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Vague Bi-Quasi-Interior Ideals of Γ-Semirings

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Abstract. In this paper, we introduce and study the concept of vague bi-quasi-interior ideals of Γ-semirings as a generalization of vague bi-ideals, vague quasi-ideals, vague interior ideals, vague bi-quasi-interior ideals, and vague bi-quasi-interior ideals of Γ-semirings.

1. Introduction

In 1965, Zadeh [13] introduced the study of fuzzy sets. Mathematically, a fuzzy set in a non-empty set X is a mapping μ from X into the interval $[0,1]$; for x in X, $\mu(x)$ is called the membership of x belonging to X. This membership function gives only an approximation for belonging, but it does not give any information about not belonging. To counter this problem and obtain a better estimation and analysis of data decision-making, Gau and Buehrer [7] have initiated the study of vague sets with the hope that they form a better tool to understand, interpret and solve real-life problems.

Further, in 1995, Rao [10] introduced the concept of Γ-semirings which is a generalization of Γ-rings, ternary semirings, and semirings, and after that, he introduced and studied the ideals of Γ-semirings. Ideals play an important role in advanced studies and using algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the

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properties of a generalization of ideals in algebraic structures. Rao et al. [9, 11, 12] introduced the concepts of left (resp., right) bi-quasi-ideals and bi-interior ideals of Γ-semirings and studied the properties of left bi-quasi ideals. However, Bhargavi et al. [1–6, 8] developed the theory of vague sets on Γ-semirings.

This paper is a sequel to our study. We introduce and study the concept of vague bi-quasi-interior ideals of Γ-semirings as a generalization of vague bi-ideals, vague quasi-ideals, and vague interior ideals and characterize the vague bi-quasi-interior ideals of Γ-semirings to the crisp bi-quasi-interior ideals of Γ-semirings.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Throughout this paper, *GSR* stands for a Γ-semiring, *VGSR* stands for a vague Γ-semiring, *VBQII* stands for a vague bi-quasi-interior ideal, and *VI* stands for a vague ideal.

Definition 2.1. *Let R and Γ be two additive commutative semigroups. Then R is called a Γ-semiring if there exists a mapping $R \times \Gamma \times R \to R, (a, \alpha, b) \mapsto a\alpha b$ for $a, b \in R$ and $\alpha \in \Gamma$, satisfying the following conditions: $\forall a, b, c \in R; \alpha, \beta \in \Gamma$,*

1. $a\alpha(b + c) = a\alpha b + a\alpha c$
2. $(a + b)\alpha c = a\alpha c + b\alpha c$
3. $a(\alpha + \beta)c = a\alpha c + a\beta c$
4. $a\alpha(b\beta c) = (a\alpha b)\beta c$.

Definition 2.2. *A non-empty subset B of a GSR R is said to be a bi-quasi-interior ideal of R if B is a Γ-subsemiring of R and $B\Gamma R\Gamma B\Gamma R\Gamma B \subseteq B$.*

Definition 2.3. *A mapping $\mu : X \to [0, 1]$ is called a fuzzy set in a universe of discourse X.*

Definition 2.4. *A vague set Φ in the universe of discourse X is a pair (t_Φ, f_Φ), where $t_\Phi : X \to [0, 1]$, $f_\Phi : X \to [0, 1]$ are mappings such that $t_\Phi(\dot{x}) + f_\Phi(\dot{x}) \leq 1, \forall \dot{x} \in X$. The functions t_Φ and f_Φ are called true membership function and false membership function, respectively.*

Definition 2.5. *For a vague set $\Phi = (t_\Phi, f_\Phi)$, the interval $[t_\Phi(\dot{x}), 1 - f_\Phi(\dot{x})]$ is called the vague value of $\dot{x}$ in Φ and it is denoted by $V_\Phi(\dot{x})$, i.e., $V_\Phi(\dot{x}) = [t_\Phi(\dot{x}), 1 - f_\Phi(\dot{x})]$.*

