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Characterizing $(\in, \in \vee q)$-Anti-Intuitionistic Fuzzy Soft UP (BCC)-Subalgebras of UP (BCC)-Algebras

Aiyared Iampan[1,*], **N. Rajesh**[2]

[1] *Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[2] *Department of Mathematics, Rajah Serfoji Government College (affiliated to Bharathidasan University), Thanjavur 613005, Tamilnadu, India*

[*] *Corresponding author:* aiyared.ia@up.ac.th

Abstract. In this paper, the concepts of $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft UP (BCC)-subalgebras and $(\in, \in)$-anti-intuitionistic fuzzy soft UP (BCC)-subalgebras of UP (BCC)-algebras are introduced and studied. The UP (BCC)-homomorphic image and inverse image are investigated in $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft UP (BCC)-subalgebras of UP (BCC)-algebras. Characterizations of $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft UP (BCC)-subalgebras of UP (BCC)-algebras are discussed.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [22]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft and rough sets, has been discussed in [1, 2, 6]. In 1996, Bhakat and Das [5] used the relation of "belongs to" and "quasi coincident with" between a fuzzy point and a fuzzy set to introduce the concepts of $(\in, \in \vee q)$-fuzzy subgroups and $(\in, \in \vee q)$-fuzzy subrings. In 1999, Molodstov [20] introduced the notion of soft sets as a new mathematical tool for dealing with uncertainties free from the difficulties that have troubled that usual theoretical approach. In 2001, Maji et al. [18] introduced the concept of fuzzy soft sets as a

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generalization of the standard soft sets. In 2001, Maji et al. [19] introduced the notion of intuitionistic fuzzy soft sets as a generalization of the standard fuzzy soft sets. Since then, many mathematicians have applied the concepts of $(\in, \in \vee q)$-fuzzy subgroups, $(\in, \in \vee q)$-fuzzy subrings, and fuzzy soft sets to other algebraic systems, see [3, 4, 7, 9, 12, 13, 17, 23]. Iampan [11] introduced a new algebraic structure called a UP-algebra. The notion of UP-algebras (see [11]) and the concept of BCC-algebras (see [16]) are the same concept, as shown by Jun et al. [15] in 2022. We shall refer to it as BCC rather than UP in this article out of respect for Komori, who initially described it in 1984.

In this paper, the concepts of $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras and $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebras of BCC-algebras are introduced and studied. The BCC-homomorphic image and inverse image are investigated in $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of BCC-algebras. Characterizations of $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of BCC-algebras are discussed.

2. Preliminaries

The concept of BCC-algebras (see [16]) can be redefined without the condition (2.6) as follows:

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *BCC-algebra* (see [10]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0) \tag{2.1}$$

$$(\forall x \in X)(0 \cdot x = x) \tag{2.2}$$

$$(\forall x \in X)(x \cdot 0 = 0) \tag{2.3}$$

$$(\forall x, y \in X)(x \cdot y = 0 = y \cdot x \Rightarrow x = y) \tag{2.4}$$

After this, we assign X instead of a BCC-algebra $(X, \cdot, 0)$ until otherwise specified.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{2.5}$$

In X, the following assertions are valid (see [11]).

$$(\forall x \in X)(x \leq x) \tag{2.6}$$

$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{2.7}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \cdot x \leq z \cdot y) \tag{2.8}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \cdot z \leq x \cdot z) \tag{2.9}$$

$$(\forall x, y, z \in X)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)) \tag{2.10}$$

$$(\forall x, y \in X)(y \cdot x \leq x \Leftrightarrow x = y \cdot x) \tag{2.11}$$

$$(\forall x, y \in X)(x \leq y \cdot y) \tag{2.12}$$

$$(\forall a, x, y, z \in X)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))) \tag{2.13}$$

$$(\forall a, x, y, z \in X)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z) \tag{2.14}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot z) \tag{2.15}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \cdot y) \tag{2.16}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)) \tag{2.17}$$

$$(\forall a, x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)) \tag{2.18}$$

Definition 2.1. *[11] A nonempty subset S of X is called a BCC-subalgebra of X if*

$$(\forall x, y \in S)(x \cdot y \in S). \tag{2.19}$$

Definition 2.2. *[21] A nonempty subset S of X is called a BCC-filter of X if*

$$0 \in S, \tag{2.20}$$

$$(\forall x, y \in X)(x \cdot y, x \in S \Rightarrow y \in S). \tag{2.21}$$

Definition 2.3. *[11] A nonempty subset S of X is called a BCC-ideal of X if* (2.20) *and*

$$(\forall x, y, z \in X)(x \cdot (y \cdot z), y \in S \Rightarrow x \cdot z \in S). \tag{2.22}$$

Definition 2.4. *[8] A nonempty subset S of X is called a strong BCC-ideal of X if* (2.20) *and*

$$(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x), y \in S \Rightarrow x \in S). \tag{2.23}$$

3. $(\in, \in \vee q)$-Anti-Intuitionistic Fuzzy Soft BCC-Subalgebras

Generalizations of $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of BCC-algebras are defined, and some important properties are presented in this section.

A *fuzzy set* [22] in a nonempty set X is defined to be a function $\lambda : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 3.1. *[14] A fuzzy set λ in a nonempty set X of the form*

$$\lambda(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support $x \in X$ and value $t \in (0, 1]$ and is denoted by x_t.

From now on, we shall let E be a set of parameters. Let $\mathrm{F}(X)$ be the set of all fuzzy sets in X. A subset A of E is called a set of statistics.

Definition 3.2. *Let $A \subseteq E$. A pair (σ_M, A) is called a fuzzy soft set over a nonempty set X if σ_M is a mapping given by $\sigma_M : A \to \mathrm{F}(X), \delta \mapsto \sigma_{M[\delta]}$, that is, a fuzzy soft set is a statistic family of fuzzy* *sets in X. In general, for every $\delta \in A, \sigma_{M[\delta]}$ is a fuzzy set in X, and it is called a fuzzy value set of a statistic δ.*

Definition 3.3. *A fuzzy soft set σ_M over a nonempty set X of the form $\sigma_{M[\delta]} = x_t$, is called a fuzzy soft point.*

Definition 3.4. *A fuzzy point x_t in a nonempty set X is said to belong to a fuzzy set $\sigma_{M[\delta]}$, written as $x_t \in \sigma_{M[\delta]}$ if $\sigma_{M[\delta]}(x) \geq t$, and be quasi-coincident with a fuzzy set $\sigma_{M[\delta]}$, written as $x_t q \sigma_{M[\delta]}$ if $\sigma_{M[\delta]}(x) + t > 1$. If $x_t \in \sigma_{M[\delta]}$ or $x_t q \sigma_{M[\delta]}$, then $x_t \in \vee q \sigma_{M[\delta]}$.*

Definition 3.5. *A fuzzy point x_t in X is said to belong to an intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$, written as $x_t \in M[\delta]$ if $\sigma_{M[\delta]}(x) \geq t$ and $\tau_{M[\delta]}(x) \leq t$, and and be quasi-coincident with an intuitionistic fuzzy soft set $M[\delta]$, written as $x_t q M[\delta]$ if $\sigma_{M[\delta]}(x) + t > 1$ and $\tau_{M[\delta]}(x) + t < 1$. If $x_t \in M[\delta]$ or $x_t q M[\delta]$, then $x_t \in \vee q M[\delta]$.*

Definition 3.6. *An intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ over X is said to be an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X if it satisfies the following conditions:*

$$(\forall x, y \in X, \delta \in A)(\sigma_{M[\delta]}(x) \leq t, \sigma_{M[\delta]}(y) \leq s \Rightarrow \sigma_{M[\delta]}(x \cdot y) \leq t \vee s \text{ or } \sigma_{M[\delta]}(x \cdot y) + t \vee s < 1), \tag{3.1}$$

$$(\forall x, y \in X, \delta \in A)(\tau_{M[\delta]}(x) \geq t, \tau_{M[\delta]}(y) \geq s \Rightarrow \tau_{M[\delta]}(x \cdot y) \geq t \wedge s \text{ or } \tau_{M[\delta]}(x \cdot y) + t \wedge s > 1), \tag{3.2}$$

where $t \vee s = \max\{t, s\}$ and $t \wedge s = \min\{t, s\}$.

Theorem 3.1. *An intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ over X is an anti-intuitionistic fuzzy soft BCC-subalgebra of X if and only if $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.*

Proof. Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an anti-intuitionistic fuzzy soft BCC-subalgebra of X. Then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y)$ and $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y)$ for all $x, y \in X$. Let $x, y \in X$ and $\delta \in A$ be such that $\sigma_{M[\delta]}(x) \leq t$, $\sigma_{M[\delta]}(y) \leq s$ and $\tau_{M[\delta]}(x) \geq t$, $\tau_{M[\delta]}(y) \geq s$. Now, $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \leq t \vee s \Rightarrow (x \cdot y)_{t \vee s} \in \sigma_{M[\delta]}$ and $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \geq t \wedge s \Rightarrow (x \cdot y)_{t \wedge s} \in \tau_{M[\delta]}$. Therefore, $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.

