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Steinhaus Type Theorem for Nörlund-(M, λ_n) and Nörlund-Euler-(M, λ_n) Methods of Summability in Non-Archimedean Fields

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Abstract. In the present research paper, an investigation is undertaken of Steinhaus type theorems for Nörlund-(M,λ_n) and Nörlund-Euler(M, λ_n) method of summability in K, a complete non-trivially valued Non-Archimedean field. The conditions for statistical summability for those matrices are discussed in such fields K. The consistency of Nörlund-(M, λ_n) method of summability is investigated when different sequences are used in the summation process. Further, the relation between Nörlund-Euler(M, λ_n) summable and its statistical summability is also established.

1. Introduction

Summability methods originated with the study of convergent and divergent series by distinguished mathematicians Euler, Gauss, Cauchy, and Abel. The theory of summability starts with the definition of Abel Convergence. Some well-known summability methods are, Nörlund transformation, Nörlund type transformation, Holder means, Cesaro means, Euler means, Hausdroff means, Taylor Exponential transformation, Borel Exponential transformation etc. G.H. Hardy developed study on Divergent series, while A. Zygmund studied on Trigonometric series.

The theory of p-adic fields, which are non-Archimedean in nature was defined by Kurt Hensel. Voronoi and Nörlund defined Nörlund method of summability, V.K. Srinivasan [16] introduced Nörlund method of summability in ultrametric fields. P.N. Natarajan developed Norlund method of summability and weighted means method in Non-Archimedean fields. Further, he introduced (M, λ_n) method,

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cauchy multiplication method of (M, λ_n) summable series [12] and verified Steinhaus type theorems [6, 8, 11, 14] in non-Archimedean fields. Suja and Srinivasan developed on Nörlund method of summability in ultrametric fields, and established the relation between (N, p_n) summability and Nörlund statistical convergence. Loku and Aljimi introduced Nörlund-Euler-λ statistical convergence for reals in their study [5]. Many researchers studied Nörlund-Euler statistical convergence [2, 3].

In this paper, Nörlund-(M,λ_n), and Nörlund-Euler-(M,λ_n) methods of summability are introduced. Steinhaus type theorems for these newly developed methods are verified in ultrametric field K. For a general reference on ultrametric analysis, it is recommended that the Book [1] be used.

Throughout this article, K denotes a non-trivially valued, non-Archimedean complete field.

2. Preliminaries

Definition 2.1. Let $x = \{x_k\}$ be a sequence of elements of K, $\{x_k\}$ is said to be statistically convergent to a limit $'\ell'$, if for any given $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0.$$

Where, vertical bars indicate the number of elements that lie outside the neighbourhood $B_\varepsilon(x_k, \ell)$ [17].

Definition 2.2. Consider $A = \{a_{nk}\}, a_{nk} \in K, k = 0, 1, 2, 3, \ldots,$ and a sequence $x = \{x_k\}, x_k \in K, k = 0, 1, 2, 3, \ldots$ A-Transform of $\{x_k\}$ denoted by $\{(Ax)_n\}$ is defined as $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k,\ k = 0, 1, 2, 3, \ldots$

If the A-Transform of $\{x_k\}$ converges to a limit $'\ell'$, it is said that the sequence $\{x_k\}$ is A-Summable to $'\ell'$ [10].

Definition 2.3. Given the sequence $p = \{p_n\}$ of elements of K, the Nörlund Method (N, p_n) is defined as the infinite matrix $\{a_{nk}\}$, where

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n}, & if\ k \leq n \\ 0\ , & if\ k > n \end{cases}$$

Where $|p_n| < |p_0|, n = 0, 1, 2, 3\ldots$ and $P_n = \sum_{k=0}^{n} p_k,\ n = 0, 1, 2, 3, \ldots$

The matrix $\{a_{nk}\}$ is denoted by (N, p_n), and it is called a Nörlund Matrix [16].

Example 2.1. *Consider the sequence $\{p_n\} = 1, 1, 1, \ldots,$ (N, p_n) Nörlund matrix for this sequence is defined as,*

$$A = \{a_{nk}\} = \begin{cases} \frac{1}{k}, & if\ k \leq n \\ 0\ , & if\ k > n \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0\dots\ \dots\ \dots & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0\dots\ \dots\ \dots & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & 0\dots\ \dots\ \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ \ddots & \vdots & \vdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\dots\ \dots\ \dots & \frac{1}{n-1} & \frac{1}{n} \end{pmatrix}$$

Definition 2.4. Given $\lambda = \{\lambda_n\}$ with $\lim_{n\to\infty} \lambda_n = 0$, the (M, λ_n) method is defined by the infinite matrix $B = \{b_{nk}\}$ where

$$b_{nk} = \begin{cases} \lambda_{n-k}, & if\ k \le n \\ 0\ , & if\ k > n \end{cases}$$

The Matrix $\{b_{nk}\}$ is denoted by (M, λ_n) [12].

