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Associative Types in a Semi-Brouwerian Almost Distributive Lattice With Respect to the Binary Operation ϱ

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Abstract. In this paper, we exhibit a detailed analysis of non-associativity and non-commutativity of the binary operation ϱ in a semi-Brouwerian almost distributive lattice and characterize the algebraic structure in terms of the different associative types.

1. Introduction

1.1. **Background and Motivation.** The idea of an almost distributive lattice [12] emerged as a generalization of the more restrictive concept of a distributive lattice [1,2,10,11]. While distributive lattices have well-defined properties and are widely studied, they impose strong conditions on the relationships between lattice operations. In a distributive lattice, the meet $\barwedge$ and join $\veebar$ operations satisfy the distributive properties. However, in certain applications or contexts, it is desirable to relax this strict requirement and consider structures where the distributive property only approximately holds or holds with some exceptions. This relaxation led to the introduction of almost distributive lattices. In an almost distributive lattice, the distributive property is not

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strictly required to hold for all elements. Instead, it is allowed to hold approximately or with some exceptions. In other words, an almost distributive lattice satisfies the distributive property "almost everywhere" but may have some specific elements or combinations of elements where the property does not hold. The relaxation of the distributive property in almost distributive lattices allows for more flexible and diverse mathematical structures that can capture real-world phenomena or scenarios that do not adhere strictly to distributivity. It provides a broader framework for modelling and analyzing situations where there might be exceptions or variations in the behaviour of lattice operations. On this almost distributive lattice, many authors [13–15] explored the pseudo-complementation, stone representation, Birkhoff center and many more, on this algebra with the two binary operations $\veebar$ and $\barwedge$.

In 2010 by introducing another binary operation ϱ on almost distributive lattices, Heyting almost distributive lattices [5] was introduced, which captures the essence of both almost distributive lattices and Heyting algebras [3]. The ϱ operation represents implication or logical implication within the lattice structure. It allows for a more expressive and powerful algebraic structure that can model reasoning and logical relationships between elements. Further, in 2014, semi-Heyting almost distributive lattices [6] and almost semi-Heyting algebra [7] extend the concept of Heyting almost distributive lattices by allowing a more flexible notion of implication, which is known as a semi-Heyting implication. The semi-Heyting implication captures a weaker form of implication, often referred to as repudiation.

Up to date, all authors have studied various algebras on almost distributive lattices with both least element 0 and maximal element m, in 2022 semi-Brouwerian almost distributive lattice [9] were studied with only a maximal element m, by having only a maximal element, provide a simplified structure that focuses on the properties and relationships associated with the maximal element. This simplicity can aid in analyzing and understanding the behaviour and implications of a single dominating element within the lattice. By excluding the least element, semi-Brouwerian almost distributive lattices possess specific properties and characteristics distinct from those found in semi-Heyting almost distributive lattices.

The major observation in an almost distributive lattice is that it fails to satisfy the one of distributive law according to the definition given in [12]. Later, it was observed that the associativity with respect to $\barwedge$ holds in an almost distributive lattice, but the associativity of $\veebar$ is still not known. It was an open problem given by Rao and Swamy in 1980. As the associativity of the binary operation with respect to $\veebar$ failed, it gave us the idea to check the associativity of the binary operation ϱ, the failure of commutativity with respect to $\veebar$ and $\barwedge$ in an almost distributive lattices lead us the way to discuss the commutativity of ϱ in semi-Brouwerian almost distributive lattices.

1.2. **Objective and Overview.** The main objective is to study the failure of the associative and commutative binary operation ϱ in semi-Brouwerian almost distributive lattices; we can recall that an associative identity contains three variables that are distinct and can occur in any order which are grouped in and around 14 different ways. We aim to identify specific elements or combinations of elements within the lattice where the 14 identities of associativity do not hold, providing a clear explanation and illustration of this failure along with the commutative property of ϱ.

In this overview, we will briefly introduce the notation of a semi-Brouwerian almost distributive lattice and its defining properties. We will then outline the significance of studying the failure of associativity and commutativity within this lattice structure and also study the peculiar behaviour of the 14 identities in associativity and how they can be obtained by considering any one of the 14 identities of associative and considering the commutative identity on semi-Brouwerian almost distributive lattices. Finally, observe that the first associative identity, the fourth associative identity and the commutative identity are distinct.

2. Preliminaries

Let us recall useful, necessary results on almost distributive lattices and semi-Brouwerian almost distributive lattices, frequently required in the sequel.

Definition 2.1. *[12] An algebra $(S, \veebar, \barwedge)$ of type $(2, 2)$ is referred to as an almost distributive lattice if it meets the conditions listed below:*

(i) $(x \veebar y) \barwedge z = (x \barwedge z) \veebar (y \barwedge z)$
(ii) $x \barwedge (y \veebar z) = (x \barwedge y) \veebar (x \barwedge z)$
(iii) $(x \veebar y) \barwedge y = y$
(iv) $(x \veebar y) \barwedge x = x$
(v) $x \veebar (x \barwedge y) = x$

for all $x, y, z \in S$.