Conversely, let $M[\delta]$ be an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Let $x, y \in X$ and $t = \sigma_{M[\delta]}(x), s = \sigma_{M[\delta]}(y)$. Then $\sigma_{M[\delta]}(x) \leq t$, $\sigma_{M[\delta]}(y) \leq s \Rightarrow x_t \in \sigma_{M[\delta]}, y_s \in \sigma_{M[\delta]} \Rightarrow$

$(x \cdot y)_{t\vee s} \in \sigma_{M[\delta]}$, since $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Then $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s$ gives

$$\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y). \tag{3.3}$$

Again, let $x, y \in X$ and $t = \tau_{M[\delta]}(x), s = \tau_{M[\delta]}(y)$. Then $\tau_{M[\delta]}(x) \geq t, \tau_{M[\delta]}(y) \geq s \Rightarrow x_t \in \tau_{M[\delta]}, y_s \in \tau_{M[\delta]} \Rightarrow (x \cdot y)_{t\wedge s} \in \tau_{M[\delta]}$, since $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Then $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s$ gives

$$\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y). \tag{3.4}$$

Hence, by (3.3) and (3.4), $M[\delta]$ is an anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.2. *If $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ is a (q, q)-anti-intuitionistic fuzzy soft BCC-subalgebra of X, then it is also an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.*

Proof. Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be a (q, q)-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Let $x, y \in X$ and $\delta \in A$ be such that $\sigma_{M[\delta]}(x) \leq t$ and $\sigma_{M[\delta]}(y) \leq s$. So $\sigma_{M[\delta]}(x) - \varepsilon < t$ and $\sigma_{M[\delta]}(y) - \varepsilon < s$, where ε is an arbitrary small positive number. Then $\sigma_{M[\delta]}(x) + 1 - \varepsilon - t < 1$ and $\sigma_{M[\delta]}(y) + 1 - \varepsilon - s < 1$ gives $x_{1-\varepsilon-t} q \sigma_{M[\delta]}$ and $y_{1-\varepsilon-s} q \sigma_{M[\delta]}$. Since $M[\delta]$ is a (q, q)-anti-intuitionistic fuzzy soft BCC-subalgebra of X, we have

$$\begin{aligned}
(x \cdot y)_{(1-\varepsilon-t)\vee(1-\varepsilon-s)} q \sigma_{M[\delta]} &\Rightarrow \sigma_{M[\delta]}(x \cdot y) + (1 - \varepsilon - t) \vee (1 - \varepsilon - s) < 1 \\
&\Rightarrow \sigma_{M[\delta]}(x \cdot y) + 1 - \varepsilon - (t \wedge s) < 1 \\
&\Rightarrow \sigma_{M[\delta]}(x \cdot y) < (t \wedge s) + \varepsilon \\
&\Rightarrow \sigma_{M[\delta]}(x \cdot y) < t \wedge s \\
&\Rightarrow \sigma_{M[\delta]}(x \cdot y) < t \wedge s < t \vee s \\
&\Rightarrow (x \cdot y)_{t\vee s} \in \sigma_{M[\delta]}.
\end{aligned}$$

Again, let $x, y \in X$ and $\delta \in A$ be such that $\tau_{M[\delta]}(x) \geq t, \tau_{M[\delta]}(y) \geq s$. Then $\tau_{M[\delta]}(x) + \varepsilon > t$ and $\tau_{M[\delta]}(y) + \varepsilon > s$, where ε is an arbitrary small positive number. Hence, $\tau_{M[\delta]}(x) + \varepsilon - t + 1 > 1$ and $\tau_{M[\delta]}(y) + \varepsilon - s + 1 > 1 \Rightarrow x_{\varepsilon-t+1} q \tau_{M[\delta]}$ and $y_{\varepsilon-s+1} q \tau_{M[\delta]}$. Since $M[\delta]$ is an (q, q)-anti-intuitionistic fuzzy soft BCC-subalgebra of X, we have

$$\begin{aligned}
(x \cdot y)_{(\varepsilon-t+1)\wedge(\varepsilon-s+1)} q \tau_{M[\delta]} &\Rightarrow \tau_{M[\delta]}(x \cdot y) + (\varepsilon - t + 1) \wedge (\varepsilon - s + 1) < 1 \\
&\Rightarrow \tau_{M[\delta]}(x \cdot y) + \varepsilon + 1 - (t \vee s) > 1 \\
&\Rightarrow \tau_{M[\delta]}(x \cdot y) > (t \vee s) - \varepsilon \\
&\Rightarrow \tau_{M[\delta]}(x \cdot y) > t \vee s \\
&\Rightarrow \tau_{M[\delta]}(x \cdot y) > t \vee s > t \wedge s \\
&\Rightarrow (x \cdot y)_{t\wedge s} \in \tau_{M[\delta]}.
\end{aligned}$$

Hence, $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.3. *An intuitionistic fuzzy soft set* $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ *over* X *is an* $(\in, \in \vee q)$*-anti-intuitionistic fuzzy soft BCC-subalgebra of* X *if and only if for all* $x, y \in X$ *and* $\delta \in A$,

(1) $\sigma_{M[\delta]}(x \cdot y) \leq \max\{\sigma_{M[\delta]}(x), \sigma_{M[\delta]}(y), 0.5\}$ *or* $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5$,

(2) $\tau_{M[\delta]}(x \cdot y) \geq \min\{\tau_{M[\delta]}(x), \tau_{M[\delta]}(y), 0.5\}$ *or* $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5$.

Proof. (1) First, let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.

Case 1: Let $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) > 0.5$ for all $x, y \in X$ and $\delta \in A$. Then $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 = \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y)$. If possible, let $\sigma_{M[\delta]}(x \cdot y) > \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y)$. Choose a real number t such that $\sigma_{M[\delta]}(x \cdot y) > t > \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \Rightarrow \sigma_{M[\delta]}(x) < t, \sigma_{M[\delta]}(y) < t$. But $\sigma_{M[\delta]}(x \cdot y) > t$ and $\sigma_{M[\delta]}(x \cdot y) + t > 2t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t > 2(\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y)) > 2 \times 0.5 = 1 \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t > 1$, which contradicts the fact that $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Therefore, $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) = \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5$.

Case 2: Let $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5$ for all $x, y \in X$ and $\delta \in A$. Then $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) = 0.5$. If possible, $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 = 0.5$. Then $\sigma_{M[\delta]}(x) \leq 0.5$ and $\sigma_{M[\delta]}(y) \leq 0.5$. But $\sigma_{M[\delta]}(x) > 0.5$ and $\sigma_{M[\delta]}(x) + 0.5 > 0.5 + 0.5 = 1$, which again contradicts that $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Hence, $\sigma_{M[\delta]}(x \cdot y) \leq 0.5 = \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5$.

(2) First, let $M[\delta]$ be an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.

Case 1: Let $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) < 0.5$ for all $x, y \in X$ and $\delta \in A$. Then $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5 = \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y)$. If possible, let $\tau_{M[\delta]}(x \cdot y) < \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y)$. Choose a real number t such that $\tau_{M[\delta]}(x \cdot y) < t < \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \Rightarrow \tau_{M[\delta]}(x) > t, \tau_{M[\delta]}(y) > t$. But $\tau_{M[\delta]}(x \cdot y) < t$ and $\tau_{M[\delta]}(x) + t = 2t \Rightarrow \tau_{M[\delta]}(x \cdot y) + t < 2(\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y)) < 2 \times 0.5 = 1 \Rightarrow \tau_{M[\delta]}(x \cdot y) + t < 1$, which contradicts the fact that $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Therefore, $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) = \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5$.

Case 2: Let $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5$ for all $x, y \in X$ and $\delta \in A$. Then $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) = 0.5$. If possible, $\tau_{M[\delta]}(x \cdot y) < \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5 = 0.5$. Then $\tau_{M[\delta]}(x) \geq 0.5$ and $\tau_{M[\delta]}(y) \geq 0.5$. But $\tau_{M[\delta]}(x \cdot y) < 0.5$ and $\tau_{M[\delta]}(x \cdot y) + 0.5 < 0.5 + 0.5 = 1$, which again contradicts that $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Hence, $\tau_{M[\delta]}(x \cdot y) \geq 0.5 = \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5$.