Example 2.2. *Consider the sequence* $\{\lambda_n\} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots,$
an $n \times n - (M, \lambda_n)$ *matrix for this sequence is ,*

$$A = \{a_{nk}\} = \begin{cases} \frac{1}{n-k+1}, & if\ k \le n \\ 0\ , & if\ k > n \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0\dots\ \dots\ \dots & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0\dots\ \dots\ \dots & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0\dots\ \dots\ \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-2} & \frac{1}{n-3} & \frac{1}{n-4}\dots\ \dots\ \dots & \frac{1}{2} & 1 \end{pmatrix}$$

Definition 2.5. Let A and B be transforms.
i) If the sequence $\{x_k\}_0^\infty$ is both A-summable and B-summable then A and B are comparable for $\{x_k\}_0^\infty$.
ii) If, given any sequence for which A and B are comparable, and if the A transform converges to the same value as B transform, then, A and B are said to be consistent [15].

Theorem 2.1. Silverman - Toeplitz theorem *A matrix* $A = \{a_{nk}\}$ *is said to be regular if and only if*
i)

$$\sup_{n,k} |a_{nk}| < \infty$$

ii)

$$\lim_{n\to\infty} a_{nk} = 0, \ \ k = 0, 1, 2, 3, \dots$$

iii)

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} = 1.$$

Definition 2.6. Given a sequence $x = \{x_k\}$, the Nörlund-(M, λ_n) Method is defined as matrix

$$A = \{a_{nk}\} = \begin{cases} \sum_{i=0}^{k} \frac{p_i x_i}{P_i}\lambda_k, & if\ k \leq n \\ 0\ , & if\ k > n \end{cases}$$

Example 2.3. *Consider the sequence* $\{p_n\} = 1, 1, 1, \ldots,$ *Given a sequence* $x = \{x_k\}$*, the Nörlund-*(M, λ_n) *Method is defined as matrix*

$$A = \{a_{nk}\} = \begin{cases} \frac{\sum_{i=1}^{k} \frac{1}{i}}{k}, & if\ k \leq n \\ 0\ , & if\ k > n \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \ldots\ \ldots\ \ldots & 0 & 0 \\ \frac{3}{4} & \frac{1}{2} & 0 & 0 \ldots\ \ldots\ \ldots & 0 & 0 \\ \frac{11}{18} & \frac{1}{2} & \frac{1}{3} & 0 \ldots\ \ldots\ \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ \ddots & \vdots & \vdots \\ \frac{\sum_{i=1}^{n}\frac{1}{i}}{n} & \frac{\sum_{i=1}^{n-1}\frac{1}{i}}{n} & \frac{\sum_{i=1}^{n-2}\frac{1}{i}}{n} & \frac{\sum_{i=1}^{n-3}\frac{1}{i}}{n} \ldots\ \ldots\ \ldots & \frac{3}{2n} & \frac{1}{n} \end{pmatrix}$$

Definition 2.7. The series $\sum_{k=0}^{\infty} x_k$, is said to be statistically summable to $'\ell'$ by Nörlund-(M, λ_n) method if $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |\sum_{i=0}^{k} \frac{p_i x_i}{P_k}\lambda_k - \ell| \geq \varepsilon\}| = 0$.

Definition 2.8. Let (X, Y) be the sequence space of elements from K.
$A = \{a_{nk}\} \in (X, Y)$, if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\}$.
Let ℓ_∞ denote the space of all bounded sequences in K, and c denote the closed subspace of ℓ_∞ consisting of all convergent sequences in K.
If $A \in c$, then A is said to be regular. The set of all regular matrices is denoted by $(c, c; P)$, where P denotes "Preservation of limits".
For any regular matrix $A = \{a_{nk}\}$, Steinhaus type theorem is stated as $(c, c; P) \cap (\ell_\infty, c) = \phi$ [11].

