B}} |u(\varphi(\zeta))|^p (1 - |\zeta|^2)^q K(1 - |\Psi_b(\zeta)|^2) dV(\zeta) \]
\[ = \int_{\mathbb{B}} |u(\zeta)|^p d\lambda_{K,q,\varphi}(\zeta) \leq \sum_{j=1}^{\infty} \int_{D(w_j, \delta)} |u(\zeta)|^p d\lambda_{K,q,\varphi}(\zeta) \]
\[ \leq \sum_{j=1}^{\infty} \sup_{\zeta \in D(w_j, \delta)} |u(\zeta)|^p \int_{D(w_j, \delta)} d\lambda_{K,q,\varphi}(\zeta) \]
\[ \leq \sum_{j=1}^{\infty} \sup_{\zeta \in D(w_j, \delta)} |u(\zeta)|^p (1 - |w_j|^2)^{q+n+1} K(1 - |\Psi_b(w_j)|^2) \]
\[ \sum_{j=1}^{\infty} \int_{D(w_j, 4\delta)} |u(\zeta)|^p (1 - |\zeta|^2)^q K(1 - |\Psi_b(\zeta)|^2) dV(\zeta) \]
\[ \leq \|u\|_{\mathcal{N}_K(p, q)}^q. \]

3.2. **Compactness.**

**Theorem 3.2.** Let \( K \in \mathcal{RC}^+ \) and \( 0 < p, q < \infty \). Then the operator \( C_\varphi \) is compact on \( \mathcal{N}_K(p, q) \) if and only if
\[ \lim_{|w| \to 1^-} \sup_{b \in \mathbb{B}} (1 - |w|^2)^p \left( \int_{\mathbb{B}} \frac{(1 - |\zeta|^2)^q}{|1 - \langle \varphi(\zeta), w \rangle|^{q+n+1}} K(1 - |\Psi_b(\zeta)|^2) dV(\zeta) \right) = 0. \]  

**Proof.** Let \( C_\varphi \) be compact on \( \mathcal{N}_K(p, q) \). Then, for any sequence \( \{\xi_j\} \subseteq \mathbb{B} \) with \( \lim_{j \to \infty} |\xi_j| = 1 \). Take
\[ h_j(\zeta) = \frac{(1 - |\xi_j|)}{(1 - \langle \zeta, \xi_j \rangle)^{\frac{q+n+1}{p}}}. \]

Since \( \{h_j\} \) is bounded on \( \mathcal{N}_K(p, q) \) and converges uniformly to 0 on any compact subset of \( \mathbb{B} \). So, by the compactness of \( C_\varphi \), we obtain
\[ (1 - |w|^2)^p \int_{\mathbb{B}} \frac{(1 - |\zeta|^2)^q K(1 - |\Psi_b(\zeta)|^2)}{|1 - \langle \varphi(\zeta), w \rangle|^{q+n+1}} dV(\zeta) \]
\[ = \|C_\varphi(h_j)\|_{\mathcal{N}_K(p, q)}^p \to 0, \; \text{as} \; j \to \infty. \]
Conversely, assume that (3.2) holds. Then, we can choose the sequence \( \{w_i\} \in \mathbb{B} \) from Lemma (2.2), such that
\[
\sup_{b \in \mathbb{B}} \frac{(1 - |w_i|^2)^p}{K(1 - |\Psi_b(w_i)|^2)^q} \int_B \frac{d\lambda_{K,a,\varphi}(\zeta)}{|1 - (\zeta, w_i)|^{q+n+p+1}} \to 0, \text{ as } i \to 0.
\]
Thus, for any \( \epsilon > 0 \), there exists a positive integer \( N_0 \) such that
\[
\sup_{b \in \mathbb{B}} \frac{(1 - |w_i|^2)^p}{K(1 - |\Psi_b(w_i)|^2)^q} \int_B \frac{d\lambda_{K,a,\varphi}(\zeta)}{|1 - (\zeta, w_i)|^{q+n+p+1}} < \epsilon, \text{ when } i > N_0. \tag{3.3}
\]
In this case, by (3.3) for all \( a \in \mathbb{B} \) when \( j > N_0 \), we have
\[
\lambda_{K,a,\varphi}(D(w_i, \delta) \leq \epsilon^p(1 - |w_i|^2)^{q+n+p+1}K(1 - |\Psi_b(\zeta)|^n). \tag{3.4}
\]

Now we let \( \{u_j\} \) be any sequence that converges to 0 uniformly on any compact subset of \( \mathbb{B} \) with \( \|u_j\|_{\mathcal{K}_r(p,q)} \leq C \). Then, the sequence \( \{u_j\} \) converges to 0 uniformly on \( M = \bigcup_{k=1}^{N_0} D(w_k, \delta) \). Thus, there exists a positive integer \( \overline{N}_0 \) such that
\[
\sup_{\zeta \in M} |u_j(\zeta)| < \epsilon \text{ when } j > \overline{N}_0. \tag{3.5}
\]
Otherwise,
\[
\lambda_{K,a,\varphi}(\mathbb{B}) \leq \int_{\mathbb{B}} (1 - |\zeta|^2)^qK((1 - |\Psi_b(\zeta)|^2)^n) dV(\zeta) \leq C. \tag{3.6}
\]
Therefore, when \( j > \overline{N}_0 \), by Lemma 2.2-2.4, (3.4)-(3.6), for all \( a \in \mathbb{B} \) we have
\[
\begin{align*}
\int_{\mathbb{B}} |u_j(\varphi(\zeta))|^p(1 - |w|^2)^qK(1 - |\Psi_b(\zeta)|^n) dV(\zeta) \\
= \int_{\mathbb{B}} |u_j(\zeta)|^p d\lambda_{K,a,\varphi} \leq \sum_{k=1}^{\infty} \int_{D(w_k, \delta)} |u_j(\zeta)|^p d\lambda_{K,a,\varphi} \\
\leq \sum_{k=1}^{\overline{N}_0} \int_{D(w_k, \delta)} |u_j(\zeta)|^p d\lambda_{K,a,\varphi} + \sum_{k=\overline{N}_0+1}^{\infty} \sup_{\zeta \in D(w_k, \delta)} |u_j(\zeta)|^p \lambda_{K,a,\varphi} \\
\lesssim N_0 \epsilon^p \lambda_{K,a,\varphi}(\mathbb{B}) + \epsilon^p \sum_{k=\overline{N}_0+1}^{\infty} \sup_{\zeta \in D(w_k, \delta)} |u_j(\zeta)|^p (1 - |\zeta|^2)^{q+n+1}K((1 - |\Psi_b(\zeta)|^2)^n) \\
\lesssim N_0 \epsilon^p \lambda_{K,a,\varphi}(\mathbb{B}) + \epsilon^p \int_{D(w_k, \delta)} |u_j|^p(1 - |\zeta|^2)^qK((1 - |\Psi_b(\zeta)|^2)^n) dV(\zeta) \\
\lesssim N_0 \epsilon^p \lambda_{K,a,\varphi}(\mathbb{B}) + \epsilon^p \|u_j\|_{\mathcal{K}_r(p,q)} \lesssim \epsilon^p,
\end{align*}
\]
which exactly
\[
\lim_{k \to \infty} \|\varphi(u_j)\|_{\mathcal{K}_r(p,q)} = 0.
\]
In this case, the operator \( C_\varphi \) is compact on \( \mathcal{K}_r(p,q) \), which completed the proof. \( \square \)
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**References**