Definition 2.6. *Let $\Phi = (t_\Phi, f_\Phi)$ and $\xi = (t_\xi, f_\xi)$ be two vague sets of a universe of discourse X. The intersection of Φ and ξ is defined as $\Phi \cap \xi = (t_{\Phi\cap\xi}, f_{\Phi\cap\xi})$, where $t_{\Phi\cap\xi} = \min\{t_\Phi, t_\xi\}$ and $f_{\Phi\cap\xi} = \max\{f_\Phi, f_\xi\}$. The union of Φ and ξ is defined as $\Phi\cup\xi = (t_{\Phi\cup\xi}, f_{\Phi\cup\xi})$, where $t_{\Phi\cup\xi} = \max\{t_\Phi, t_\xi\}$ and $f_{\Phi\cup\xi} = \min\{f_\Phi, f_\xi\}$. A vague set Φ is contained in another vague set ξ, $\Phi \subseteq \xi$ if and only if $V_\Phi(\dot{x}) \leq V_\xi(\dot{x})$, i.e., $t_\Phi(\dot{x}) \leq t_\xi(\dot{x})$ and $1 - f_\Phi(\dot{x}) \leq 1 - f_\xi(\dot{x})$, $\forall \dot{x} \in X$.*

Definition 2.7. *Let $\Phi = (t_\Phi, f_\Phi)$ be a vague set of a universe of discourse X. For $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, the (α,β)-cut or vague cut of Φ is the crisp subset of X is given by $\Phi_{(\alpha,\beta)} = \{\dot{x} \in X \mid V_\Phi(\dot{x}) \geq [\alpha,\beta]\}$, i.e., $\Phi_{(\alpha,\beta)} = \{\dot{x} \in X \mid t_\Phi(\dot{x}) \geq \alpha$ and $1 - f_\Phi(\dot{x}) \geq \beta\}$.*

Definition 2.8. *For any subset S of a GSR R, the vague characteristic set of S is a vague set $\delta_S = (t_{\delta_S}, f_{\delta_S})$ given by*

$$V_{\delta_S}(\dot{x}) = \begin{cases} [1,1] & \text{if } \dot{x} \in S, \\ [0,0] & \text{otherwise.} \end{cases}$$

Then δ_S is called the vague characteristic set of S in $[0,1]$.

Definition 2.9. *If $\Phi = (t_\Phi, f_\Phi)$ and $\xi = (t_\xi, f_\xi)$ are vague sets of a GSR R, then the product $\Phi\Gamma\xi$ of Φ and ξ is defined as follows:*

$$V_{\Phi\Gamma\xi}(\dot{x}) = \begin{cases} \sup\{\min\{V_\Phi(\dot{y}), V_\xi(\dot{z})\} \mid \dot{x} = \dot{y}\gamma\dot{z}, \text{where } \dot{y}, \dot{z} \in R; \gamma \in \Gamma\}, \\ [0,0] \text{ if for any } \dot{y}, \dot{z} \in R; \gamma \in \Gamma, \dot{y}\gamma\dot{z} \neq \dot{x}. \end{cases}$$

Definition 2.10. *A vague set $\Phi = (t_\Phi, f_\Phi)$ of a GSR R is said to be a VGSR if for all $\dot{x}, \dot{y} \in R$ and $\gamma \in \Gamma$,*

1. $V_\Phi(\dot{x} + \dot{y}) \geq \min\{V_\Phi(\dot{x}), V_\Phi(\dot{y})\}$
2. $V_\Phi(\dot{x}\gamma\dot{y}) \geq \min\{V_\Phi(\dot{x}), V_\Phi(\dot{y})\}$.

Definition 2.11. *A vague set $\Phi = (t_\Phi, f_\Phi)$ of a GSR R is said to be a left (resp., right) VI if for all $\dot{x}, \dot{y} \in R$ and $\gamma \in \Gamma$,*

1. $V_\Phi(\dot{x} + \dot{y}) \geq \min\{V_\Phi(\dot{x}), V_\Phi(\dot{y})\}$
2. $V_\Phi(\dot{x}\gamma\dot{y}) \geq V_\Phi(\dot{y})$ (resp., $V_\Phi(\dot{x}\gamma\dot{y}) \geq V_\Phi(\dot{x})$).

If Φ is both a left and a right VI of R, then Φ is called a VI of R.