Definition 2.9. Let $x = \{x_k\}$ be sequence of elements of K.
The series $\sum_{k=0}^{\infty} x_k$ is summable to S, by Nörlund-Euler(M, λ_n) method, and it is denoted by $S_n \to S(NE(M, \lambda_n), p)$ if,

$$\sum_{k=0}^{n} b_k = S, \text{where } b_k = t_k\lambda_0 + t_{k-1}\lambda_1 + t_{k-2}\lambda_2 + \ldots \tag{2.1}$$

Where, $t_n^{(NE(M,\lambda_n),p)(E,r)} = \sum_{k=0}^{\infty} \left(\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}x_k\right),\ n = 0, 1, 2, 3, \ldots$
where $d_{ni} = \frac{p_n}{P_n}$,

$$e_{ij}^{(q)} = \begin{cases} \binom{i}{j} q^j (1-q)^{i-j}, & \text{if i } \le j \\ 0\ , & if\ i > j \end{cases}$$

Definition 2.10. The series $\sum_{k=0}^{\infty} x_k$, is said to be statistically summable to $'\ell'$ by Nörlund-Euler(M, λ_n) method if

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \frac{1}{P_n} \sum_{j=k}^{\infty} \sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}x_k - \ell| \ge \epsilon \right\} \right| = 0. \tag{2.2}$$

By $A = \{a_{nk}\} \in (c, (NE(M, \lambda_n), p))$ it is denoted that $(Ax)_n \in (NE(M, \lambda_n), p)$, whenever $x = \{x_k\} \in X$.

3. Steinhaus type theorem for Nörlund-(M, λ_n) method of summability

Theorem 3.1. Nörlund-(M, λ_n) Method is regular if and only if

$$i) \quad \lim_{n\to\infty} \frac{\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n} = 0. \tag{3.1}$$

$$ii) \quad \sup_{n,k} \left| \frac{\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n} \right| < \infty. \tag{3.2}$$

$$iii) \quad \lim_{n\to\infty} \frac{\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n} = 1. \tag{3.3}$$

Proof. Let Nörlund-(M, λ_n) be regular.
Let

$$a_{nk} = \frac{\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n}$$

$$Since \lim_{n\to\infty} \lambda_n = 0, \sum_{n=0}^{\infty} \lambda_n = 1.$$

$$\lim_{n\to\infty} \frac{\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n} = 1$$

Now

$$\lim_{n\to\infty} a_{nk} = 1,$$

$$i.e., \lim_{n\to\infty} \frac{\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n} = 1$$

$$i.e., \lim_{n\to\infty} \frac{\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-i}\lambda_i}{P_n} = 1$$

Conversely, if

$$\frac{\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-i}\lambda_i}{P_n} = 1$$

$$\lim_{n\to\infty}\sum_{n=0}^{\infty} a_{nk} = 1$$

.

$$\lim_{n\to\infty}\frac{\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-i}\lambda_i}{P_n} = 1$$

$$Since \lim_{n\to\infty}\frac{\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n} = 1$$

$$\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty}\frac{\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n} = 1, k = 0, 1, 2, 3\ldots$$

Since $\frac{\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n}$ is bounded, it follows that

$$\sup_{n,k}\left|\frac{\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i}{P_n}\right| < \infty$$

By Theorem 2.1 Nörlund-(M, λ_n) is Regular. □

Theorem 3.2. $A = \{a_{nk}\} \in (c, (N(M, \lambda_n), p))$ if and only if

$$i) \quad \underset{n,\,k}{\text{Sup}}\left|\frac{1}{P_n}\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i\right| < \infty. \tag{3.4}$$

$$ii) \quad \lim_{n\to\infty}\frac{1}{P_n}\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i = 0. \tag{3.5}$$

$$iii) \quad \lim_{n\to\infty}\frac{1}{P_n}\sum_{n=0}^{\infty}\sum_{i=0}^{n-k} p_{n-k-i}\lambda_i = 1. \tag{3.6}$$

Proof. A matrix $A = \{a_{nk}\}$ is regular iff

$$i)\sup_{n,k}|a_{nk}| < \infty$$

$$\text{ii)} \lim_{n\to\infty} a_{nk} = 0, k = 0, 1, 2, 3, \ldots$$

$$\text{iii)} \lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} = 1.$$

Sufficiency part

Let Equations 3.4-3.6 hold true, and let $A = \{a_{nk}\}$ denote the A-transform of $\{x_k\}$.