Example 2.1. *[12] If S is a non-empty set, then for any $x, y \in S$. Define $x \barwedge y = y, x \veebar y = x$. Then $(S, \veebar, \barwedge)$ is an ADL, and it is classified as a discrete ADL.*

Throughout the preliminaries section, by S, we mean an almost distributive lattice $(S, \veebar, \barwedge)$, until otherwise mentioned. Given $x, y \in S$, we say that x is less than or equal to y if and only if $x = x \barwedge y$; or equivalently $x \veebar y = y$, and it is denoted by $x \leq y$. Therefore $\leq$ is a partial ordering on S. An element m is considered maximal if no element x exists, such as $m < x$.

Theorem 2.1. *[12] For any $m \in S$, the below conditions are interchangeable,*

(i) m is a maximal element
(ii) $m \veebar x = m$, *for all* $x \in S$
(iii) $m \barwedge x = x$, *for all* $x \in S$.

Theorem 2.2. *[12] For any $x, y, z \in S$,*

(i) $x \veebar y = x \Leftrightarrow x \barwedge y = x$
(ii) $x \veebar y = y \Leftrightarrow x \barwedge y = x$
(iii) $x \barwedge y = y \barwedge x = x$ *whenever* $x \leq y$
(iv) $\barwedge$ *is associative in L*

(v) $x \,\overline{\wedge}\, y \,\overline{\wedge}\, z = y \,\overline{\wedge}\, x \,\overline{\wedge}\, z$

(vi) $(x \,\underline{\vee}\, y) \,\overline{\wedge}\, z = (y \,\underline{\vee}\, x) \,\overline{\wedge}\, z$

(vii) $x \,\overline{\wedge}\, y \leq y$ *and* $x \leq x \,\underline{\vee}\, y$

(viii) $x \,\overline{\wedge}\, x = x$ *and* $x \,\underline{\vee}\, x = x$

(ix) *if* $x \leq z$ *and* $y \leq z$, *then* $x \,\overline{\wedge}\, y = y \,\overline{\wedge}\, x$ *and* $x \,\underline{\vee}\, y = y \,\underline{\vee}\, x$.

Definition 2.2. *[9] S with m as its maximal element is considered as a semi-Brouwerian almost distributive lattice (SBADL) if there is a binary operation* ϱ *on S with the following identities:*

(N_1) $(x \,\varrho\, x) \,\overline{\wedge}\, m = m$

(N_2) $x \,\overline{\wedge}\, (x \,\varrho\, y) = x \,\overline{\wedge}\, y \,\overline{\wedge}\, m$

(N_3) $x \,\overline{\wedge}\, (y \,\varrho\, z) = x \,\overline{\wedge}\, [(x \,\overline{\wedge}\, y) \,\varrho\, (x \,\overline{\wedge}\, z)]$

(N_4) $(x \,\varrho\, y) \,\overline{\wedge}\, m = [(x \,\overline{\wedge}\, m) \,\varrho\, (y \,\overline{\wedge}\, m)]$

for all $x, y, z \in S$.

Theorem 2.3. *[9] If S is an SBADL, then these are equivalent to one another:*

(i) $(x \,\varrho\, y) \,\overline{\wedge}\, m = (y \,\varrho\, x) \,\overline{\wedge}\, m$

(ii) $(x \,\varrho\, m) \,\overline{\wedge}\, m = x \,\overline{\wedge}\, m$

(iii) $y \,\overline{\wedge}\, (x \,\varrho\, y) \,\overline{\wedge}\, m = x \,\overline{\wedge}\, y \,\overline{\wedge}\, m$

for all $x, y \in S$.

3. Identities of Associative Type

In this section, we provide a good number of counter-examples for an SBADL in which the binary operation ϱ is not associative as well as commutative. We present different identities of associative types of length three with respect to the binary operation ϱ and characterize SBADLs through these identities.

Lemma 3.1. *If S is an SBADL, with m as its maximal element then, for any* $x, y \in S$,

(i) $(x \,\varrho\, y) \,\overline{\wedge}\, m = m \Rightarrow x \,\overline{\wedge}\, m \leq y \,\overline{\wedge}\, m$

(ii) $x \,\overline{\wedge}\, m \leq y \,\overline{\wedge}\, m \Rightarrow x \,\overline{\wedge}\, m \leq x \,\varrho\, y$

(iii) $m \,\varrho\, x = x \,\overline{\wedge}\, m$.

Proof. Let x, y be any two elements in an SBADL S with a maximal element m.