Conversely, let

$$\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5, \forall x, y \in X, \delta \in A. \tag{3.5}$$

Let $x, y \in X$ and $\delta \in A$ be such that $\sigma_{M[\delta]}(x) \leq t$ and $\sigma_{M[\delta]}(y) \leq s$. Then $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \leq t \vee s$. By (3.5), we have $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s \vee 0.5$. Now, if $t \vee s \geq 0.5$, then $t \vee s \vee 0.5 = t \vee s$. Therefore,

$$\sigma_{M[\delta]}(x \cdot y) \leq t \vee s. \tag{3.6}$$

Again, if $t \vee s > 0.5$, then $t \vee s \vee 0.5 = 0.5$. Therefore, $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s \vee 0.5 = 0.5 \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t \vee s < 0.5 + 0.5 = 1$. Then

$$\sigma_{M[\delta]}(x \cdot y) + t \vee s < 1. \tag{3.7}$$

From (3.6) and (3.7), we have (3.1). Let

$$\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5, \forall x, y \in X, \delta \in A. \tag{3.8}$$

Let $x, y \in X$ and $\delta \in A$ be such that $\tau_{M[\delta]}(x) \geq t$ and $\tau_{M[\delta]}(y) \geq s$. Therefore, $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \geq t \wedge s$. By (3.8), we have $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s \vee 0.5$. Now, if $t \wedge s \leq 0.5$, then $t \wedge s \wedge 0.5 = t \wedge s$. Therefore,

$$\tau_{M[\delta]}(x \cdot y) \geq t \wedge s. \tag{3.9}$$

Again, if $t \wedge s > 0.5$, then $t \wedge s \wedge 0.5 = 0.5$. Therefore, $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s \wedge 0.5 = 0$, that is, $\tau_{M[\delta]}(x \cdot y) + t \wedge s > 0.5 + 0.5 = 1$. Then

$$\tau_{M[\delta]}(x \cdot y) + t \wedge s > 1. \tag{3.10}$$

From (3.9) and (3.10), we have (3.2). Hence, $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.4. *An intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ over X is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X and if $\sigma_{M[\delta]}(x) > 0.5$, $\tau_{M[\delta]}(x) < 0.5$ for all $x, y \in X$ and $\delta \in A$, then $M[\delta]$ is also an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.*

Proof. Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X and $\sigma_{M[\delta]}(x) > 0.5$ and $\tau_{M[\delta]}(x) < 0.5$ for all $x, y \in X$ and $\delta \in A$. Let $x, y \in X$ and $\delta \in A$ be such that $0.5 < \sigma_{M[\delta]}(x) \leq t$ and $0.5 < \sigma_{M[\delta]}(y) \leq s$. Therefore, $t \vee s > 0.5$ and also $\sigma_{M[\delta]}(x \cdot y) > 0.5$. Thus $\sigma_{M[\delta]}(x \cdot y) + t \vee s > 0.5 + 0.5 = 1$. Hence, $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s$ or $\sigma_{M[\delta]}(x \cdot y) + t \vee s < 1$. Therefore,

$$\sigma_{M[\delta]}(x \cdot y) \leq t \vee s. \tag{3.11}$$

Again, let $x, y \in X$ and $\delta \in A$ be such that $t \leq \tau_{M[\delta]}(x) < 0.5$ and $s \leq \tau_{M[\delta]}(y) < 0.5$. Therefore, $t \wedge s < 0.5$ and also $\tau_{M[\delta]}(x \cdot y) < 0.5$. Thus $\tau_{M[\delta]}(x \cdot y) + t \wedge s < 0.5 + 0.5 = 1$. Hence, $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s$ or $\tau_{M[\delta]}(x \cdot y) + t \wedge s > 1$. Therefore,

$$\tau_{M[\delta]}(x \cdot y) \geq t \wedge s. \tag{3.12}$$

Hence, by (3.11) and (3.12), we have $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.5. *An intuitionistic fuzzy soft set* $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ *over* X *is an* $(\in, \in \vee q)$*-anti-intuitionistic fuzzy soft BCC-subalgebra of* X *if and only if the sets* $(\sigma_{M[\delta]})_t = \{x \in X \mid \sigma_{M[\delta]}(x) < t, t \in (0.5, 1], \sigma_{M[\delta]}(0) < t\}$ *and* $(\tau_{M[\delta]})_s = \{x \in X \mid \tau_{M[\delta]}(x) \geq s, s \in (0, 0.5), \tau_{M[\delta]}(0) \geq s\}$ *are BCC-subalgebras of* X.

Proof. Assume $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Clearly, $0 \in (\sigma_{M[\delta]})_t$ and $0 \in (\tau_{M[\delta]})_s$ since $\sigma_{M[\delta]}(0) \leq t$ and $\tau_{M[\delta]}(0) \geq s$. Let $x, y \in X$ and $\delta \in A$ be such that $x, y \in (\sigma_{M[\delta]})_t$, where $t \in (0.5, 1]$. Then $\sigma_{M[\delta]}(x) < t$ and $\sigma_{M[\delta]}(y) < t$. Then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 < t \vee s \vee 0.5 = t \Rightarrow \sigma_{M[\delta]}(x \cdot y) < t \Rightarrow x \cdot y \in (\sigma_{M[\delta]})_t$. Hence, $(\sigma_{M[\delta]})_t$ is a BCC-subalgebra of X. Again, let $x, y \in X$ and $\delta \in A$ be such that $x, y \in (\tau_{M[\delta]})_s$, where $s \in (0, 0.5)$. Then $\tau_{M[\delta]}(x) \geq s$ and $\tau_{M[\delta]}(y) \geq s$. Then $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5 > s \wedge s \wedge 0.5 = s \Rightarrow \tau_{M[\delta]}(x \cdot y) \geq s \Rightarrow x \cdot y \in (\tau_{M[\delta]})_s$. Hence, $(\tau_{M[\delta]})_s$ is a BCC-subalgebra of X.

Conversely, let $(\sigma_{M[\delta]})_t = \{x \in X \mid \sigma_{M[\delta]}(x) < t, t \in (0.5, 1], \sigma_{M[\delta]}(0) < t\}$ and $(\tau_{M[\delta]})_s = \{x \in X \mid \tau_{M[\delta]}(x) \geq s, s \in (0, 0.5), \tau_{M[\delta]}(0) \geq s\}$ are BCC-subalgebras of X. Suppose then there exist $a, b \in X$ such that at least one of $\sigma_{M[\delta]}(a \cdot b) > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee 0.5$ and $\tau_{M[\delta]}(a \cdot b) < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge 0.5$ hold. Suppose $\sigma_{M[\delta]}(a \cdot b) > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee 0.5$ holds. Let $t = \frac{1}{2}(\sigma_{M[\delta]}(a \cdot b) + (\sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee 0.5))$. Then $t \in (0.5, 1]$ and

$$\sigma_{M[\delta]}(a \cdot b) > t > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee 0.5. \tag{3.13}$$

Then $\sigma_{M[\delta]}(a) < t, \sigma_{M[\delta]}(b) < t \Rightarrow a \in (\sigma_{M[\delta]})_t, b \in (\sigma_{M[\delta]})_t \Rightarrow a \cdot b \in (\sigma_{M[\delta]})_t$ since $(\sigma_{M[\delta]})_t$ is a BCC-subalgebra of X. Therefore, $\sigma_{M[\delta]}(a \cdot b) < t$, which contradicts (3.13). Hence,

$$\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5. \tag{3.14}$$

Next, let $\tau_{M[\delta]}(a \cdot b) < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge 0.5$ holds. Let $s = \frac{1}{2}(\tau_{M[\delta]}(a \cdot b) + (\tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge 0.5))$. Then $s \in (0, 0.5)$ and

$$\tau_{M[\delta]}(a \cdot b) < s < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge 0.5. \tag{3.15}$$

Then $\tau_{M[\delta]}(a) > s, \tau_{M[\delta]}(b) > s \Rightarrow a \in (\tau_{M[\delta]})_s, b \in (\tau_{M[\delta]})_s \Rightarrow a \cdot b \in (\tau_{M[\delta]})_s$ since $(\tau_{M[\delta]})_s$ is a BCC-subalgebra of X. Therefore, $\tau_{M[\delta]}(a \cdot b) > s$, which contradicts (3.15). Hence,

$$\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5. \tag{3.16}$$

Hence, by (3.14) and (3.16), we have $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Definition 3.7. *Let* $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ *be an intuitionistic fuzzy soft set over a nonempty set* X *and* $t \in (0, 1]$. *Then let*

(1) $(\sigma_{M[\delta]})_t = \{x \in X \mid \sigma_{M[\delta]}(x) \leq t\}$,

(2) $\langle\sigma_{M[\delta]}\rangle_t = \{x \in X \mid \sigma_{M[\delta]}(x) + t < 1\}$,

(3) $[\sigma_{M[\delta]}]_t = \{x \in X \mid \sigma_{M[\delta]} \le t \text{ or } \sigma_{M[\delta]}(x) + t < 1\}$,

where $(\sigma_{M[\delta]})_t$ *is called a t-level set of* $\sigma_{M[\delta]}$, $\langle\sigma_{M[\delta]}\rangle_t$ *is called a q-level set of* $\sigma_{M[\delta]}$, *and* $[\sigma_{M[\delta]}]_t$ *is called an* $\in \vee q$*-level set of* $\sigma_{M[\delta]}$. *Clearly,* $[\sigma_{M[\delta]}]_t = \langle\sigma_{M[\delta]}\rangle_t \cup (\sigma_{M[\delta]})_t$. *Let*

(1) $(\tau_{M[\delta]})_t = \{x \in X \mid \tau_{M[\delta]}(x) \ge t\}$,

(2) $\langle\tau_{M[\delta]}\rangle_t = \{x \in X \mid \tau_{M[\delta]} + t > 1\}$,

(3) $[\tau_{M[\delta]}]_t = \{x \in X \mid \tau_{M[\delta]} \ge \tau \text{ or } \tau_{M[\delta]}(x) + t > 1\}$,

where $(\tau_{M[\delta]})_t$ *is called a t-level set of* $\tau_{M[\delta]}$, $\langle\tau_{M[\delta]}\rangle_t$ *is called a q-level set of* $\tau_{M[\delta]}$, *and* $[\tau_{M[\delta]}]_t$ *is called an* $\in \vee q$*-level set of* $\tau_{M[\delta]}$. *Clearly,* $[\tau_{M[\delta]}]_t = \langle\tau_{M[\delta]}\rangle_t \cup (\tau_{M[\delta]})_t$.