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$$

For the $N\ddot{o}$rlund-Euler(M,λ_n) matrix of $\{a_{nk}\}$
where

$$a_{nk}=\begin{cases}\sum_{i=0}^{n-k}\frac{p_{n-k-i}\lambda_i}{P_n}, & if\ k\leq n\\ 0\ , & if\ k>n\end{cases}$$

$B=\{b_{nk}\}=\sum_{i=0}^{n-k}\frac{p_{n-k-i}\lambda_i}{P_n}$.
Using equations 3.2 & 3.3 of Theorem 3.1, and Theorem 2.1, the following is obtained.

$$\left\{\sum_{k=0}^{\infty}b_{nk}x_k\right\}_{n=0}^{\infty}\in c,$$

i.e.,

$$\left\{\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{i=0}^{n-k}p_{n-k-i}\lambda_i\right)x_k\right\}_{n=0}^{\infty}\in c,$$

i.e.,

$$\left\{\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{j=0}^{n}\sum_{i=0}^{n-j}p_{n-j-i}\lambda_i x_j\right)\right\}_{n=0}^{\infty}\in c,$$

$$\left\{\frac{1}{P_n}\sum_{k=0}^{\infty}a_{nk}x_k\right\}_{n=0}^{\infty}\in c,$$

$$(Ax)_n\in(N(M,\lambda_n),p)$$

Thus

$$A\in(c,(N(M,\lambda_n),p)).$$

Necessity part

Let

$$A\in(c,(N(M,\lambda_n),p))$$

So far

$$x=\{x_k\}\in c,\{(Ax)_n\}_{n=0}^{\infty}\in(N(M,\lambda_n),p)$$

$$i.e.,\left\{\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{j=0}^{n}\sum_{i=0}^{n-j}p_{n-j-i}\lambda_i x_j\right)\right\}_{n=0}^{\infty}\in c,$$

$$i.e.,\left\{\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{i=0}^{n-k}p_{n-k-i}\lambda_i\right)x_k\right\}_{n=0}^{\infty}\in c,$$

$$\left\{\sum_{k=0}^{\infty}b_{nk}x_k\right\}_{n=0}^{\infty}\in c,$$

$B=\{b_{nk}\}\in(c,c)$.

Using Theorem 2.1, it is seen that equations 3.4- 3.6 hold good.

The A-Transform of the convergent sequence $\{x_k\}$, is considered where $x_k = k, 0 \leq k \leq p-1$.

It is noted that $A = \{a_{nk}\}$ converges for $n = 0, 1, 2, 3 \ldots$

Hence, proved. □

Corollary 3.1. $A \in (c, (N(M, \lambda_n), p); P)$

i.e., $A \in (c, (N(M, \lambda_n), p))$ with

$$\lim_{n\to\infty} \frac{1}{P_n}\left\{\sum_{i=0}^{n} p_{n-i}\lambda_i x_0 + \sum_{i=0}^{n-1} p_{n-i-1}\lambda_i x_1 + \ldots + \sum_{i=0}^{1} p_{1-i}\lambda_i x_{n-1} + p_0\lambda_0 x_n\right\} = \lim_{k\to\infty} x_k$$

Rearranging the terms, the following is obtained.

$$\lim_{n\to\infty} \frac{1}{P_n}\left\{p_0\sum_{i=0}^{n} \lambda_{n-i}x_i + p_1\sum_{i=0}^{n-1} \lambda_{n-i}x_i + \ldots + p_{n-1}\sum_{i=0}^{1} \lambda_{n-i}x_i + p_n\lambda_0 x_0\right\} = \lim_{k\to\infty} x_k$$

$$\lim_{n\to\infty} \frac{1}{P_n}\{p_0(Ax)_n + p_1(Ax)_{n-1} + \ldots + p_n(Ax)_0\} = \lim_{k\to\infty} x_k$$

$x = \{x_k\}$ holds good if and only if 3.4-3.6 of Theorem 3.2 hold true.

Theorem 3.3. $A = \{a_{nk}\} \in (\ell_\infty, (N(M, \lambda_n), p))$ if and only if equations 3.4 & 3.5 of Theorem 3.2 are true and

$$\lim_{n\to\infty} Sup_k\Big|p_i\big(p_{n-j-i}\lambda_i - p_{n-j-i+1}\lambda_i\big)\Big| = 0 \tag{3.7}$$

Proof. Proof follows directly from Theorem 3.2. □

The following theorem is Steinhaus type theorem in non-Archimedean field using Nörlund-(M, λ_n) method of summability.

Theorem 3.4. $(c, (N(M, \lambda_n), p); P) \cap (\ell_\infty, (N(M, \lambda_n), p)) = \phi$.

Proof. Let $A = \{a_{nk}\} \in (c, (N(M, \lambda_n), p); P) \cap (\ell_\infty, (N(M, \lambda_n), p))$

$$\lim_{n\to\infty} \sup_{n,k\to\infty} |a_{nk} - 1| = 0.$$

$$\lim_{n\to\infty} \sup_{n,k\to\infty} |\frac{\lambda_n p_k}{P_n} - 1| = 0.$$

$$\implies |\frac{\lambda_n p_k}{P_n} - 1| = 0.\ \forall\ n, k = 0, 1, 2, 3, \ldots$$

$$\implies \lambda_n p_k = P_n.\ \forall\ n, k = 0, 1, 2, 3, \ldots$$

which is a $\Rightarrow\Leftarrow$.