(i) Assume that $(x \,\varrho\, y) \,\overline{\wedge}\, m = m$. Then $x \,\overline{\wedge}\, (x \,\varrho\, y) \,\overline{\wedge}\, m = x \,\overline{\wedge}\, m$. Now,

$$\begin{aligned}(x \,\overline{\wedge}\, m) \,\overline{\wedge}\, (y \,\overline{\wedge}\, m) &= x \,\overline{\wedge}\, y \,\overline{\wedge}\, m \,\overline{\wedge}\, m \\ &= x \,\overline{\wedge}\, (x \,\varrho\, y) \,\overline{\wedge}\, m \quad \text{(by } N_2 \text{ of Definition 2.2)} \\ &= x \,\overline{\wedge}\, m.\end{aligned}$$

Therefore, $x \,\overline{\wedge}\, m \leq y \,\overline{\wedge}\, m$.

(ii) Assume that $x \barwedge m \leq y \barwedge m$. Then $x \barwedge m = x \barwedge m \barwedge y \barwedge m$. Now,

$$\begin{aligned} (x \barwedge m) \barwedge (x \varrho y) &= x \barwedge (x \varrho y) && \text{(since } m \text{ is maximal)} \\ &= x \barwedge y \barwedge m && \text{(by } N_2 \text{ of Definition 2.2)} \\ &= x \barwedge m. && \text{(by our assumption)} \end{aligned}$$

Therefore, $x \barwedge m \leq (x \varrho y)$.

(iii) Consider,

$$\begin{aligned} m \varrho x &= m \barwedge (m \varrho x) \\ &= m \barwedge x \barwedge m && \text{(by } N_2 \text{ of Definition 2.2)} \\ &= x \barwedge m. \end{aligned}$$

□

In the following, we give a counter-example for an SBADL in which the binary operation ϱ does not satisfy commutative and associative identities.

Example 3.1. *Consider a five-element chain $S = \{w, x, y, z, m\}$ in which the binary operation ϱ is given as follows:*

ϱ	w	x	y	z	m
w	m	m	m	m	m
x	w	m	y	z	m
y	x	x	m	z	m
z	y	y	y	m	m
m	w	x	y	z	m

Clearly, $(S, \veebar, \barwedge, \varrho, m)$ is an SBADL. For $w, x, y \in S$, it is straight forward to observe that $[(w \varrho x) \varrho y] \barwedge m \neq [w \varrho (x \varrho y)] \barwedge m$, and also $[(w \varrho x)] \barwedge m \neq [(x \varrho w)] \barwedge m$. Therefore, ϱ is not associative and commutative in S.

If S is an SBADL, m_1 is a maximal element, and ϱ is the binary operation, then let us state and name around 14 identities of length 3 of an associative kind with respect to the binary operation ϱ, as follows: for any $x, y, z \in S$,

(SA_1) $[(x \varrho y) \varrho z] \barwedge m = [x \varrho (y \varrho z)] \barwedge m$ (Associative law)
(SA_2) $[x \varrho (y \varrho z)] \barwedge m = [x \varrho (z \varrho y)] \barwedge m$
(SA_3) $[x \varrho (y \varrho z)] \barwedge m = [(x \varrho z) \varrho y] \barwedge m$
(SA_4) $[x \varrho (y \varrho z)] \barwedge m = [y \varrho (x \varrho z)] \barwedge m$
(SA_5) $[x \varrho (y \varrho z)] \barwedge m = [(y \varrho x) \varrho z] \barwedge m$
(SA_6) $[x \varrho (y \varrho z)] \barwedge m = [y \varrho (z \varrho x)] \barwedge m$
(SA_7) $[x \varrho (y \varrho z)] \barwedge m = [(y \varrho z) \varrho x] \barwedge m$
(SA_8) $[x \varrho (y \varrho z)] \barwedge m = [(z \varrho x) \varrho y] \barwedge m$
(SA_9) $[x \varrho (y \varrho z)] \barwedge m = [z \varrho (y \varrho x)] \barwedge m$
(SA_{10}) $[x \varrho (y \varrho z)] \barwedge m = [(z \varrho y) \varrho x] \barwedge m$
(SA_{11}) $[(x \varrho y) \varrho z] \barwedge m = [(x \varrho z) \varrho y] \barwedge m$
(SA_{12}) $[(x \varrho y) \varrho z] \barwedge m = [(y \varrho x) \varrho z] \barwedge m$
(SA_{13}) $[(x \varrho y) \varrho z] \barwedge m = [(y \varrho z) \varrho x] \barwedge m$

(SA_{14}) $[(x \varrho y) \varrho z] \barwedge m = [(z \varrho y) \varrho x] \barwedge m$.

Throughout this section, by S, we mean that $(S, \veebar, \barwedge, \varrho, m)$ is an SBADL in which m is a maximal element and ϱ is the binary operation.

Definition 3.1. *S is said to be associative with respect to the binary operation ϱ if it satisfies SA_1.*

Definition 3.2. *S said to be commutative with respect to the binary operation ϱ, if it holds the property $(x \varrho y) \barwedge m = (y \varrho x) \barwedge m$, for all $x, y \in S$.*

It can be easily observed that the following Theorem 3.1 follows from Definitions 3.1, 3.2 and 2.2.