Definition 2.12. *A VGSR $\Phi = (t_\Phi, f_\Phi)$ of a GSR R is said to be a vague bi-ideal of R if for all $\dot{x}, \dot{y}, \dot{z} \in R$ and $\alpha, \beta \in \Gamma$, $V_\Phi(\dot{x}\alpha\dot{y}\beta\dot{z}) \geq \min\{V_\Phi(\dot{x}), V_\Phi(\dot{z})\}$.*

Definition 2.13. *A vague set $\Phi = (t_\Phi, f_\Phi)$ of a GSR R is said to be a vague quasi-ideal of R if for all $\dot{x}, \dot{y} \in R$,*

1. $V_\Phi(\dot{x} + \dot{y}) \geq \min\{V_\Phi(\dot{x}), V_\Phi(\dot{y})\}$
2. $(\Phi\Gamma\delta) \cap (\delta\Gamma\Phi) \subseteq \Phi$, *where δ is a vague characteristic set of R.*

Definition 2.14. *A vague set $\Phi = (t_\Phi, f_\Phi)$ of a GSR R is said to be a vague interior ideal of R if for all $\dot{x}, \dot{y}, \dot{z} \in R$ and $\alpha, \beta \in \Gamma$,*

1. $V_\Phi(\dot{x} + \dot{y}) \geq \min\{V_\Phi(\dot{x}), V_\Phi(\dot{y})\}$
2. $V_\Phi(\dot{x}\alpha\dot{y}\beta\dot{z}) \geq V_\Phi(\dot{y})$.

3. Vague Bi-Quasi-Interior Ideals of Γ-Semirings

In this section, we introduce and study *VBQII* as a generalization of vague bi-ideals, vague quasi-ideals, and vague interior ideals of *GSR*s and characterize the vague bi-quasi-interior ideals of *GSR*s to crisp bi-quasi-interior ideals of *GSR*s.

Throughout this section, $\dot{R}$ stands for a *GSR* with unity, and δ stands for the vague characteristic set of $\dot{R}$ unless otherwise mentioned.

Definition 3.1. *A VGSR* $\Phi = (t_\Phi, f_\Phi)$ *of* $\dot{R}$ *is called a VBQII if* $\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi \subseteq \Phi$.

Every *VBQII* of $\dot{R}$ need not be a vague bi-ideal, a vague quasi-ideal, a vague interior ideal, a bi-interior ideal, and a bi-quasi-ideals of $\dot{R}$ in general.

Example 3.1. *Let* $\dot{R}$ *be the set of all negative integers and* Γ *be the set of all negative even integers. Then* $\dot{R}$ *and* Γ *are additive commutative semigroups. Define the mapping* $\dot{R}\times\Gamma\times\dot{R}\to\dot{R}$ *by* $\dot{x}\alpha\dot{y}$ *usual product of* $\dot{x},\alpha,\dot{y},\forall\dot{x},\dot{y}\in\dot{R};\alpha\in\Gamma$. *Then* $\dot{R}$ *is a GSR. Let* $\Phi = (t_\Phi, f_\Phi)$, *where* $t_\Phi:\dot{R}\to[0,1]$ *and* $f_\Phi:\dot{R}\to[0,1]$ *defined by*

$$t_\Phi(\dot{x}) = \begin{cases} 0.5 & if\ \dot{x}=-1 \\ 0.6 & if\ \dot{x}=-2 \\ 0.8 & if\ \dot{x}<-2 \end{cases} \text{ and } f_\Phi(\dot{x}) = \begin{cases} 0.5 & if\ \dot{x}=-1 \\ 0.15 & if\ \dot{x}=-2 \\ 0.1 & if\ \dot{x}<-2. \end{cases}$$

Then Φ *is a VBQII of* $\dot{R}$.

Theorem 3.1. *Every vague bi-ideal of* $\dot{R}$ *is a VBQII of* $\dot{R}$.

Proof. Let $\Phi = (t_\Phi, f_\Phi)$ be a vague bi-ideal of $\dot{R}$. Then Φ is a *VGSR* of $\dot{R}$. Since Φ is a vague bi-ideal of $\dot{R}$, we have $\Phi\Gamma\delta\Gamma\Phi\subseteq\Phi$. Thus $\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi\subseteq\Phi\Gamma\delta\Gamma\Phi\subseteq\Phi$. Hence, Φ is a *VBQII* of $\dot{R}$. □

Theorem 3.2. *Every vague interior ideal of* $\dot{R}$ *is a VBQII of* $\dot{R}$.

Proof. Let $\Phi = (t_\Phi, f_\Phi)$ be a vague interior ideal of $\dot{R}$. Then Φ is a *VGSR* of $\dot{R}$. Since Φ is a vague interior ideal of $\dot{R}$, we have $\delta\Gamma\Phi\Gamma\delta\subseteq\Phi$. Thus $\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi\subseteq\delta\Gamma\Phi\Gamma\delta\subseteq\Phi$. Thus Φ is a *VBQII* of $\dot{R}$. □

Theorem 3.3. *Every left VI of* $\dot{R}$ *is a VBQII of* $\dot{R}$.