As per our assumption, $\{P_n\}$ is a sequence of non-decreasing elements of K.

Hence, $(c, (N(M, \lambda_n), p); P) \cap (\ell_\infty, (N(M, \lambda_n), p)) = \phi$. □

Theorem 3.5. If a sequence is Nörlund-(M, λ_n) summable to $'\ell'$ then it is statistically summable to $'\ell'$ by Nörlund-(M, λ_n) method.

Proof. Given $x = \{x_k\}$ is Nörlund-(M, λ_n) summable to $'\ell'$
The partial sum $S_n = \sum_{k=0}^{n} x_k$ is summable to $'\ell'$ by Nörlund-(M, λ_n) method.
consider,

$$\left|t_n^{(N(M,\lambda_n),p)} - \ell\right| = \left|\sum_{i=0}^{k} \frac{p_i x_i}{P_k}\lambda_k - \ell\right| \tag{3.8}$$

Fix an $m \leq k$ such that, $|\sum_{i=0}^{k} \frac{\lambda_n p_i}{P_k}(x_k - \ell)| \leq M$, $M > 0$ is a real value for all k.
To prove,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |\sum_{i=0}^{k} \frac{p_i x_i}{P_k}\lambda_k - \ell| \geq \varepsilon\}| = 0.$$

Equation (3.8) can be rewritten as,
$\left|t_n^{(N(M,\lambda_n),p)} - \ell\right| = \left|\sum_{i=0}^{m} \frac{\lambda_n p_i}{P_k}(x_k - \ell) + \sum_{i=m+1}^{k} \frac{\lambda_n p_i}{P_k}(x_k - \ell)\right|$

$$\leq \max\left\{|\sum_{i=0}^{m} \frac{\lambda_n p_i}{P_k}(x_k - \ell)|, \sum_{i=m+1}^{k} \frac{\lambda_n p_i}{P_k}(x_k - \ell)|\right\} \tag{3.9}$$

Now $\{x_k\}$ is summable to $'\ell'$ by Nörlund-(M, λ_n) method.

$$\implies \lim_{n\to\infty} |t_n^{(N(M,\lambda_n),p)} - \ell| = 0, \ as\ n \to \infty.$$

$\therefore$ By (3.9), Max $\left\{\left|\sum_{i=0}^{m} \frac{\lambda_n p_i}{P_k}(x_k - \ell)|, |\sum_{i=m+1}^{k} \frac{\lambda_n p_i}{P_k}(x_k - \ell)\right|\right\} = 0.$
i.e.,

$$\lim_{n\to\infty} \text{Max}\left\{\sum_{i=0}^{m} \frac{\lambda_n p_i}{P_k}|x_k - \ell|, |\sum_{i=m+1}^{k} \frac{\lambda_n p_i}{P_k}|x_k - \ell|\right\} = 0.$$

In particular, for a given $\varepsilon > 0$, we have,

$$\lim_{n\to\infty} \frac{1}{P_k} \sum_{i=m+1}^{k} \lambda_n p_i |x_k - \ell| = 0.$$

i.e., $\lim\limits_{n\to\infty} \frac{1}{P_k} \sum_{i=m+1}^{\infty} \lambda_n p_i |x_k - \ell| = 0.$ $\forall\ k \geq m.$
$\implies$

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |\sum_{i=0}^{k} \frac{p_i x_i}{P_k}\lambda_k - \ell| \geq \varepsilon\}| = 0.$$

$\implies \{x_k\}$ is statistically summable to $'\ell'$ by Nörlund-(M, λ_n) method. □

4. Steinhaus type theorem using Nörlund-Euler-(M, λ_n) method of summability

Theorem 4.1. *$A = \{a_{nk}\} \in (c, (NE(M, \lambda_n), p))$ if and only if*

$$i) \quad \lim_{n\to\infty} \sum_{j=m}^{\infty}\sum_{i=j}^{\infty} a_{ni} e_{ij} \lambda_{j-m} = 0. \tag{4.1}$$

$$ii) \quad Sup_{n,k} \Big|\sum_{j=m}^{\infty}\sum_{i=j}^{\infty} a_{ni} e_{ij} \lambda_{j-m}\Big| = 0. \tag{4.2}$$

$$iii) \quad \lim_{n\to\infty} \sum_{m=0}^{\infty}\sum_{j=m}^{\infty}\sum_{i=j}^{\infty} a_{ni} e_{ij} \lambda_{j-m} = 0. \tag{4.3}$$