Theorem 3.1. *If S is commutative and associative with regard to the binary operation ϱ, then SA_i if and only if SA_j, for all $i, j \in \{1, 2, 3, \ldots, 14\}$.*

Lemma 3.2. *If S is associative, then $p \barwedge m = (p \varrho m) \barwedge m$, for all $p \in S$.*

Proof. Suppose S is an associative SBADL, with m as its maximal element. Then, we have $[(x \varrho y) \varrho z] \barwedge m = [x \varrho (y \varrho z)] \barwedge m$, for all $x, y, z \in S$. Replacing x, y, z with p in above, we get

$$[(p \varrho p) \varrho p] \barwedge m = [p \varrho (p \varrho p)] \barwedge m$$
$$\Rightarrow (m \varrho p) \barwedge m = (p \varrho m) \barwedge m \quad \text{(by } N_1 \text{ of Definition 2.2)}$$
$$\Rightarrow m \barwedge (m \varrho p) \barwedge m = (p \varrho m) \barwedge m$$
$$\Rightarrow m \barwedge p \barwedge m = (p \varrho m) \barwedge m \quad \text{(by } N_2 \text{ of Definition 2.2)}$$
$$\Rightarrow p \barwedge m = (p \varrho m) \barwedge m. \quad \text{(by (v) of Theorem 2.2)}$$

Therefore, $p \barwedge m = (p \varrho m) \barwedge m$. □

We provide a counter-example for an SBADL in which the identity in Lemma 3.2 does not hold in the paragraphs that follow. In the following, we give a counter-example for an SBADL in which the identity in Lemma 3.2 does not hold.

Example 3.2. *Consider a three-element chain $S = \{x, y, m\}$, in which the binary operation ϱ is given as follows:*

ϱ	x	y	m
x	m	x	x
y	x	m	m
m	x	y	m

Then $(S, \veebar, \barwedge, \varrho, m)$ is a semi-Brouwerian almost distributive lattice. Moreover $y \barwedge m = y \neq m = (y \varrho m) \barwedge m$.

Lemma 3.3. *If S satisfies identities SA_5 or SA_8 or SA_{10} or SA_{12} or SA_{13} or SA_{14}, then $p \barwedge m = (p \varrho m) \barwedge m$, for all $p \in S$.*

Proof. Suppose that S satisfies SA_5. Then

$$\begin{aligned}
(p \varrho m) \barwedge m &= [(p \barwedge m) \varrho (p \varrho p) \barwedge m] && \text{(by } N_1 \text{ and } N_4 \text{ of Definition 2.2)} \\
&= [p \varrho (p \varrho p)] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= [(p \varrho p) \varrho p] \barwedge m && \text{(by } SA_5\text{)} \\
&= (m \varrho p) \barwedge m && \text{(by } N_1 \text{ of Definition 2.2)} \\
&= p \barwedge m. && \text{(by (iii) of Lemma 3.1)}
\end{aligned}$$

SA_8:

$$\begin{aligned}
p \barwedge m &= p \barwedge m \barwedge m \\
&= (m \varrho p) \barwedge m && \text{(by (iii) of Lemma 3.1)} \\
&= [(m \varrho m) \varrho p] \barwedge m && \text{(by } N_1 \text{ of Definition 2.2)} \\
&= [m \varrho (p \varrho m)] \barwedge m && \text{(by } SA_8\text{)} \\
&= (p \varrho m) \barwedge m \barwedge m && \text{(by (iii) of Lemma 3.1)} \\
&= (p \varrho m) \barwedge m.
\end{aligned}$$

SA_{10}:

$$\begin{aligned}
(p \varrho m) \barwedge m &= [p \varrho (m \varrho m)] \barwedge m && \text{(by } N_1 \text{ of Definition 2.2)} \\
&= [(m \varrho m) \varrho p] \barwedge m && \text{(by } SA_{10}\text{)} \\
&= (m \varrho p) \barwedge m && \text{(by } N_1 \text{ of Definition 2.2)} \\
&= p \barwedge m. && \text{(by (iii) of Lemma 3.1)}
\end{aligned}$$

SA_{12}:

From Lemma 3.1 (ii), since $p \barwedge m \leq m$, we have $p \barwedge m \leq (p \varrho m) \barwedge m$. On the other hand, consider

$$\begin{aligned}
[(p \varrho m) \varrho p] \barwedge m &= [(m \varrho p) \varrho p] \barwedge m && \text{(by } SA_{12}\text{)} \\
&= [(p \barwedge m) \varrho p] \barwedge m && \text{(by (iii) of Lemma 3.1)} \\
&= (p \varrho p) \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= m.
\end{aligned}$$

Hence $(p \varrho m) \barwedge m \leq p \barwedge m$ from (i) of Lemma 3.1.

Therefore $p \barwedge m = (p \varrho m) \barwedge m$.