Proof. Let $\Phi = (t_\Phi, f_\Phi)$ be a left *VI* of $\dot{R}$. Let $\dot{x}\in\dot{R}$. Then

$$\begin{aligned} V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}) &= \sup\{\min\{V_{(\Phi\Gamma\delta\Gamma\Phi)}(p\alpha q), V_{\delta\Gamma A}(r)\}, \text{where } \dot{x}=\dot{p}\alpha\dot{q}\beta\dot{r}; \dot{p},\dot{q},\dot{r}\in R;\alpha,\beta\in\Gamma\} \\ &\leq \sup\{\min\{V_\Phi(\dot{p}),V_\Phi(\dot{q}),V_\Phi(\dot{r})\}\} \\ &\leq \sup\{V_\Phi(\dot{p}\alpha\dot{q}\beta\dot{r})\} \\ &= \sup\{V_\Phi(\dot{x})\} \\ &= V_\Phi(\dot{x}). \end{aligned}$$

This implies $\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi \subseteq \Phi$. Hence, Φ is a *VBQII* of $\dot{R}$. □

Theorem 3.4. *Every right VI of $\dot{R}$ is a VBQII of $\dot{R}$.*

Proof. The proof is similar to the above theorem. □

Corollary 3.1. *Every VI of $\dot{R}$ is a vague bi-interior ideal of $\dot{R}$.*

Theorem 3.5. *The intersection of a VBQII and a right VI of $\dot{R}$ is always a VBQII of $\dot{R}$.*

Proof. Let Φ be a *VBQII* and ξ be a right *VI* of $\dot{R}$. Let $C = \Phi \cap \xi$. Obviously, $C = \Phi \cap \xi$ is a *VGSR* of $\dot{R}$. Now, $C\Gamma\delta\Gamma C\Gamma\delta\Gamma C \subseteq \Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi \subseteq \Phi$ and $C\Gamma\delta\Gamma C\Gamma\delta\Gamma C \subseteq \xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi \subseteq \xi$. Therefore, $C\Gamma\delta\Gamma C\Gamma\delta\Gamma C \subseteq \Phi \cap \xi = C$. Hence, C is a *VBQII* of $\dot{R}$. □

Theorem 3.6. *The intersection of a vague bi-ideal and a vague interior ideal of $\dot{R}$ is always VBQII of $\dot{R}$.*

Proof. Let Φ be a vague bi-ideal and ξ be a vague interior ideal of $\dot{R}$. Let $C = \Phi \cap \xi$. Now, $C\Gamma\delta\Gamma C\Gamma\delta\Gamma C \subseteq \Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi \subseteq \Phi\Gamma\delta\Gamma\Phi \subseteq \Phi$ and $C\Gamma\delta\Gamma C\Gamma\delta\Gamma C \subseteq \xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi \subseteq \xi\Gamma\xi \subseteq \xi$. Therefore, $C\Gamma\delta\Gamma C\Gamma\delta\Gamma C \subseteq \Phi \cap \xi = C$. Hence, C is a *VBQII* of $\dot{R}$. □

Theorem 3.7. *A vague set $\Phi = (t_\Phi, f_\Phi)$ is a VBQII of $\dot{R}$ if and only if its vague cut $\Phi_{(\alpha,\beta)}$ is a bi-quasi-interior ideal of $\dot{R}$.*

Proof. Suppose $\Phi = (t_\Phi, f_\Phi)$ is a *VBQII* of $\dot{R}$. Let $\dot{x} \in \Phi_{(\alpha,\beta)}\Gamma R\Gamma\Phi_{(\alpha,\beta)}\Gamma R\Gamma\Phi_{(\alpha,\beta)}$. Then $\dot{x} = \dot{a}\gamma\dot{p}\eta\dot{b}\zeta\dot{q}\epsilon\dot{c}$, where $\dot{a}, \dot{b}, \dot{c} \in \Phi_{(\alpha,\beta)}$; $\dot{p}, \dot{q}, \dot{r} \in \dot{R}$; $\gamma, \eta, \zeta, \epsilon \in \Gamma$. Now,

$$\begin{aligned} V_\Phi(\dot{x}) &\geq V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}) \\ &= \sup\{\min\{V_{\Phi\Gamma\delta\Gamma\Phi}(\dot{a}\gamma\dot{p}\eta\dot{b}), V_{\delta\Gamma\Phi}(\dot{q}\epsilon\dot{c})\}\} \\ &= \sup\{\min\{V_\Phi(\dot{a}), V_\Phi(\dot{b}), V_\Phi(\dot{c})\}\} \\ &\geq [\alpha, \beta] \end{aligned}$$

$\dot{x} \in \Phi_{(\alpha,\beta)}$. Therefore, $\Phi_{(\alpha,\beta)}\Gamma\dot{R}\Gamma\Phi_{(\alpha,\beta)}\Gamma\dot{R}\Gamma\Phi_{(\alpha,\beta)} \subseteq \Phi$. Hence, $\Phi_{(\alpha,\beta)}$ is bi-quasi-interior ideal of $\dot{R}$.