Proof. **Sufficiency part** Let 4.1,4.2 & 4.3 hold good.
Let $A = \{h_{nk}\}$ denote the Nörlund-Euler (M, λ_n) matrix of $\{x_k\}$,
where $\{h_{nk}\} = \sum_{n=0}^{\infty}\sum_{j=m}^{\infty}\sum_{i=j}^{\infty} a_{ni} e_{ij} \lambda_{j-m}$, $k = 0, 1, 2, 3 \ldots$ is defined.
Using equations 4.2, 4.3 and Theorem 2.1, the following result is obtained

$$B = \{b_{nk}\},\ b_{nk} = \frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} p_n e_{ij} \lambda_{j-k} x_k, n, k = 0, 1, 2, 3, \ldots$$

It is seen that,

$$\left\{\sum_{k=0}^{\infty} b_{nk} x_k\right\}_0^{\infty} \in c$$

i.e.,

$$\left\{\sum_{k=0}^{\infty} \Big(\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} p_n e_{ij} \lambda_{j-k}\Big) x_k\right\}_{n=0}^{\infty} \in c$$

i.e.,

$$\left\{\frac{1}{P_n}\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} p_n e_{ij} \lambda_{j-k} x_k\right\}_{n=0}^{\infty} \in c$$

$$\left\{\frac{1}{P_n}\sum_{k=0}^{\infty} h_{nk} x_k\right\}_{n=0}^{\infty} \in c$$

$$(Ax_n) \in (c, (NE(M, \lambda_n), p))$$

Thus

$$A \in (c, (NE(M, \lambda_n), p)).$$

Necessity part

$$A \in (c, (NE(M, \lambda_n), p))$$

$$(Ax_n) \in (c, (NE(M, \lambda_n), p))$$

$$\left\{\frac{1}{P_n}\sum_{k=0}^{\infty} h_{nk} x_k\right\}_{n=0}^{\infty} \in c$$

$$\left\{\frac{1}{P_n}\left(\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}p_ne_{ij}\lambda_{j-k}\right)x_k\right\}_{n=0}^{\infty}\in c$$

$$\left\{\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}p_ne_{ij}\lambda_{j-k}\right)x_k\right\}_{n=0}^{\infty}\in c$$

$$\left\{\sum_{k=0}^{\infty}b_{nk}x_k\right\}_{n=0}^{\infty}\in c,$$

$$B=\{b_{nk}\}\in(c,c)$$

Using Theorem 2.1, it is noted that equations 4.1-4.3 hold good.

Considering the A-Transform of the convergent sequence $\{x_k\}$ where $x_k = k,\ 0 \le k \le p-1$, it is found that $A = \{a_{nk}\}$ converges for $n = 0, 1, 2, 3\ldots$

i.e., $B = \{b_{nk}\} \in (c, c)$.

Hence, proved. □

Corollary 4.1.

$$A\in(c,(NE(M,\lambda_n),p);P)$$

i.e.,

$$A\in(c,(NE(M,\lambda_n),p))$$

with

$$\lim_{n\to\infty}\frac{1}{P_n}\{p_0(Ax)_n+p_1(Ax)_{n-1}+\ldots+p_n(Ax)_0\}=\lim_{k\to\infty}x_k$$

$x = \{x_k\}$ holds true if and only if equations 4.1-4.3 holds good.

Theorem 4.2. $A = \{ae\lambda_{nk}\} \in (\ell_\infty, (NE(M, \lambda_n), p))$ if and only if 4.1 & 4.3 of Theorem 4.1 are true and

$$\lim_{n\to\infty}Sup_{n,k}\left|p_i\left(\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}a_{n-i+1,i}e_{ij}\lambda_{j-k}-\sum_{j=k}^{\infty}\sum_{i=j}^{\infty}a_{n-i,i}e_{ij}\lambda_{j-k}\right)\right|=0 \tag{4.4}$$

Proof. Proof of theorem is direct. □

Theorem 4.3. $(c, (NE(M, \lambda_n), p); P) \cap (\ell_\infty, (NE(M, \lambda_n), p)) = \phi$

Proof. Let $A = \{ae\lambda_{nk}\} \in (c, (NE(M, \lambda_n), p); P) \cap (\ell_\infty, (NE(M, \lambda_n), p))$

using Theorem 4.1 , the following is obtained

$|p_n| = |p_0|,\ n = 0, 1, 2, 3, \ldots$

$$\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{i=0}^{n}p_ih_{n-i,k}\right)$$

converges uniformly to 0.