Theorem 3.6. *An intuitionistic fuzzy soft set* $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ *over* X *is an* $(\in, \in \vee q)$*-anti-intuitionistic fuzzy soft BCC-subalgebra of* X *if and only if* $[\sigma_{M[\delta]}]_t$ *and* $[\tau_{M[\delta]}]_t$ *are BCC-subalgebras of* X *for all* $t \in (0, 1]$. *We call* $[\sigma_{M[\delta]}]_t$ *and* $[\tau_{M[\delta]}]_t$ *as* $\in \vee q$*-level BCC-subalgebra of* X.

Proof. Assume that $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Let $x, y \in [\sigma_{M[\delta]}]_t$ for $t \in (0, 1]$. Then $\sigma_{M[\delta]}(x) \le t$ or $\sigma_{M[\delta]}(x) + t < 1$ and $\sigma_{M[\delta]}(y) \le t$ or $\sigma_{M[\delta]}(y) + t < 1$. Since $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X, we have $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5$ for any $x, y \in X$ and $\delta \in A$. Now we have the following cases.

Case 1: Let $\sigma_{M[\delta]}(x) \le t, \sigma_{M[\delta]}(y) \le t$, let $t < 0.5$. Then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 = t \vee t \vee 0.5 = 0.5$. Then $\sigma_{M[\delta]}(x \cdot y) \le 0.5 \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 0.5 + 0.5 = 1$. Again, if $t \ge 0.5$, then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 \le t \vee t \vee 0.5 = t$.

Case 2: Let $\sigma_{M[\delta]}(x) \le t, \sigma_{M[\delta]}(y) + t < 1$, let $t < 0.5$. Then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 < t \vee (1 - t) \vee 0.5 = 1 - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) < 1 - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 1$. Again, if $t \ge 0.5$, then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 \le t \vee (1 - t) \vee 0.5 = t \Rightarrow \sigma_{M[\delta]}(x \cdot y) \le t$.

Case 3: Let $\sigma_{M[\delta]}(x) + t < 1, \sigma_{M[\delta]}(y) \le t$, let $t < 0.5$. Then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 < (1 - t) \vee t \vee 0.5 = 1 - t, \Rightarrow \sigma_{M[\delta]}(x \cdot y) < 1 - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 1$. Again, if $t \ge 0.5$, then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 = (1 - t) \vee t \vee 0.5 = t \Rightarrow \sigma_{M[\delta]}(x \cdot y) \le t$.

Case 4: Let $\sigma_{M[\delta]}(x) + t < 1, \sigma_{M[\delta]}(y) + t < 1$, let $t < 0.5$. Then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 < (1 - t) \vee (1 - t) \vee 0.5 = 1 - t, \Rightarrow \sigma_{M[\delta]}(x \cdot y) < 1 - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 1$. Again, if $t \ge 0.5$, then $\sigma_{M[\delta]}(x \cdot y) \le \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5 = (1 - t) \vee (1 - t) \vee 0.5 = 0.5 \le t \Rightarrow \sigma_{M[\delta]}(x \cdot y) \le t$. Hence, from above four cases, $[\sigma_{M[\delta]}]_t$ is a BCC-subalgebra of X. Similarly, we can prove $[\tau_{M[\delta]}]_t$ is a BCC-subalgebra of X.

Conversely, let $[\sigma_{M[\delta]}]_t$ and $[\tau_{M[\delta]}]_t$ be BCC-subalgebras of X for all $t \in (0, 1]$. Suppose $M[\delta]$ is not an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Then there exist $a, b \in X$ such that at least one of $\sigma_{M[\delta]}(a \cdot b) > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(y) \vee 0.5$ and $\tau_{M[\delta]}(a \cdot b) < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge 0.5$

hold. Suppose $\sigma_{M[\delta]}(a \cdot b) > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee 0.5$ is true, then choose $t \in (0, 1]$ such that

$$\sigma_{M[\delta]}(a \cdot b) > t > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee 0.5. \tag{3.17}$$

Then $\sigma_{M[\delta]}(a) < t, \sigma_{M[\delta]}(b) < t \Rightarrow a, b \in (\sigma_{M[\delta]})_t \subseteq [\sigma_{M[\delta]}]_t$, which is a BCC-subalgebra of X. Therefore, $a \cdot b \in [\sigma_{M[\delta]}]_t \Rightarrow \sigma_{M[\delta]}(a \cdot b) \leq t$ or $\sigma_{M[\delta]}(a \cdot b) + t < 1$, which contradicts (3.17). Again, if $\tau_{M[\delta]}(a \cdot b) < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge 0.5$ is true, then choose $t \in (0, 1]$, such that

$$\tau_{M[\delta]}(a \cdot b) < t < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge 0.5. \tag{3.18}$$

Then $\tau_{M[\delta]}(a) > t,\ \tau_{M[\delta]}(b) > t \Rightarrow a, b \in (\tau_{M[\delta]})_t \subseteq [\tau_{M[\delta]}]_t$, which is a BCC-subalgebra of X. Therefore, $a \cdot b \in [\tau_{M[\delta]}]_t \Rightarrow \tau_{M[\delta]}(a \cdot b) \geq t$ or $\tau_{M[\delta]}(a \cdot b) + t > 1$ which contradicts (3.18). Hence, $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5$ and $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5$ for all $x, y \in X$ and $\delta \in A$. Hence, $M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Definition 3.8. *If $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$, and $N[\eta] = \{(x, \sigma_{N[\eta]}(x), \tau_{N[\eta]}(x)) \mid x \in X, \eta \in B\}$ are any two $(\in, \in \vee q)$-intuitionistic fuzzy soft sets over a nonempty set X, then*

$$M[\delta] \cap N[\eta] = \{(x, \sigma_{M[\delta] \cap N[\eta]}(x), \tau_{M[\delta] \cap N[\eta]}(x)) \mid x \in X, \delta, \eta \in A \cap B\},$$

where $\sigma_{M[\delta] \cap N[\eta]}(x) = \sigma_{M[\delta]}(x) \wedge \sigma_{N[\eta]}(x)$ and $\tau_{M[\delta] \cap N[\eta]}(x) = \tau_{M[\delta]}(x) \vee \tau_{N[\eta]}(x)$.

Theorem 3.7. *Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$, and $N[\eta] = \{(x, \sigma_{N[\eta]}(x), \tau_{N[\eta]}(x)) \mid x \in X, \eta \in B\}$ be two $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of X. Then*

$$M[\delta] \cap N[\eta] = \{(x, \sigma_{M[\delta] \cap N[\eta]}(x), \tau_{M[\delta] \cap N[\eta]}(x)) \mid x \in X, \delta, \eta \in A \cap B\}$$

is also an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.

Proof. Let $x, y \in X$ and $\delta, \eta \in A \cap B$. Then

$$\begin{aligned} \sigma_{M[\delta] \cap N[\eta]}(x \cdot y) &= \sigma_{M[\delta]}(x \cdot y) \vee \sigma_{N[\eta]}(x \cdot y) \\ &\leq (\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5) \vee (\sigma_{N[\eta]}(x) \vee \sigma_{N[\eta]}(y) \vee 0.5) \\ &= (\sigma_{M[\delta]}(x) \vee \sigma_{N[\eta]}(x)) \vee (\sigma_{M[\delta]}(y) \vee \sigma_{N[\eta]}(y)) \vee 0.5 \\ &\leq \sigma_{M[\delta] \cap N[\eta]}(x) \vee \sigma_{M[\delta] \cap N[\eta]}(y) \vee 0.5. \end{aligned}$$

and

$$\begin{aligned} \tau_{M[\delta] \cap N[\eta]}(x \cdot y) &= \tau_{M[\delta]}(x \cdot y) \wedge \tau_{N[\eta]}(x \cdot y) \\ &\geq (\tau_{M[\alpha]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5) \wedge (\tau_{N[\eta]}(x) \wedge \tau_{N[\eta]}(y) \wedge 0.5) \\ &= (\tau_{M[\alpha]}(x) \wedge \tau_{M[\delta]}(x)) \wedge (\tau_{N[\eta]}(y) \wedge \tau_{N[\eta]}(y)) \wedge 0.5 \\ &\geq \tau_{M[\delta] \cap N[\eta]}(x) \wedge \tau_{M[\delta] \cap N[\eta]}(y) \wedge 0.5. \end{aligned}$$

Hence, $M[\delta] \cap N[\eta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Definition 3.9. *Let $M[\delta_i] = \{(x, \sigma_{M[\delta_i]}, \tau_{M[\delta_i]}) \mid x \in X, \delta_i \in A_i\}$ be an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of X for all $i = 1, 2, \ldots$. Then $\bigcap_{i=1}^{n} M[\delta_i]$ is also an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X, where $\bigcap_{i=1}^{n} M[\delta_i](x) = \{(x, \bigvee_{i=1}^{n} \sigma_{M[\delta_i]}(x), \bigwedge_{i=1}^{n} \tau_{M[\delta_i]}(x)) \mid x \in X, \delta_i \in \bigcap_{i=1}^{n} A_i\}$.*

We define two intuitionistic fuzzy soft sets $\oplus M[\delta]$ and $\otimes M[\delta]$ induced by an intuitionistic fuzzy soft set $M[\delta]$ as follows:

Definition 3.10. *Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an intuitionistic fuzzy soft set over a nonempty set X. The two intuitionistic fuzzy soft sets $\oplus M[\delta]$ and $\otimes M[\delta]$ over X are defined as follows:*

(1) $\oplus M[\delta] = \{(x, \sigma_{M[\delta]}(x), \overline{\sigma}_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$,

(2) $\otimes M[\delta] = \{(x, \overline{\tau}_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$.