Conversely, suppose $\Phi_{(\alpha,\beta)}$ is a bi-quasi-interior ideal of $\dot{R}$. Obviously, Φ is a *VGSR* of $\dot{R}$. Suppose if possible $\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi \nsubseteq \Phi$. This implies that there exists $\dot{x} \in \dot{R}$ such that $V_\Phi(\dot{x}) < V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x})$. Let $\alpha, \beta \in [0, 1]$ be such that

$$V_\Phi(\dot{x}) < [\alpha, \beta] < V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}). \tag{3.1}$$

Now, suppose for any $\dot{a}, \dot{b}, \dot{c}, \dot{p}, \dot{q}, \dot{r} \in \dot{R}$ and $\gamma, \zeta, \eta, \epsilon \in \Gamma$, $\dot{x} = \dot{a}\gamma\dot{p}\zeta\dot{b}\eta\dot{q}\epsilon\dot{c}$ such that $\dot{a}, \dot{b}, \dot{c} \notin \Phi_{(\alpha,\beta)}$. Thus $V_\Phi(\dot{a}) < [\alpha,\beta]$, $V_\Phi(\dot{b}) < [\alpha,\beta]$, $V_\Phi(\dot{c}) < [\alpha,\beta]$. Now,

$$\begin{aligned} V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}) &= \sup\{\min\{V_{\Phi\Gamma\delta\Gamma\Phi}(\dot{a}\gamma\dot{p}\zeta\dot{b}), V_{\delta\Gamma\Phi}(\dot{q}\epsilon\dot{c})\}\} \\ &= \sup\{\min\{V_\Phi(\dot{a}), V_\Phi(\dot{b}), V_\Phi(\dot{c})\}\} \\ &< [\alpha,\beta]. \end{aligned}$$

This implies $V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}) < [\alpha,\beta]$, which is a contradiction to (3.1). Therefore there exist $\dot{a}, \dot{b}, \dot{c}, \dot{p}, \dot{q}, \dot{r} \in \dot{R}$ and $\gamma, \zeta, \eta, \epsilon \in \Gamma$, $\dot{x} = \dot{a}\gamma\dot{p}\zeta\dot{b}\eta\dot{q}\epsilon\dot{c}$ such that $\dot{a}, \dot{b}, \dot{c} \in \Phi_{(\alpha,\beta)}$. Hence, $\dot{x} \in \Phi_{(\alpha,\beta)}\Gamma\dot{R}\Gamma\Phi_{(\alpha,\beta)}\Gamma\dot{R}\Gamma\Phi_{(\alpha,\beta)}$. Thus $\dot{x} \in \Phi_{(\alpha,\beta)}$. But from (3.1), $\dot{x} \notin \Phi_{(\alpha,\beta)}$, which is a contradiction to $\Phi_{(\alpha,\beta)}$ is a bi-quasi-interior ideal of $\dot{R}$. Thus $\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi \subseteq \Phi$. Hence, Φ is a *VBQII* of $\dot{R}$. □

Theorem 3.8. *Let $\dot{B}$ be a non-empty subset of $\dot{R}$ and $\delta_{\dot{B}} = (t_{\delta_{\dot{B}}}, f_{\delta_{\dot{B}}})$ be a vague characteristic set of $\dot{R}$. Then $\dot{B}$ is a bi-quasi-interior ideal of $\dot{R}$ if and only if $\delta_{\dot{B}}$ is VBQII of $\dot{R}$.*

Proof. Suppose $\dot{B}$ is a bi-quasi-interior ideal of $\dot{R}$. Obviously, $\delta_{\dot{B}}$ is a *VGSR* of $\dot{R}$. Since $\dot{B}$ is a bi-quasi-interior ideal of $\dot{R}$, we have $\dot{B}\Gamma\dot{R}\Gamma\dot{B}\Gamma\dot{R}\Gamma\dot{B} \subseteq \dot{B}$. Now, $\delta_{\dot{B}}\Gamma\delta\Gamma\delta_{\dot{B}}\Gamma\delta\Gamma\delta_{\dot{B}} = \delta_{\dot{B}\Gamma\dot{R}\Gamma\dot{B}\Gamma\dot{R}\Gamma\dot{B}} \subseteq \delta_{\dot{B}}$. Thus $\delta_{\dot{B}}$ is a *VBQII* of $\dot{R}$.