$$\implies \lim_{n\to\infty} \sum_{k=0}^{\infty} \left(\frac{1}{P_n} \sum_{i=0}^{n} p_i h_{n-i,k}\right)$$
$$= \sum_{k=0}^{\infty} \lim_{n\to\infty} \left(\frac{1}{P_n} \sum_{i=0}^{n} p_i h_{n-i,k}\right)$$
$$= 0$$

which is a contradiction to our assumption that $A \in (c, (NE(M, \lambda_n), p))$.
Hence, $(c, (NE(M, \lambda_n), p); P) \cap (\ell_\infty, (NE(M, \lambda_n), p)) = \phi$ □

Theorem 4.4. *Any two regular methods $NE(M, \lambda_n)$ and $NE(M, \mu_n)$ are consistent.*

Proof. Let $NE(M, \lambda_n)$ and $NE(M, \mu_n)$ be regular.

$$\implies \lim_{n\to\infty} \sum b_i \lambda_i = 0$$
$$\lim_{n\to\infty} \sum b_i \mu_i = 0$$

Let γ_n be the product of λ_n, μ_n matrices. So,

$$\gamma_n = \lambda_0\mu_n + \lambda_1\mu_{n-1} + \ldots + \lambda_n\mu_0,\ \ for\ n = 0, 1, 2, 3, \ldots \tag{4.5}$$
$$u_n = \lambda_n b_0 + \lambda_{n-1} b_1 + \ldots + \lambda_1 b_{n-1} + \lambda_0 b_n,\ \ for\ n = 0, 1, 2, 3, \ldots \tag{4.6}$$
$$v_n = \mu_n b_0 + \mu_{n-1} b_1 + \ldots + \mu_1 b_{n-1} + \mu_0 b_n,\ \ for\ n = 0, 1, 2, 3, \ldots \tag{4.7}$$

Consider,
$w_n = \gamma_n b_0 + \gamma_{n-1} b_1 + \ldots + \gamma_1 b_{n-1} + \gamma_0 b_n$
$= (\lambda_0\mu_n + \lambda_1\mu_{n-1} + \ldots + \lambda_n\mu_0)b_0 + (\lambda_0\mu_{n-1} + \lambda_1\mu_{n-2} + \ldots + \lambda_{n-2}\mu_1 + \lambda_{n-1}\mu_0)b_1 + \ldots + (\lambda_0\mu_1 + \lambda_1\mu_0)b_{n-1} + \lambda_0\mu_0 b_n$
$= \lambda_0(\mu_n b_0 + \mu_{n-1} b_1 + \ldots + \mu_1 b_{n-1} + \mu_0 b_n) + \lambda_1(\mu_{n-1} b_0 + \mu_{n-2} b_1 + \ldots + \mu_1 b_{n-2} + \mu_0 b_{n-1}) + \ldots + \lambda_{n-1}(\mu_1 b_0 + \mu_0 b_1) + \lambda_n\mu_0 b_0$
$= \lambda_0 v_n + \lambda_1 v_{n-1} + \ldots + \lambda_{n-1} v_1 + \lambda_n v_0$

$$w_n = \lambda_0(v_n - x) + \lambda_1(v_{n-1} - x) + \ldots + \lambda_{n-1}(v_1 - x) + \lambda_n(v_0 - x) + x(\lambda_0 + \lambda_1 + \ldots + \lambda_{n-1} + \lambda_n)$$

$$\lim_{n\to\infty} \Big[\lambda_0(v_n - x) + \lambda_1(v_{n-1} - x) + \ldots + \lambda_{n-1}(v_1 - x) + \lambda_n(v_0 - x)\Big] = 0$$

using similar proof and logic of proof discussed by P.N. Natarajan in [12], the following is obtained,

$$\lim_{n\to\infty} w_n = x.\sum_{n=0}^{\infty} = x.1 = x$$
$$i.e., \lim_{n\to\infty} w_n = x$$

similarly, we can prove

$$\lim_{n\to\infty} w_n = y$$

It is noted that $x = y$. Hence, $NE(M, \lambda_n)$ and $NE(M, \mu_n)$ are consistent. □

Theorem 4.5. If a sequence is Nörlund-Euler(M, λ_n) Summable to $'\ell'$, then it is Statistically summable to $'\ell'$.