SA_{13}:

$$\begin{aligned}
a \barwedge m &= p \barwedge m \barwedge m \\
&= (m \varrho p) \barwedge m && \text{(by (iii) of Lemma 3.1)} \\
&= [(m \varrho m) \varrho p] \barwedge m && \text{(by } N_1 \text{ of Definition 2.2)} \\
&= [(m \varrho p) \varrho m] \barwedge m && \text{(by } SA_{13}\text{)} \\
&= (p \varrho m) \barwedge m. && \text{(by (iii) of Lemma 3.1)}
\end{aligned}$$

SA_{14}:

$$\begin{aligned}
(p \varrho m) \barwedge m &= (p \varrho m) \barwedge (p \varrho m) \barwedge m \barwedge m \\
&= (p \varrho m) \barwedge [(p \barwedge m) \varrho m] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= (p \varrho m) \barwedge [(m \varrho p) \varrho m] \barwedge m && \text{(by (iii) of Lemma 3.1)} \\
&= (p \varrho m) \barwedge [(m \varrho m) \varrho p] \barwedge m && \text{(by } SA_{14}\text{)} \\
&= (p \varrho m) \barwedge (m \varrho p) \barwedge m && \text{(by } N_1 \text{ of Definition 2.2)} \\
&= (p \varrho m) \barwedge p \barwedge m \barwedge m. && \text{(by (iii) of Lemma 3.1)}
\end{aligned}$$

Therefore $(p\ \varrho\ m)\,\overline{\wedge}\,m \leq p\,\overline{\wedge}\,m$. On the other hand, we know that $p\,\overline{\wedge}\,m \leq (p\ \varrho\ m)\,\overline{\wedge}\,m$. Thus $p\,\overline{\wedge}\,m = (p\ \varrho\ m)\,\overline{\wedge}\,m$. □

Remark 3.1. *If S satisfies SA_3 or SA_4 or SA_7 or SA_9 or SA_{11}, then S need not satisfies the identity in Lemma 3.2. For, see Example 3.2, we have $y\,\overline{\wedge}\,m \neq (y\ \varrho\ m)\,\overline{\wedge}\,m$.*

Remark 3.2. *If S satisfies SA_2, then S need not satisfy the identity in Lemma 3.2. For, take the example below.*

Example 3.3. *Let $S = \{x, y, m\}$, with $\veebar, \overline{\wedge}$ and ϱ defined as follows:*

$\veebar$	x	y	m
x	x	y	m
y	y	y	y
m	m	m	m

$\overline{\wedge}$	x	y	m
x	x	x	x
y	x	y	m
m	x	y	m

ϱ	x	y	m
x	m	m	m
y	x	m	m
m	x	x	m

It is evident that $(S, \veebar, \overline{\wedge}, \varrho, m)$ is an SBADL which satisfies SA_2. It is clear to observe that $x\,\overline{\wedge}\,m \neq (x\ \varrho\ m)\,\overline{\wedge}\,m$.

Recall the following lemma from [9].

Lemma 3.4. *The following are equivalent in S:*

(i) $(x\ \varrho\ y)\,\overline{\wedge}\,m = (y\ \varrho\ x)\,\overline{\wedge}\,m$, *for all* $x, y \in S$

(ii) $(x\ \varrho\ m)\,\overline{\wedge}\,m = x\,\overline{\wedge}\,m$, *for all* $x \in S$

(iii) $y\,\overline{\wedge}\,(x\ \varrho\ y)\,\overline{\wedge}\,m = x\,\overline{\wedge}\,y\,\overline{\wedge}\,m$, *for all* $x, y \in S$.

We establish a sufficient condition for an SBADL to satisfy condition Lemma 3.4(i) in the results that follow.

Theorem 3.2. *If S satisfies the identities SA_1 or SA_5 or SA_8 or SA_{10} or SA_{12} or SA_{13} or SA_{14}, then S is commutative.*

Proof. It is easy to prove the result from Lemmas 3.2, 3.3 and 3.4. □

Theorem 3.3. *If S satisfies the identities SA_2 or SA_3 or SA_6 or SA_7 or SA_9 or SA_{11}, then S is commutative.*

Proof. Consider an SBADL with

SA_2:

$$
\begin{aligned}
(x\ \varrho\ y)\,\overline{\wedge}\,m &= m\ \varrho\ (x\ \varrho\ y)\,\overline{\wedge}\,m && \text{(since } x\,\overline{\wedge}\,y\,\overline{\wedge}\,m = y\,\overline{\wedge}\,x\,\overline{\wedge}\,m\text{)}\\
&= m\ \varrho\ [(x\,\overline{\wedge}\,m)\ \varrho\ (y\,\overline{\wedge}\,m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= m\ \varrho\ [(y\,\overline{\wedge}\,m)\ \varrho\ (x\,\overline{\wedge}\,m)] && \text{(by } SA_2\text{)}\\
&= m\ \varrho\ [(y\ \varrho\ x)\,\overline{\wedge}\,m] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (y\ \varrho\ x)\,\overline{\wedge}\,m. && \text{(by (iii) of Lemma 3.1)}
\end{aligned}
$$