Conversely, suppose $\delta_{\dot{B}}$ is a *VBQII* of $\dot{R}$. Then clearly, $\dot{x} + \dot{y} \in \dot{B}$ for all $\dot{x}, \dot{y} \in \dot{B}$. Since $\delta_{\dot{B}}$ is a *VBQII* of $\dot{R}$, we have $\delta_{\dot{B}}\Gamma\delta\Gamma\delta_{\dot{B}}\Gamma\delta\Gamma\delta_{\dot{B}} \subseteq \delta_{\dot{B}}$. Thus $\delta_{\dot{B}\Gamma\dot{R}\Gamma\dot{B}\Gamma\dot{R}\Gamma\dot{B}} \subseteq \delta_{\dot{B}}$. Therefore, $\dot{B}\Gamma\dot{R}\Gamma\dot{B}\Gamma\dot{R}\Gamma\dot{B} \subseteq \dot{B}$. Hence, $\dot{B}$ is a bi-quasi-interior ideal of $\dot{R}$. □

Theorem 3.9. *If $\Phi = (t_\Phi, f_\Phi)$ and $\xi = (t_\xi, f_\xi)$ are vague bi-quasi-interior ideals of $\dot{R}$, then $\Phi \cap \xi$ is a VBQII of $\dot{R}$.*

Proof. Let Φ and ξ be vague bi-quasi-interior ideals of $\dot{R}$. Then $\Phi \cap \xi$ is a *VGSR* of $\dot{R}$. Let $\dot{x} \in \dot{R}$. Then

$$\begin{aligned} V_{\delta\Gamma(\Phi\cap\xi)}(\dot{x}) &= \sup\{\min\{V_\delta(\dot{y}), V_{\Phi\cap\xi}(\dot{z}), \dot{x} = \dot{y}\alpha\dot{z}, \text{where } \dot{y}, \dot{z} \in \dot{R}; \alpha \in \Gamma\}\} \\ &= \sup\{\min\{V_\delta(\dot{y}), \min\{V_\Phi(\dot{z}), V_\xi(\dot{z})\}\}\} \\ &= \sup\{\min\{\min\{V_\delta(\dot{y}), V_\Phi(\dot{z})\}, \min\{V_\delta(\dot{y}), V_\xi(\dot{z})\}\} \\ &= \min\{\sup\{\min\{V_\delta(\dot{y}), V_\Phi(\dot{z})\}\}, \sup\{\min\{V_\delta(\dot{y}), V_\xi(\dot{z})\}\}\} \\ &= \min\{V_{\delta\Gamma\Phi}(\dot{x}), V_{\delta\Gamma\xi}(\dot{x})\} \\ &= V_{(\delta\Gamma\Phi)\cap(\delta\Gamma\xi)}(\dot{x}). \end{aligned}$$

This implies $\delta\Gamma(\Phi\cap\xi)=(\delta\Gamma\Phi)\cap(\delta\Gamma\xi)$. Also,