Theorem 3.8. *Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Then both $\oplus M[\delta]$ and $\otimes M[\delta]$ are $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of X.*

Proof. For $\oplus M[\delta]$, it is sufficient to show that $\overline{\sigma}_{M[\delta]}$ satisfies $\overline{\sigma}_{M[\delta]}(x \cdot y) \geq \overline{\sigma}_{M[\delta]}(x) \wedge \overline{\sigma}_{M[\delta]}(y) \wedge 0.5$ for all $x, y \in X$ and $\delta \in A$. Let $x, y \in X$ and $\delta \in A$. Then

$$\begin{aligned} \overline{\sigma}_{M[\delta]}(x \cdot y) &= 1 - \sigma_{M[\delta]}(x \cdot y) \\ &\geq 1 - (\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee 0.5) \\ &= (1 - \sigma_{M[\delta]}(x)) \wedge (1 - \sigma_{M[\delta]}(y)) \wedge (1 - 0.5) \\ &= \overline{\sigma}_{M[\delta]}(x) \wedge \overline{\sigma}_{M[\delta]}(y) \wedge 0.5. \end{aligned}$$

Hence, $\oplus M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.

For $\otimes M[\delta]$, it is sufficient to show that $\overline{\tau}_{M[\delta]}$ satisfies $\overline{\tau}_{M[\delta]}(x \cdot y) \leq \overline{\tau}_{M[\delta]}(x) \vee \overline{\tau}_{M[\delta]}(y) \vee 0.5$ for all $x, y \in X$ and $\delta \in A$. Let $x, y \in X$ and $\delta \in A$. Then

$$\begin{aligned} \overline{\tau}_{M[\delta]}(x \cdot y) &= 1 - \tau_{M[\delta]}(x \cdot y) \\ &\leq 1 - (\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge 0.5) \\ &= (1 - \tau_{M[\delta]}(x)) \vee (1 - \tau_{M[\delta]}(y)) \vee (1 - 0.5) \\ &= \overline{\tau}_{M[\delta]}(x) \vee \overline{\tau}_{M[\delta]}(y) \vee 0.5. \end{aligned}$$

Hence, $\otimes M[\delta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Definition 3.11. *Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$,
and $N[\eta] = \{(x, \sigma_{N[\eta]}(x), \tau_{N[\eta]}(x)) \mid x \in X, \eta \in B\}$ be two $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of X. Then their cartesian product $M[\delta] \times N[\eta]$ over $X \times X$ is defined by*

$$M[\delta] \times N[\eta] = \{((x, y), \sigma_{M[\delta] \times N[\eta]}(x, y), \tau_{M[\delta] \times N[\eta]}(x, y)) \mid (x, y) \in X \times X, (\delta, \eta) \in A \times B\},$$

where $\sigma_{M[\delta]\times N[\eta]}(x,y) = \sigma_{M[\delta]}(x) \vee \sigma_{N[\eta]}(y)$ *and* $\tau_{M[\delta]\times N[\eta]}(x,y) = \tau_{M[\delta]}(x) \wedge \tau_{N[\eta]}(y)$.

Theorem 3.9. *Let* $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$, *and* $N[\eta] = \{(x, \sigma_{N[\eta]}(x), \tau_{N[\eta]}(x)) \mid x \in X, \eta \in B\}$ *be two* $(\in, \in \vee q)$*-anti-intuitionistic fuzzy soft BCC-subalgebras of* X. *Then* $M[\delta] \times N[\eta]$ *is also an* $(\in, \in \vee q)$*-anti-intuitionistic fuzzy soft BCC-subalgebra of* $X \times X$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times X$ and $(\delta, \eta) \in A \times B$. Then

$$\begin{aligned}
\sigma_{M[\delta]\times N[\eta]}((x_1, y_1)\cdot(x_2, y_2)) &= \sigma_{M[\delta]\times N[\eta]}(x_1 \cdot x_2, y_1 \cdot y_2) \\
&= \sigma_{M[\delta]}(x_1 \cdot x_2) \vee \sigma_{M[\eta]}(y_1 \cdot y_2) \\
&\leq (\sigma_{M[\delta]}(x_1) \vee \sigma_{M[\delta]}(x_2) \vee 0.5) \vee (\sigma_{N[\eta]}(y_1) \vee \sigma_{N[\eta]}(y_2) \vee 0.5) \\
&= (\sigma_{M[\delta]}(x_1) \vee \sigma_{N[\eta]}(y_1)) \vee (\sigma_{M[\delta]}(x_2) \vee \sigma_{N[\eta]}(y_2)) \vee 0.5 \\
&\leq \sigma_{M[\delta]\times N[\eta]}(x_1, y_1) \vee \sigma_{M[\delta]\times N[\eta]}(x_2, y_2) \vee 0.5
\end{aligned}$$

and

$$\begin{aligned}
\tau_{M[\delta]\times N[\eta]}((x_1, y_1)\cdot(x_2, y_2)) &= \tau_{M[\delta]\times N[\eta]}(x_1 \cdot x_2, y_1 \cdot y_2) \\
&= \tau_{M[\delta]}(x_1 \cdot x_2) \wedge \tau_{M[\eta]}(y_1 \cdot y_2) \\
&\geq (\tau_{M[\delta]}(x_1) \wedge \tau_{M[\delta]}(x_2) \wedge 0.5) \wedge (\tau_{N[\eta]}(y_1) \wedge \tau_{N[\eta]}(y_2) \wedge 0.5) \\
&= (\tau_{M[\delta]}(x_1) \wedge \tau_{N[\eta]}(y_1)) \wedge (\tau_{M[\delta]}(x_2) \wedge \tau_{N[\eta]}(y_2)) \wedge 0.5 \\
&\geq \tau_{M[\delta]\times N[\eta]}(x_1, y_1) \wedge \tau_{M[\delta]\times N[\eta]}(x_2, y_2) \wedge 0.5.
\end{aligned}$$

Hence, $M[\delta] \times N[\eta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of $X \times X$. □

Example 3.1. *Let* $X = \{0, 1, 2, 3\}$ *with the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then X *is a BCC-algebra. We consider two* $(\in, \in \vee q)$*-intuitionistic fuzzy soft sets* $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ *and and* $N[\eta] = \{(x, \sigma_{N[\eta]}(x), \tau_{N[\eta]}(x)) \mid x \in X, \eta \in B\}$ *as follows:*

$M[\delta](x)$	0	1	2	3
$(\sigma_{M[\delta]}(x), \tau_{M[\delta]}(x))$	(0.2, 0.7)	(0.2, 0.7)	(0.5, 0.5)	(0.5, 0.5)

$N[\eta](x)$	0	1	2	3
$(\sigma_{N[\eta]}(x), \tau_{N[\eta]}(x))$	(0.3, 0.6)	(0.4, 0.5)	(0.3, 0.6)	(0.3, 0.6)

Then $M[\delta]$ and $N[\eta]$ are $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of X. Now,

$$\sigma_{M[\delta]} \times \sigma_{N[\eta]}(0,0) = 0.3, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(0,1) = 0.4, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(0,2) = 0.3,$$

$$\sigma_{M[\delta]} \times \sigma_{N[\eta]}(0,3) = 0.3, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(1,0) = 0.4, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(1,1) = 0.3,$$

$$\sigma_{M[\delta]} \times \sigma_{N[\eta]}(1,2) = 0.3, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(1,3) = 0.4, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(2,0) = 0.5,$$

$$\sigma_{M[\delta]} \times \sigma_{N[\eta]}(2,1) = 0.5, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(2,2) = 0.5, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(2,3) = 0.5,$$

$$\sigma_{M[\delta]} \times \sigma_{N[\eta]}(3,0) = 0.5, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(3,1) = 0.5, \sigma_{M[\delta]} \times \sigma_{N[\eta]}(3,2) = 0.5,$$

$$\sigma_{M[\delta]} \times \sigma_{N[\eta]}(3,3) = 0.5.$$

$$\tau_{M[\delta]} \times \tau_{N[\eta]}(0,0) = 0.6, \tau_{M[\delta]} \times \tau_{N[\eta]}(0,1) = 0.5, \tau_{M[\delta]} \times \tau_{N[\eta]}(0,2) = 0.6,$$

$$\tau_{M[\delta]} \times \tau_{N[\eta]}(0,3) = 0.6, \tau_{M[\delta]} \times \tau_{N[\eta]}(1,0) = 0.6, \tau_{M[\delta]} \times \tau_{N[\eta]}(1,1) = 0.5,$$

$$\tau_{M[\delta]} \times \tau_{N[\eta]}(1,2) = 0.6, \tau_{M[\delta]} \times \tau_{N[\eta]}(1,3) = 0.5, \tau_{M[\delta]} \times \tau_{N[\eta]}(2,0) = 0.5,$$

$$\tau_{M[\delta]} \times \tau_{N[\eta]}(2,1) = 0.5, \tau_{M[\delta]} \times \tau_{N[\eta]}(2,2) = 0.5, \tau_{M[\delta]} \times \tau_{N[\eta]}(2,3) = 0.5,$$

$$\tau_{M[\delta]} \times \tau_{N[\eta]}(3,0) = 0.5, \tau_{M[\delta]} \times \tau_{N[\eta]}(3,1) = 0.5, \tau_{M[\delta]} \times \tau_{N[\eta]}(3,2) = 0.5,$$

$$\tau_{M[\delta]} \times \tau_{N[\eta]}(3,3) = 0.5.$$

Hence, $M[\delta] \times N[\eta]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of $X \times X$.