SA_3:

$$\begin{aligned}(x \varrho y) \barwedge m &= m \varrho [(x \varrho y) \barwedge m] && \text{(by (iii) of Lemma 3.1)}\\ &= m \varrho [(x \barwedge m) \varrho (y \barwedge m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\ &= [m \varrho (y \barwedge m)] \varrho (x \barwedge m) && \text{(by } SA_3\text{)}\\ &= (y \barwedge m) \varrho (x \barwedge m) && \text{(by (iii) of Lemma 3.1)}\\ &= (y \varrho x) \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}\end{aligned}$$

SA_6:

$$\begin{aligned}(x \varrho y) \barwedge m &= (x \barwedge m) \varrho (y \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)}\\ &= (x \barwedge m) \varrho (m \varrho y) && \text{(by (iii) of Lemma 3.1)}\\ &= m \varrho [y \varrho (x \barwedge m)] && \text{(by } SA_6\text{)}\\ &= [y \varrho (x \barwedge m)] \barwedge m && \text{(by (iii) of Lemma 3.1)}\\ &= (y \barwedge m) \varrho (x \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)}\\ &= (y \varrho x) \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}\end{aligned}$$

SA_7:

$$\begin{aligned}(x \varrho y) \barwedge m &= (x \barwedge m) \varrho (y \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)}\\ &= (x \barwedge m) \varrho (m \varrho y) && \text{(by (iii) of Lemma 3.1)}\\ &= (m \varrho y) \varrho (x \barwedge m) && \text{(by } SA_7\text{)}\\ &= (y \barwedge m) \varrho (x \barwedge m) && \text{(by (iii) of Lemma 3.1)}\\ &= (y \varrho x) \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}\end{aligned}$$

SA_9:

$$\begin{aligned}(x \varrho y) \barwedge m &= (x \varrho y) \barwedge m \barwedge m\\ &= [(x \barwedge m) \varrho (y \barwedge m)] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)}\\ &= [(x \barwedge m) \varrho (m \varrho y)] \barwedge m && \text{(by (iii) of Lemma 3.1)}\\ &= (x \barwedge m) \varrho [m \varrho (y \barwedge m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\ &= (y \barwedge m) \varrho [m \varrho (x \barwedge m)] && \text{(by } SA_9\text{)}\\ &= (y \barwedge m) \varrho (x \barwedge m) && \text{(by (iii) of Lemma 3.1)}\\ &= (y \varrho x) \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}\end{aligned}$$

SA_{11} :

$$\begin{aligned}(x \varrho y) \barwedge m &= (x \barwedge m) \varrho (y \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)}\\ &= (m \varrho x) \varrho (y \barwedge m) && \text{(by (iii) of Lemma 3.1)}\\ &= [m \varrho (y \barwedge m)] \varrho (x \barwedge m) && \text{(by } SA_{11}\text{)}\\ &= (y \barwedge m) \varrho (x \barwedge m) && \text{(by (iii) of Lemma 3.1)}\\ &= (y \varrho x) \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}\end{aligned}$$

□

Remark 3.3. *Every SBADL with SA_4 may not be commutative. For, see Example 3.2, $(m \varrho y) \barwedge m = y \neq m = (y \varrho m) \barwedge m$.*

Theorem 3.4. *If S satisfies the identities SA_3 or SA_5 or SA_6 or SA_8 or SA_9 or SA_{11} or SA_{13} or SA_{14}, then $[(x \varrho y) \varrho z] \barwedge m = [x \varrho (y \varrho z)] \barwedge m$, for all $x, y, z \in S$.*

Proof. Consider an SBADL with

SA_3 :

$$
\begin{aligned}
[x \varrho (y \varrho z)] \barwedge m &= (x \barwedge m) \varrho [(y \varrho z) \barwedge m] && \text{(by (iii) of Lemma 3.1)}\\
&= (x \barwedge m) \varrho [(z \varrho y) \barwedge m] && \text{(by Theorem 3.3)}\\
&= (x \barwedge m) \varrho [(z \barwedge m) \varrho (y \barwedge m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(x \barwedge m) \varrho (y \barwedge m)] \varrho (z \barwedge m) && \text{(by } SA_3\text{)}\\
&= [(x \varrho y) \barwedge m] \varrho (z \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(x \varrho y) \varrho z] \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2).}
\end{aligned}
$$

SA_5 :

$$
\begin{aligned}
[x \varrho (y \varrho z)] \barwedge m &= (x \barwedge m) \varrho [(y \varrho z) \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (x \barwedge m) \varrho [(y \barwedge m) \varrho (z \barwedge m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(y \barwedge m) \varrho (x \barwedge m)] \varrho (z \barwedge m) && \text{(by } SA_5\text{)}\\
&= [(y \varrho x) \barwedge m] \varrho (z \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(x \varrho y) \barwedge m] \varrho (z \barwedge m) && \text{(by Theorem 3.2)}\\
&= [(x \varrho y) \varrho z] \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}
\end{aligned}
$$