$$
\begin{aligned}
&V_{(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)}(\dot{x})\\
&=\sup\{\min\{V_{\Phi\cap\xi}(\dot{p}),V_{\delta\Gamma(\Phi\cap\xi)}(\dot{q}),V_{\delta\Gamma(\Phi\cap\xi)}(\dot{r}),\dot{x}=\dot{p}\alpha\dot{q}\beta\dot{r},\text{where } \dot{p},\dot{q},\dot{r}\in\dot{R};\alpha,\beta\in\Gamma\}\}\\
&=\sup\{\min\{V_{\Phi\cap\xi}(\dot{p}),V_{(\delta\Gamma\Phi)\cap(\delta\Gamma\xi)}(\dot{q}),V_{(\delta\Gamma\Phi)\cap(\delta\Gamma\xi)}(\dot{r})\}\}\\
&=\sup\{\min\{\min\{V_{\Phi}(\dot{p}),V_{\xi}(\dot{p})\},\min\{V_{\delta\Gamma\Phi}(\dot{q}),V_{\delta\Gamma\xi}(\dot{q})\},\min\{V_{\delta\Gamma A}(\dot{r}),V_{\delta\Gamma\xi}(\dot{r})\}\}\}\\
&=\sup\{\min\{\min\{V_{\Phi}(\dot{p}),V_{\delta\Gamma\Phi}(\dot{q}),V_{\delta\Gamma\Phi}(\dot{r})\},\min\{V_{\xi}(\dot{p}),V_{\delta\Gamma\xi}(\dot{q}),V_{\delta\Gamma\xi}(\dot{r})\}\}\}\\
&=\min\{\sup\{\min\{V_{\Phi}(\dot{p}),V_{\delta\Gamma\Phi}(\dot{q}),V_{\delta\Gamma\Phi}(\dot{r})\},\sup\{\min\{V_{\xi}(\dot{p}),V_{\delta\Gamma\xi}(\dot{q}),V_{\delta\Gamma\xi}(\dot{r})\}\}\}\\
&=\min\{V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}),V_{\xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi}(\dot{x})\}\\
&=V_{(\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi)\cap(\xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi)}(\dot{x}).
\end{aligned}
$$

This implies $((\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)=(\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi)\cap(\xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi)$. Therefore, $((\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)=(\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi)\cap(\xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi)\subseteq\Phi\cap\xi$. Hence, $\Phi\cap\xi$ is a *VBQII* of $\dot{R}$. □

Theorem 3.10. *If $\Phi=(t_{\Phi},f_{\Phi})$ is a minimal left VI and $\xi=(t_{\xi},f_{\xi})$ is a minimal right VI of $\dot{R}$, then $C=\Phi\Gamma\xi$ is a minimal VBQII of $\dot{R}$.*

Proof. Suppose Φ is a minimal left *VI* and ξ is a minimal right *VI* of $\dot{R}$. Let $x\in R$. Then $V_{C\Gamma\delta\Gamma C\Gamma\delta\Gamma C}(\dot{x})=V_{(\Phi\Gamma\xi)\Gamma\delta\Gamma(\Phi\Gamma\xi)\Gamma\delta\Gamma(\Phi\Gamma\xi)}(\dot{x})\le V_{\Phi\Gamma\xi}(\dot{x})=V_C(\dot{x})$. This implies $C\Gamma\delta\Gamma C\Gamma\delta\Gamma C\subseteq C$. Hence, C is a *VBQII* of $\dot{R}$.

Let G be a *VBQII* of $\dot{R}$ such that $G\subseteq C$. Then $\delta\Gamma G\subseteq\delta\Gamma C=\delta\Gamma\Phi\Gamma\xi\subseteq\xi$. Similarly, we can prove $G\Gamma\delta\subseteq\Phi$. Since Φ and ξ are minimal, we have $\delta\Gamma G=\xi$ and $G\Gamma\delta=\Phi$. Also, $C=\Phi\Gamma\xi=G\Gamma\delta\Gamma\delta\Gamma G\subseteq G\Gamma\delta\Gamma G\Gamma\delta\Gamma G\subseteq G$. This implies $C=G$. Hence, C is a minimal *VBQII* of $\dot{R}$. □

Theorem 3.11. *The intersection of a VBQII and a VGSR of $\dot{R}$ is also a VBQII of $\dot{R}$.*

Proof. Let $\Phi=(t_{\Phi},f_{\Phi})$ be a *VBQII* and $\xi=(t_{\xi},f_{\xi})$ be a *VGSR* of $\dot{R}$. Let $\dot{x}\in\dot{R}$. Then

$$
\begin{aligned}
V_{\delta\Gamma(\Phi\cap\xi)\Gamma\delta}(\dot{x})&=\sup\{\min\{V_{\delta}(\dot{p}),V_{(\Phi\cap\xi)\Gamma\delta}(\dot{q}\beta\dot{r}),\dot{x}=\dot{p}\alpha\dot{q}\beta\dot{r},\text{where } \dot{p},\dot{q},\dot{r}\in\dot{R};\alpha,\beta\in\Gamma\}\}\\
&=\sup\{\min\{V_{\delta}(\dot{p}),\sup\{\min\{V_{\Phi\cap\xi}(\dot{q}),V_{\delta}(\dot{r})\}\}\}\}\\
&=\sup\{\min\{V_{\delta}(\dot{p}),\sup\{\min\{\min\{V_{\Phi}(\dot{q}),V_{\xi}(\dot{q})\},V_{\delta}(\dot{r})\}\}\}\}\\
&\le\sup\{\min\{V_{\delta}(\dot{p}),\sup\{\min\{V_{\Phi}(\dot{q}),V_{\delta}(\dot{r})\}\}\}\}\\
&=V_{\delta\Gamma\Phi\Gamma\delta}(\dot{x}).
\end{aligned}
$$