Theorem 3.10. *Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ and $N[\eta] = \{(x, \sigma_{N[\eta]}(x), \tau_{N[\eta]}(x)) \mid x \in X, \eta \in B\}$ be two $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of X. Then $\oplus(M[\delta] \times N[\eta])$ and $\otimes(M[\delta] \times N[\eta])$ are also $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of $X \times X$.*

Proof. It's straightforward from Theorems 3.8 and 3.9. □

Let k denote an arbitrary element of $[0,1)$ unless otherwise specified. To say that $x_t q_k \sigma_{M[\delta]}$, we mean $\sigma_{M[\delta]}(x) + t > 1 - k$. To say that $x_t \in \vee q_k \sigma_{M[\delta]}$, we mean $x_t \in \sigma_{M[\delta]}$ or $\sigma_{M[\delta]} q_k \sigma_{M[\delta]}$.

Definition 3.12. *An intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ over X is said to be an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X if it satisfies the following conditions:*

$$(\forall x, y \in X, \delta \in A, t, s \in (0,1])(\sigma_{M[\delta]}(x) \leq t, \sigma_{M[\delta]}(y) \leq s \Rightarrow \sigma_{M[\delta]}(x \cdot y) \leq t \vee s \text{ or } \sigma_{M[\delta]}(x \cdot y) + t \vee s < 1 - k), \tag{3.19}$$

$$(\forall x, y \in X, \delta \in A, t, s \in (0,1])(\tau_{M[\delta]}(x) \geq t, \tau_{M[\delta]}(y) \geq s \Rightarrow \tau_{M[\delta]}(x \cdot y) \geq t \wedge s \text{ or } \tau_{M[\delta]}(x \cdot y) + t \wedge s > 1 - k). \tag{3.20}$$

Theorem 3.11. *An intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ over X is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X if and only if for all $x, y \in X$ and $\delta \in A$,*

(1) $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}$,

(2) $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2}$.

Proof. Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.

Case 1: Let $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) > \frac{k-1}{2}$ for all $x, y \in X$ and $\delta \in A$. Then $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2} = \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y)$. If possible, let $\sigma_{M[\delta]}(x \cdot y) > \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y)$. Choose a real number t such that $\sigma_{M[\delta]}(x \cdot y) > t > \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \Rightarrow \sigma_{M[\delta]}(x) < t, \sigma_{M[\delta]}(y) < t$ for some $t \in (0, 1)$. But $\sigma_{M[\delta]}(x \cdot y) > t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t > 2t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t > 2(\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y)) > 2 \times \frac{k-1}{2} = 1 - k \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 1 - k$, which contradicts the fact that $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Therefore, $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) = \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}$.

Case 2: Let $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \leq \frac{k-1}{2}$ for all $x, y \in X$ and $\delta \in A$. Then $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) = \frac{k-1}{2}$. If possible, let $\sigma_{M[\delta]}(x \cdot y) > \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2} = \frac{k-1}{2}$. Then $\sigma_{M[\delta]}(x) \leq \frac{k-1}{2}$ and $\sigma_{M[\delta]}(y) \leq \frac{k-1}{2}$. But $\sigma_{M[\delta]}(x \cdot y) > \frac{k-1}{2}$. Also $\sigma_{M[\delta]}(x \cdot y) + \frac{k-1}{2} > \frac{k-1}{2} + \frac{k-1}{2} = 1 - k$, which again a contradiction that $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Therefore, $\sigma_{M[\delta]}(x \cdot y) \leq \frac{k-1}{2} = \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}$.

Case 3: Let $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) < \frac{k-1}{2}$ for all $x, y \in X$ and $\delta \in A$. Then $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2} = \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y)$. If possible, let $\tau_{M[\delta]}(x \cdot y) < \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y)$. Choose a real number t such that $\tau_{M[\delta]}(x \cdot y) < t < \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \Rightarrow \tau_{M[\delta]}(x) > t, \tau_{M[\delta]}(y) > t$ for some $t \in (0, 1)$. But $\tau_{M[\delta]}(x \cdot y) < t \Rightarrow \tau_{M[\delta]}(x \cdot y) + t < 2t \Rightarrow \tau_{M[\delta]}(x \cdot y) + t < 2(\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y)) < 2 \times \frac{k-1}{2} = 1 - k \Rightarrow \tau_{M[\delta]}(x \cdot y) + t < 1 - k$, which contradicts the fact that $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Therefore, $\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) = \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2}$.

Case 4: Let $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \geq \frac{k-1}{2}$ for all $x, y \in X$ and $\delta \in A$. Then $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) = \frac{k-1}{2}$. If possible, let $\tau_{M[\delta]}(x \cdot y) < \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2} = \frac{k-1}{2}$. Then $\tau_{M[\delta]}(x) \geq \frac{k-1}{2}$ and $\tau_{M[\delta]}(y) \geq \frac{k-1}{2}$. But $\tau_{M[\delta]}(x \cdot y) + \frac{k-1}{2} < \frac{k-1}{2}2 + \frac{k-1}{2} = 1 - k$, which again a contradiction that $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Therefore, $\tau_{M[\delta]}(x \cdot y) \geq \frac{k-1}{2} = \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2}$.

Conversely, let for all $x, y \in X$ and $\delta \in A$,

$$\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}. \tag{3.21}$$

Let $x, y \in X, \delta \in A$ and $t, s \in (0, 1]$ be such that $\sigma_{M[\delta]}(x) \leq t$ and $\sigma_{M[\delta]}(y) \leq s$. Therefore, $\sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \leq t \vee s$. Then $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s \vee \frac{k-1}{2}$. Now, if $t \vee s \geq \frac{k-1}{2}$, then $t \vee s \vee \frac{k-1}{2} = t \vee s$. Therefore,

$$\sigma_{M[\delta]}(x \cdot y) \leq t \vee s. \tag{3.22}$$

Again, if $t \vee s > \frac{k-1}{2}$, then $t \vee s \vee \frac{k-1}{2} = \frac{k-1}{2}$. Therefore, $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s \vee \frac{k-1}{2} = \frac{k-1}{2} \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t \vee s < \frac{k-1}{2} + \frac{k-1}{2} = 1 - k$. Then

$$\sigma_{M[\delta]}(x \cdot y) + t \vee s < 1 - k. \tag{3.23}$$

Conversely, let for all $x, y \in X$ and $\delta \in A$,

$$\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2}. \tag{3.24}$$

Let $x, y \in X, \delta \in A$ and $t, s \in (0, 1]$ be such that $\tau_{M[\delta]}(x) \geq t$ and $\tau_{M[\delta]}(y) \geq s$. Therefore, $\tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \geq t \wedge s$. Then $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s \wedge \frac{k-1}{2}$. Now, if $t \wedge s \leq \frac{k-1}{2}$, then $t \wedge s \wedge \frac{k-1}{2} = t \wedge s$. Therefore,

$$\tau_{M[\delta]}(x \cdot y) \geq t \wedge s. \tag{3.25}$$

Again, if $t \wedge s > \frac{k-1}{2}$, then $t \wedge s \wedge \frac{k-1}{2} = \frac{k-1}{2}$. Therefore, $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s \wedge \frac{k-1}{2} = \frac{k-1}{2} \Rightarrow \tau_{M[\delta]}(x \cdot y) + t \wedge s > \frac{k-1}{2} + \frac{k-1}{2} = 1 - k$. Then

$$\tau_{M[\delta]}(x \cdot y) + t \wedge s > 1 - k. \tag{3.26}$$

Hence by (3.22), (3.23), (3.25), and (3.26), we have $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.12. *An intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ over X is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X if and only if the level subset $L(M[\delta], t) = \{x \in X \mid \sigma_{M[\delta]} \leq t\}$ and $U(M[\delta], t) = \{x \in X \mid \tau_{M[\delta]} \geq t\}$ are BCC-subalgebras of X for all $t \in (0, \frac{k-1}{2})$.*

Proof. Assume that $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Let $t \in (0, \frac{k-1}{2})$ and $x, y \in L(M[\delta], t)$. Then $\sigma_{M[\delta]}(x) \leq t$. It follows from (3.21) that $\sigma_{M[\delta]}(x \cdot y) \leq t \vee t \vee \frac{k-1}{2} = t$. Thus $x \cdot y \in L(M[\delta], t)$. Hence, $L(M[\delta], t)$ is a BCC-subalgebra of X. Let $t \in (0, \frac{k-1}{2})$ and $x, y \in U(M[\delta], t)$. Then $\tau_{M[\delta]}(x) \leq t$. It follows from (3.24) that $\tau_{M[\delta]}(x \cdot y) \geq t \wedge t \wedge \frac{k-1}{2} = t$ so that $x \cdot y \in U(M[\delta], t)$. Hence, $U(M[\delta], t)$ is a BCC-subalgebra of X.