SA_6 :

$$
\begin{aligned}
[x \varrho (y \varrho z)] \barwedge m &= (x \barwedge m) \varrho [(y \varrho z) \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (x \barwedge m) \varrho [(z \varrho y) \barwedge m] && \text{(by Theorem 3.3)}\\
&= (x \barwedge m) \varrho [(z \barwedge m) \varrho (y \barwedge m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (z \barwedge m) \varrho [(y \barwedge m) \varrho (x \barwedge m)] && \text{(by } SA_6\text{)}\\
&= (z \barwedge m) \varrho [(y \varrho x) \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (z \barwedge m) \varrho [(x \varrho y) \barwedge m] && \text{(by Theorem 3.3)}\\
&= [z \varrho (x \varrho y)] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(x \varrho y) \varrho z] \barwedge m. && \text{(by Theorem 3.3)}
\end{aligned}
$$

SA_8 :

$$
\begin{aligned}
[(x \varrho (y \varrho z)] \barwedge m &= (x \barwedge m) \varrho [(y \varrho z) \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (x \barwedge m) \varrho [(z \varrho y) \barwedge m] && \text{(by Theorem 3.2)}\\
&= (x \barwedge m) \varrho [(z \barwedge m) \varrho (y \barwedge m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(y \barwedge m) \varrho (x \barwedge m)] \varrho (z \barwedge m) && \text{(by } SA_8\text{)}\\
&= [(y \varrho x) \barwedge m] \varrho (z \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(x \varrho y) \barwedge m] \varrho (z \barwedge m) && \text{(by Theorem 3.2)}\\
&= [(x \varrho y) \varrho z] \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}
\end{aligned}
$$

SA_9 :

$$
\begin{aligned}
[(x \varrho (y \varrho z)] \barwedge m &= (x \barwedge m) \varrho [(y \barwedge m) \varrho (z \barwedge m)] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (z \barwedge m) \varrho [(y \barwedge m) \varrho (x \barwedge m)] && \text{(by } SA_9\text{)}\\
&= (z \barwedge m) \varrho [(y \varrho x) \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= (z \barwedge m) \varrho [(x \varrho y) \barwedge m] && \text{(by Theorem 3.3)}\\
&= [z \varrho (x \varrho y)] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)}\\
&= [(x \varrho y) \varrho z] \barwedge m. && \text{(by Theorem 3.3)}
\end{aligned}
$$

SA_{11} :

$$\begin{aligned}
[(x \varrho y) \varrho z] \barwedge m &= [(x \varrho y) \barwedge m] \varrho (z \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= [(y \varrho x) \barwedge m] \varrho (z \barwedge m) && \text{(by Theorem 3.3)} \\
&= [(y \varrho x) \varrho z] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= [(y \varrho z) \varrho x] \barwedge m && \text{(by } SA_{11}\text{)} \\
&= [x \varrho (y \varrho z)] \barwedge m. && \text{(by Theorem 3.3)}
\end{aligned}$$

SA_{13} :

$$\begin{aligned}
[(x \varrho y) \varrho z] \barwedge m &= [(y \varrho z) \varrho x] \barwedge m && \text{(by } SA_{13}\text{)} \\
&= [x \varrho (y \varrho z)] \barwedge m. && \text{(by Theorem 3.2)}
\end{aligned}$$

SA_{14} :

$$\begin{aligned}
[(x \varrho y) \varrho z] \barwedge m &= [(z \varrho y) \varrho x] \barwedge m && \text{(by } SA_{14}\text{)} \\
&= [x \varrho (z \varrho y)] \barwedge m && \text{(by Theorem 3.2)} \\
&= (x \barwedge m) \varrho [(z \varrho y) \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= (x \barwedge m) \varrho [(y \varrho z) \barwedge m] && \text{(by Theorem 3.2)} \\
&= [x \varrho (y \varrho z)] \barwedge m. && \text{(by } N_4 \text{ of Definition 2.2)}
\end{aligned}$$

□

Theorem 3.5. *Every commutative SBADL satisfies the identities SA_2, SA_7, SA_{10} and SA_{12}.*

Proof. Suppose that S satisfies the property $[(x \varrho y)] \barwedge m = [(y \varrho x)] \barwedge m$. Now, consider

$$\begin{aligned}
[x \varrho (y \varrho z)] \barwedge m &= (x \barwedge m) \varrho [(y \varrho z) \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= (x \barwedge m) \varrho [(z \varrho y) \barwedge m] && \text{(by Definition 3.2)} \\
&= [x \varrho (z \varrho y)] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)}
\end{aligned}$$

which is SA_2.

SA_7 is clear from the commutative property.

Consider

$$\begin{aligned}
[x \varrho (y \varrho z)] \barwedge m &= [(y \varrho z) \varrho x] \barwedge m \\
&= [(y \varrho z) \barwedge m] \varrho [x \barwedge m] && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= [(z \varrho y) \barwedge m] \varrho [x \barwedge m] && \text{(by Definition 3.2)} \\
&= [(z \varrho y) \varrho x] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)}
\end{aligned}$$

which is SA_{10}.