This implies $\delta\Gamma(\Phi\cap\xi)\Gamma\delta\subseteq\delta\Gamma\Phi\Gamma\delta$. Also,

$$
\begin{aligned}
&V_{(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)}(\dot{x})\\
&=\sup\{\min\{V_{\Phi\cap\xi}(\dot{p}),V_{\delta\Gamma(\Phi\cap\xi)}(\dot{q}),V_{\delta\Gamma(\Phi\cap\xi)}(\dot{r}),\dot{x}=\dot{p}\alpha\dot{q}\beta\dot{r},\text{where } \dot{p},\dot{q},\dot{r}\in\dot{R};\alpha,\beta\in\Gamma\}\}\\
&=\sup\{\min\{V_{\Phi\cap\xi}(\dot{p}),V_{(\delta\Gamma\Phi)\cap(\delta\Gamma\xi)}(\dot{q}),V_{(\delta\Gamma\Phi)\cap(\delta\Gamma\xi)}(\dot{r})\}\}
\end{aligned}
$$

$$= \sup\{\min\{\min\{V_\Phi(\dot{p}), V_\xi(\dot{p})\}, \min\{V_{\delta\Gamma\Phi}(\dot{q}), V_{\delta\Gamma\xi}(\dot{q})\}, \min\{V_{\delta\Gamma\Phi}(\dot{r}), V_{\delta\Gamma\xi}(\dot{r})\}\}\}$$

$$= \sup\{\min\{\min\{V_\Phi(\dot{p}), V_{\delta\Gamma\Phi}(\dot{q}), V_{\delta\Gamma\Phi}(\dot{r})\}, \min\{V_\xi(\dot{p}), V_{\delta\Gamma\xi}(\dot{q}), V_{\delta\Gamma\xi}(\dot{r})\}\}\}$$

$$= \min\{\sup\{\min\{V_\Phi(\dot{p}), V_{\delta\Gamma\Phi}(\dot{q}), V_{\delta\Gamma\Phi}(\dot{r})\}, \sup\{\min\{V_B(\dot{p}), V_{\delta\Gamma\xi}(\dot{q}), V_{\delta\Gamma\xi}(\dot{r})\}\}\}$$

$$= \min\{V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}), V_{\xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi}(\dot{x})\}$$

$$= V_{(\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi)\cap(\xi\Gamma\delta\Gamma\xi\Gamma\delta\Gamma\xi)}(\dot{x})$$

$$\leq V_{\Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi}(\dot{x}).$$

This implies $(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi) \subseteq \Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi$. Now, $(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi) \subseteq \Phi\Gamma\delta\Gamma\Phi\Gamma\delta\Gamma\Phi \subseteq \Phi$. Moreover,

$$V_{(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)\Gamma\delta\Gamma(\Phi\cap\xi)}(\dot{x})$$

$$= \sup\{\min\{V_{(\Phi\cap\xi)}(\dot{p}), V_{\Phi\cap\xi}(\dot{q}), V_{\Phi\cap\xi}(\dot{r})\}, \dot{x} = \dot{p}\alpha\dot{q}\beta\dot{r}, \text{where } \dot{p}, \dot{q}, \dot{r} \in \dot{R}; \alpha, \beta \in \Gamma\}\}$$

$$= \sup\{\min\{\min\{V_\Phi(\dot{p}), V_\xi(\dot{p})\}, \min\{V_\Phi(\dot{q}), V_\xi(\dot{q})\}, \{\min\{V_\Phi(\dot{r}), V_\xi(\dot{r})\}\}\}$$

$$\leq \sup\{\min\{V_\xi(\dot{p}), V_\xi(\dot{q}), V_\xi(\dot{r})\}\}$$

$$\leq \sup\{V_\xi(\dot{x})\}$$

$$= V_\xi(\dot{x}).$$

This implies $(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi) \subseteq \xi$. Therefore, $(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi)\Gamma\delta\Gamma(\Phi \cap \xi) \subseteq \Phi \cap \xi$. Hence, $\Phi \cap \xi$ is a *VBQII* of $\dot{R}$. □

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