Conversely, assume that $L(M[\delta], t)$ and $U(M[\delta], t)$ are BCC-subalgebras of X for all $t \in (0, \frac{k-1}{2})$. If (3.21) is not valid, then there exist $x, y \in X$ and $\delta \in A$ such that $\sigma_{M[\delta]}(x \cdot y) > \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}$. Hence, we can choose $t \in (0, 1)$ such that $\sigma_{M[\delta]}(x \cdot y) > t \geq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}$. Then $t \in (0, \frac{k-1}{2})$ and $x, y \in L(M[\delta], t)$. Since $L(M[\delta], t)$ is a BCC-subalgebra of X, it follows that $x \cdot y \in L(M[\delta], t)$ so that $\sigma_{M[\delta]}(x \cdot y) \leq t$. This is a contradiction. Therefore (3.21) is valid. If (3.24) is not valid, then there exist $x, y \in X$ and $\delta \in A$ such that $\tau_{M[\delta]}(x \cdot y) < \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2}$. Hence, we can choose $t \in (0, 1)$ such that $\tau_{M[\delta]}(x \cdot y) < t \leq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(y) \wedge \frac{k-1}{2}$. Then $t \in (0, \frac{k-1}{2})$ and $x, y \in U(M[\delta], t)$. Since $U(M[\delta], t)$ is a BCC-subalgebra of X, it follows that

$x \cdot y \in U(M[\delta], t)$ so that $\tau_{M[\delta]}(x \cdot y) \geq t$. This is a contradiction. Therefore (3.24) is valid. Hence, $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.13. *An anti-intuitionistic fuzzy soft set $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ over X is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X and if $\sigma_{M[\delta]}(x) > \frac{k-1}{2}$ and $\tau_{M[\delta]}(x) < \frac{k-1}{2}$ for all $x, y \in X$ and $\delta \in A$, then $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.*

Proof. Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X and $\sigma_{M[\delta]}(x) > \frac{k-1}{2}$ and $\tau_{M[\delta]}(x) < \frac{k-1}{2}$ for all $x, y \in X$ and $\delta \in A$. Let $\frac{k-1}{2} < \sigma_{M[\delta]}(x) \leq t$ and $\frac{k-1}{2} < \sigma_{M[\delta]}(y) \leq s$. Then $t \vee s > \frac{k-1}{2}$. Also $\sigma_{M[\delta]}(x \cdot y) > \frac{k-1}{2}$. Thus $\sigma_{M[\delta]}(x \cdot y) + t \vee s > \frac{k-1}{2} + \frac{k-1}{2} = 1 - k$. Since $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X, we have either $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s$ or $\sigma_{M[\delta]}(x \cdot y) + t \vee s < 1 - k$. So we must have $\sigma_{M[\delta]}(x \cdot y) \leq t \vee s$. Again, let $t \leq \tau_{M[\delta]}(x) < \frac{k-1}{2}$ and $s \leq \tau_{M[\delta]}(y) < \frac{k-1}{2}$. Then $t \wedge s < \frac{k-1}{2}$ and also $\tau_{M[\delta]}(x \cdot y) < \frac{k-1}{2}$. Thus $\tau_{M[\delta]}(x \cdot y) + t \wedge s < \frac{k-1}{2} + \frac{k-1}{2} = 1 - k$. Since $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X, we have either $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s$ or $\tau_{M[\delta]}(x \cdot y) + t \wedge s > 1 - k$. So we must have $\tau_{M[\delta]}(x \cdot y) \geq t \wedge s$. Hence, $M[\delta]$ is an $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.14. *Let $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ be an intuitionistic fuzzy soft set over X. Then $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X if and only if $[\sigma_{M[\delta]}]_t$ and $[\tau_{M[\delta]}]_t$ are BCC-subalgebras of X for all $t \in (0, 1]$. We call $[\sigma_{M[\delta]}]_t$ and $[\tau_{M[\delta]}]_t$ as $(\in, \in \vee q_k)$-level BCC-subalgebras of X.*

Proof. Assume that $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X, \delta \in A\}$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. Let $x, y \in [\sigma_{M[\delta]}]_t$ for $t \in (0, 1]$. Then $\sigma_{M[\delta]}(x) \leq t$ or $\sigma_{M[\delta]}(x) + t < 1$ and $\sigma_{M[\delta]}(y) \leq t$ or $\sigma_{M[\delta]}(y) + t < 1$. Since $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X, we have $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}$ for any $x, y \in X$ and $\delta \in A$. Now we have the following cases.

Case 1: $\sigma_{M[\delta]}(x) \leq t, \sigma_{M[\delta]}(y) \leq t$, let $t < \frac{k-1}{2}$. Then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2} = t \vee t \vee \frac{k-1}{2} = \frac{k-1}{2} \Rightarrow \sigma_{M[\delta]}(x \cdot y) \leq \frac{k-1}{2} \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < \frac{k-1}{2} + \frac{k-1}{2} = 1 - k$. Again, if $t \geq \frac{k-1}{2}$, then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2} \leq t \vee t \vee \frac{k-1}{2} = t \Rightarrow \sigma_{M[\delta]}(x \cdot y) \leq t$.

Case 2: $\sigma_{M[\delta]}(x) \leq t, \sigma_{M[\delta]}(y) + t < 1 - k$, let $t < \frac{k-1}{2}$. Then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}2 < t \vee (1 - k - t) \vee \frac{k-1}{2}2 = 1 - k - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) < 1 - k - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 1 - k$. Again, if $t \geq \frac{k-1}{2}$, then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2} \leq t \vee (1 - k - t) \vee \frac{k-1}{2} = t \Rightarrow \sigma_{M[\delta]}(x \cdot y) \leq t$.

Case 3: $\sigma_{M[\delta]}(x) + t < 1 - k, \sigma_{M[\delta]}(y) \leq t$, let $t < \frac{k-1}{2}$. Then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}2 < (1 - k - t) \vee t \vee \frac{k-1}{2} = 1 - k - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) < 1 - k - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 1 - k$. Again, if $t \geq \frac{k-1}{2}$, then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2} = (1 - k - t) \vee t \vee \frac{k-1}{2} = t \Rightarrow \sigma_{M[\delta]}(x \cdot y) \leq t$.

Case 4: $\sigma_{M[\delta]}(x) + t < 1 - k, \sigma_{M[\delta]}(y) + t < 1 - k$, let $t < \frac{k-1}{2}$. Then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2} < (1 - k - t) \vee (1 - k - t) \vee \frac{k-1}{2}2 = 1 - k - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) < 1 - k - t \Rightarrow \sigma_{M[\delta]}(x \cdot y) + t < 1 - k$. Again, if $t \geq \frac{k-1}{2}$, then $\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(y) \vee \frac{k-1}{2}2 = (1 - k - t) \vee (1 - k - t) \vee \frac{k-1}{2}2 = \frac{k-1}{2} \leq t \Rightarrow \sigma_{M[\delta]}(x \cdot y) \leq t$.

Hence, from above four cases, $[\sigma_{M[\delta]}]_t$ is a BCC-subalgebra of X. Similarly, we can prove $[\tau_{M[\delta]}]_t$ is a BCC-subalgebra of X.

Conversely, let $[\sigma_{M[\delta]}]_t$ and $[\tau_{M[\delta]}]_t$ are BCC-subalgebras of X for all $t \in (0, 1]$. Suppose $M[\delta]$ is not an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X, then there exist $a, b \in X$ such that at least one of $\sigma_{M[\delta]}(a \cdot b) > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee \frac{k-1}{2}$ or $\tau_{M[\delta]}(a \cdot b) < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge \frac{k-1}{2}$ hold. Suppose $\sigma_{M[\delta]}(a \cdot b) > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee \frac{k-1}{2}$ is true, then choose $t \in (0, 1]$ such that

$$\sigma_{M[\delta]}(a \cdot b) > t > \sigma_{M[\delta]}(a) \vee \sigma_{M[\delta]}(b) \vee \frac{1-k}{2}. \tag{3.27}$$

Then $\sigma_{M[\delta]}(a) < t, \sigma_{M[\delta]}(b) < t \Rightarrow a, b \in (\sigma_{M[\delta]})_t \subseteq [\sigma_{M[\delta]}]_t$, which is a BCC-subalgebra of X. Therefore, $a \cdot b \in [\sigma_{M[\delta]}]_t \Rightarrow \sigma_{M[\delta]}(a \cdot b) \leq t$ or $\sigma_{M[\delta]}(a \cdot b) + t < 1 - k$, which contradict (3.27). Again, if $\tau_{M[\delta]}(a \cdot b) < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge \frac{k-1}{2}$ is true, then choose $t \in (0, 1]$ such that

$$\tau_{M[\delta]}(a \cdot b) < t < \tau_{M[\delta]}(a) \wedge \tau_{M[\delta]}(b) \wedge \frac{1-k}{2}. \tag{3.28}$$

Then $\tau_{M[\delta]}(a) > t, \tau_{M[\delta]}(b) > t \Rightarrow a, b \in (\tau_{M[\delta]})_t \subseteq [\tau_{M[\delta]}]_t$, which is a BCC-subalgebra of X. Therefore, $a \cdot b \in [\tau_{M[\delta]}]_t \Rightarrow \tau_{M[\delta]}(a \cdot b) \geq t$ or $\tau_{M[\delta]}(a \cdot b) + t > 1 - k$, which contradict (3.28). Hence,

$$\sigma_{M[\delta]}(x \cdot y) \leq \sigma_{M[\delta]}(x) \vee \sigma_{M[\delta]}(x) \vee \frac{k-1}{2}$$

and

$$\tau_{M[\delta]}(x \cdot y) \geq \tau_{M[\delta]}(x) \wedge \tau_{M[\delta]}(x) \wedge \frac{k-1}{2}$$

for all $x, y \in X$ and $\delta \in A$. Hence, $M[\delta]$ is an $(\in, \in \vee q_k)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Definition 3.13. *Let X and X' be two BCC-algebras. Then a mapping $f : X \to X'$ is said to be a BCC-homomorphism if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$.*