Consider

$$\begin{aligned}
[(x \varrho y) \varrho z] \barwedge m &= [(x \varrho y) \barwedge m] \varrho (z \barwedge m) && \text{(by } N_4 \text{ of Definition 2.2)} \\
&= [(y \varrho x) \barwedge m] \varrho (z \barwedge m) && \text{(by Definition 3.2)} \\
&= [(y \varrho x) \varrho z] \barwedge m && \text{(by } N_4 \text{ of Definition 2.2)}
\end{aligned}$$

which is SA_{12}. □

Corollary 3.1. *In an associative SBADL, we have $SA_3 = SA_5 = SA_6 = SA_8 = SA_9 = SA_{11} = SA_{13} = SA_{14}$.*

Proof. The proof follows directly from Theorems 3.1, 3.2, 3.3 and 3.4. □

Corollary 3.2. *In a commutative SBADL, we have $SA_2 = SA_7 = SA_{10} = SA_{12}$.*

Proof. Theorems 3.3, 3.4 and 3.5 all directly lead to the proof. □

Theorem 3.6. *Every commutative SBADL with the identity SA_4 is associative.*

Proof. Let $x, y, z \in S$. Then

$$\begin{aligned} [x \varrho (y \varrho z)] \barwedge m &= [x \varrho (z \varrho y)] \barwedge m \quad \text{(by Definition 3.2)} \\ &= [z \varrho (x \varrho y)] \barwedge m \quad \text{(by } SA_4\text{)} \\ &= [(x \varrho y) \varrho z] \barwedge m. \quad \text{(by Definition 3.2)} \end{aligned}$$

Therefore, S is associative. □

4. Conclusion

This paper extensively studied the associativity and commutativity properties of ϱ in a semi-Brouwerian almost distributive lattice. We provided a good number of counter-examples to demonstrate that associativity and commutativity can fail in a semi-Brouwerian almost distributive lattice, to highlight the non-trivial nature of this algebraic structure. Overall, the paper's contributions lie in characterising semi-Brouwerian almost distributive lattices as a novel class of almost distributive lattices.

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References

[1] G. Birkhoff, Lattice Theory, Third Edition, Colloquium Publications, Vol. 25, American Mathematical Society, Providence, (1967).

[2] G. Boole, An Investigation Into the Laws of Thought, London, 1854 (Reprinted by Open Court Publishing Co.), Chesla, (1940).

[3] S. Burris, H.P. Sankappanavar, A First Course in Universal Algebra, Spinger-Verlag, New York, Heidelberg, Berlin, (1981).

[4] J.M. Cornejo, H.P. Sankappanavar, Semi-Heyting Algebras and Identities of Associative Type, Bull. Sect. Logic. 48 (2019), 177–198. https://doi.org/10.18778/0138-0680.48.2.03.

[5] G.C. Rao, B. Assaye, M.V. Ratnamani, Heyting Almost Distributive Lattices, Int. J. Comput. Cognit. 8 (2010), 89–93.

[6] G.C. Rao, M.V. Ratnamani, K.P. Shum, B. Assaye, Semi Heyting Almost Distributive Lattices, Lobachevskii J. Math. 36 (2015), 184–189. https://doi.org/10.1134/s1995080215020158.

[7] G.C. Rao, M.V. Ratnamani, K.P. Shum, Almost Semi Heyting Algebra, Southeast Asian Bull. Math. 42 (2018), 95–110.

[8] H.P. Sankappanavar, Semi-Heyting Algebras: An Abstraction From Heyting Algebras, In: Actas del IX Congreso Antonio Monteiro, Universidad Nacional del Sur, Bahia Blanca, Argentina, 2008, pp. 33–66.

[9] V.V.V.S.S.P.S. Srikanth, S. Ramesh, M.V. Ratnamani, Semi-Brouwerian Almost Distributive Lattices, Southeast Asian Bull. Math. 45 (2022), 849–860.

[10] M.H. Stone, The Theory of Representation for Boolean Algebras, Trans. Amer. Math. Soc. 40 (1936), 37–111. https://doi.org/10.2307/1989664.

[11] M.H. Stone, Topological Representation of Distributive Lattices and Brouwerian Logics, Čas. Mat. Fys. 67 (1937), 1–25. http://eudml.org/doc/27235.

[12] U.M. Swamy, G.C. Rao, Almost Distributive Lattices, J. Aust. Math. Soc. A. 31 (1981), 77–91. https://doi.org/10.1017/s1446788700018498.

[13] U.M. Swamy, G.C. Rao, G.N. Rao, Pseudo-Complementation on Almost Distributive Lattices, Southeast Asian Bull. Math. 24 (2000), 95–104.

[14] U.M. Swamy, G.C. Rao, G.N. Rao, Stone Almost Distributive Lattices, Southeast Asian Bull. Math. 27 (2003), 513–526.

[15] U.M. Swamy, S. Ramesh, Birkhoff Centre of an Almost Distributive Lattice, Int. J. Algebra, 3 (2009), 539–546.