Proof. Given $x = \{x_k\}$ is Nörlund-Euler(M, λ_n) summable to $'\ell'$.
The partial sum $S_n = \sum_{k=0}^{n} x_k$ is summable to $'\ell'$ by Nörlund-Euler(M, λ_n) method.
i.e.,

$$\sum_{k=0}^{n} b_k = \ell, \text{where } b_k = t_k\lambda_0 + t_{k-1}\lambda_1 + t_{k-2}\lambda_2 + \ldots$$

Where,

$$t_n^{(NE(M,\lambda_n),p)(E,r)} = \sum_{k=0}^{\infty} \Big(\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{nj}e_{ij}\lambda_{j-k}x_k\Big) \ n = 0, 1, 2, 3, \ldots Where\ d_{nj} = \frac{p_n}{P_n},$$

$$e_{ij}^{(q)} = \begin{cases} \binom{i}{j} q^j (1-q)^{i-j}, & if\ i \le j \\ 0\ , & if\ i > j \end{cases}$$

consider,

$$\left|t_n^{(NE(M,\lambda_n),p)(E,r)} - \ell\right| = \left|\left\{\sum_{k=0}^{\infty}\left(\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}(x_k-\ell)\right)\right\}\right|$$

$$= \left|\left\{\frac{1}{P_n}\sum_{k=0}^{\infty}\Big(\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}\Big)x_k - \ell\right\}\right| \tag{4.8}$$

Fix an $m \le k$ such that, $|\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}(x_k-\ell)| \le M$,
$M > 0$ is a real value for all k.
To prove ,
$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : |\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{nj}e_{ij}\lambda_{j-k}x_k - \ell| \ge \varepsilon\right\}\right| = 0$, for some ℓ.
4.8 can be rewritten as,

$$\left|t_n^{(NE(M,\lambda_n),p)(E,r)} - \ell\right| = \left|\frac{1}{P_n}\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}(x_k - x_m + x_m - \ell)\right|$$

$$= \left|\frac{1}{P_n}\left(\sum_{k=0}^{m}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}(x_k - x_m) + \sum_{k=m+1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}(x_m-\ell)\right)\right|$$

$$\leq \max\left\{\frac{1}{P_n}\left|\sum_{k=0}^{m}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}(x_k - x_m)\right|, \frac{1}{P_n}\left|\sum_{k=m+1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}(x_m - \ell)\right|\right\} \tag{4.9}$$

Now $\{x_k\}$ is summable to $'\ell'$ by Nörlund-Euler(M, λ_n) method.
$\implies \lim_{n\to\infty}\left|t_n^{(NE(M,\lambda_n),p)(E,r)} - \ell\right| = 0$, as n $\to \infty$.
$\therefore$ By (4.9),

$$\max\left\{\frac{1}{P_n}\sum_{k=0}^{m}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}|x_k - x_m|, \frac{1}{P_n}\sum_{k=m+1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}|x_m - \ell|\right\} = 0.$$

i.e.,

$$\lim_{n\to\infty} Max\left\{\frac{M}{P_n}, \frac{1}{P_n}\sum_{k=m+1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}|x_m - \ell|\right\} = 0.$$

In particular, for a given $\varepsilon > 0$, it is seen that,

$$\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=m+1}^{\infty}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}|x_m - \ell| = 0.$$

i.e.,

$$\lim_{n\to\infty}\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{ni}e_{ij}\lambda_{j-k}|x_m - \ell| = 0. \ \forall\ k \ \geq \ m.$$

$$\implies \lim_{n\to\infty}\frac{1}{n}|\{k \leq n : |\frac{1}{P_n}\sum_{j=k}^{\infty}\sum_{i=j}^{\infty} d_{nj}e_{ij}\lambda_{j-k}x_k - \ell| \geq \varepsilon\}| = 0, \text{for some } \ell$$

. $\implies$ $\{x_k\}$ is statistically summable to $'\ell'$ by Nörlund-Euler(M, λ_n) method. □

5. Conclusions

In this research paper, Nörlund-(M, λ_n) and Nörlund-Euler-(M,λ_n) of summability are introduced. Steinhaus type theorems are investigated in a non-trivially valued non-Archimedean field. The relation between Nörlund-Euler(M, λ_n) summability, Nörlund-Euler(M, λ_n) method statistical summability, and properties of Nörlund-(M, λ_n) method of summability is proved. It is also proved that Nörlund-(M, λ_n) method of summability is consistent, when different sequences $\{\lambda_n\}$ or $\{\mu_n\}$ are used in the summation process. Further, the relation betweeen Nörlund-Euler(M, λ_n) summability and statistical summability in non-Archimedean field is discussed.

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