Theorem 3.15. *Let X and X' be two BCC-algebras and $f : X \to X'$ be a BCC-homomorphism. If $M[\delta] = \{(x, \sigma_{M[\delta]}(x), \tau_{M[\delta]}(x)) \mid x \in X', \delta \in A\}$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X', then $f^{-1}(M[\delta]) = \{(x, f^{-1}(\sigma_{M[\delta]})(x), f^{-1}(\sigma_{M[\delta]})(x)) \mid x \in X, \delta \in A\}$ is also an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X.*

Proof. Let $M[\delta]=\{(x,\sigma_{M[\delta]}(x),\tau_{M[\delta]}(x))\mid x\in X',\delta\in A\}$ be an $(\in,\in\vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X'. Let $x,y\in X$. Then

$$\begin{aligned} x_t,y_s\in f^{-1}(\sigma_{M[\delta]}) &\Rightarrow f^{-1}(\sigma_{M[\delta]})(x)\le t \text{ and } f^{-1}(\sigma_{M[\delta]})(y)\le s\\ &\Rightarrow \sigma_{M[\delta]}f(x)\le t \text{ and } \sigma_{M[\delta]}f(y)\le s\\ &\Rightarrow (f(x))_t\in\sigma_{M[\delta]} \text{ and } (f(y))_s\in\sigma_{M[\delta]}\\ &\Rightarrow (f(x)\cdot f(y))_{t\vee s}\in\sigma_{M[\delta]}\\ &\Rightarrow (f(x\cdot y))_{t\vee s}\in\sigma_{M[\delta]}\\ &\Rightarrow \sigma_{M[\delta]}(f(x\cdot y))\le t\vee s \text{ or } \sigma_{M[\delta]}(f(x\cdot y))+t\vee s<1\\ &\Rightarrow f^{-1}(\sigma_{M[\delta]}(x\cdot y))\le t\vee s \text{ or } f^{-1}(\sigma_{M[\delta]}(x\cdot y))+t\vee s<1\\ &\Rightarrow (x\cdot y)_{t\vee s}\in f^{-1}(\sigma_{M[\delta]}) \text{ or } (x\cdot y)_{t\vee s}\in qf^{-1}(\sigma_{M[\delta]})\\ &\Rightarrow (x\cdot y)_{t\vee s}\in\vee qf^{-1}(\sigma_{M[\delta]}) \end{aligned}$$

and

$$\begin{aligned} x_t,y_s\in f^{-1}(\tau_{M[\delta]}) &\Rightarrow f^{-1}(\tau_{M[\delta]})(x)\ge t \text{ and } f^{-1}(\tau_{M[\delta]})(y)\ge s\\ &\Rightarrow \tau_{M[\delta]}f(x)\ge t \text{ and } \tau_{M[\delta]}f(y)\ge s\\ &\Rightarrow (f(x))_t\in\tau_{M[\delta]} \text{ and } (f(y))_s\in\tau_{M[\delta]}\\ &\Rightarrow (f(x)\cdot f(y))_{t\wedge s}\in\tau_{M[\delta]}\\ &\Rightarrow (f(x\cdot y))_{t\wedge s}\in\tau_{M[\delta]}\\ &\Rightarrow \tau_{M[\delta]}(f(x\cdot y))\ge t\wedge s \text{ or } \tau_{M[\delta]}(f(x\cdot y))+t\wedge s>1\\ &\Rightarrow f^{-1}(\tau_{M[\delta]}(x\cdot y))\ge t\wedge s \text{ or } f^{-1}(\tau_{M[\delta]}(x\cdot y))+t\wedge s>1\\ &\Rightarrow (x\cdot y)_{t\wedge s}\in f^{-1}(\tau_{M[\delta]}) \text{ or } (x\cdot y)_{t\wedge s}\in qf^{-1}(\tau_{M[\delta]})\\ &\Rightarrow (x\cdot y)_{t\wedge s}\in\vee qf^{-1}(\tau_{M[\delta]}). \end{aligned}$$

Therefore, $f^{-1}(M[\delta])$ is an $(\in,\in\vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X. □

Theorem 3.16. *Let X and X' be two BCC-algebras and $f:X\to X'$ be an onto BCC-homomorphism. If $M[\delta]=\{(x,\sigma_{M[\delta]}(x),\tau_{M[\delta]}(x))\mid x\in X',\delta\in A\}$ is an intuitionistic fuzzy soft set over X' such that $f^{-1}(M[\delta])=\{(x,f^{-1}(\sigma_{M[\delta]})(x),f^{-1}(\sigma_{M[\delta]})(x))\mid x\in X,\delta\in A\}$ is an $(\in,\in\vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X, then $M[\alpha]$ is also an $(\in,\in\vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X'.*

Proof. Let $x',y'\in X'$ and $\alpha\in A$ be such that $x'_t,y'_s\in\sigma_{M[\alpha]}$ and $x'_t,y'_s\in\tau_{M[\delta]}$, where $t,s\in[0,1]$. Then $\sigma_{M[\alpha]}(x')\le t$, $\sigma_{M[\alpha]}(y')\le s$ and $\tau_{M[\alpha]}(x')\ge t$, $\tau_{M[\alpha]}(y')\ge s$. Since f is onto, there exist $x,y\in X$ such that $f(x)=x',f(y)=y'$. Since f is a BCC-homomorphism, $f(x\cdot y)=f(x)\cdot f(y)=x'\cdot y'$. Thus

$$
\begin{aligned}
x'_t, y'_s \in \sigma_{M[\alpha]} &\Rightarrow \sigma_{M[\alpha]}(f(x)) \leq t \text{ and } \sigma_{M[\alpha]}(f(y)) \leq s \\
&\Rightarrow f^{-1}(\sigma_{M[\alpha]})(x) \leq t \text{ and } f^{-1}(\sigma_{M[\alpha]})(y) \leq s \\
&\Rightarrow (x)_t \in f^{-1}(\sigma_{M[\alpha]}) \text{ and } y_s \in f^{-1}(\sigma_{M[\alpha]}) \\
&\Rightarrow (x \cdot y)_{t \vee s} \in \vee q f^{-1}(\sigma_{M[\alpha]}) \\
&\Rightarrow f^{-1}(\sigma_{M[\alpha]})(x \cdot y) \leq t \vee s \text{ or } f^{-1}(\sigma_{M[\alpha]})(x \cdot y) + t \vee s < 1 \\
&\Rightarrow \sigma_{M[\alpha]}(f(x \cdot y)) \leq t \vee s \text{ or } \sigma_{M[\alpha]}(f(x \cdot y)) + t \vee s < 1 \\
&\Rightarrow \sigma_{M[\alpha]}(x' \cdot y') \leq t \vee s \text{ or } \sigma_{M[\alpha]}(x' \cdot y') + t \vee s < 1 \\
&\Rightarrow (x' \cdot y')_{t \vee s} \in \vee q \sigma_{M[\alpha]}
\end{aligned}
$$

and

$$
\begin{aligned}
x'_t, y'_s \in \tau_{M[\alpha]} &\Rightarrow \tau_{M[\alpha]}(f(x)) \geq t \text{ and } \tau_{M[\alpha]}(f(y)) \geq s \\
&\Rightarrow f^{-1}(\tau_{M[\alpha]})(x) \geq t \text{ and } f^{-1}(\tau_{M[\delta]})(y) \geq s \\
&\Rightarrow (x)_t \in f^{-1}(\tau_{M[\delta]}) \text{ and } y_t \in f^{-1}(\tau_{M[\delta]}) \\
&\Rightarrow (x \cdot y)_{t \wedge s} \in \vee q f^{-1}(\tau_{M[\alpha]}) \\
&\Rightarrow f^{-1}(\tau_{M[\alpha]})(x \cdot y) \geq t \wedge s \text{ or } f^{-1}(\tau_{M[\alpha]})(x \cdot y) + t \wedge s > 1 \\
&\Rightarrow \tau_{M[\delta]}(f(x \cdot y)) \geq t \vee s \text{ or } \tau_{M[\alpha]}(f(x \cdot y)) + t \wedge s > 1 \\
&\Rightarrow \tau_{M[\delta]}(x' \cdot y') \geq t \wedge s \text{ or } \tau_{M[\alpha]}(x' \cdot y') + t \wedge s > 1 \\
&\Rightarrow (x' \cdot y')_{t \wedge s} \in \vee q \tau_{M[\alpha]}.
\end{aligned}
$$

Hence, $M[\alpha]$ is an $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebra of X'. □

4. Conclusion

In the present paper, we have introduced the concepts of $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras and $(\in, \in)$-anti-intuitionistic fuzzy soft BCC-subalgebras of BCC-algebras. The BCC-homomorphic image and inverse image are investigated in $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of BCC-algebras. Characterizations of $(\in, \in \vee q)$-anti-intuitionistic fuzzy soft BCC-subalgebras of BCC-algebras are provided.

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