Int. J. Anal. Appl. (2023), 21:124

International Journal of Analysis and Applications


ISSN: 2291-8639 DOI: 10.28924/2291-8639-19-2021-812

International Journal of Analysis and Applications

Vol. 20 (2022)

Some Subsets of Bitopological Spaces

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Abstract. In this paper, we introduce the notions of minimal u-ω-closed sets, maximal u-ω-open sets, u-ω-paraopen sets, u-ω-paraclosed sets, u-ω-mean open sets and u-ω-mean closed sets in bitopological spaces and obtain several characterizations and some of its properties.

1. Introduction

The notion of biotopological spaces was first introduced by Kelly [7]. Then A large number of topologists have directed their attention to generalizing different well known concepts of a topological space and trying to study them in biotopological spaces. The importance of generalized open sets in general topology is well known. and are now research topics of many topologists around the world. In fact, a significant topic in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms, etc. using generalized open sets. Recently, as a generalization of closed sets, the notion of ω-closed sets was introduced and studied by Hdeib [5]. Several characterizations and properties of ω-closed sets were provided in [2–6]. In this paper, we introduce and study the notions of minimal u-ω-closed sets, maximal u-ω-open sets, u-ω-paraopen sets, u-ω-paraclosed sets, u-ω-mean

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Received: Sep. 13, 2023.
2020 *Mathematics Subject Classification.* 54A40.
Key words and phrases. bitopological spaces; u-ω-open sets; u-ω-closed sets.

https://doi.org/10.28924/2291-8639-21-2023-124
ISSN: 2291-8639

open sets and u-ω-mean closed sets in bitopological spaces and obtain several characterizations and some of its properties.

2. Preliminaries

Throughout this paper, (X, τ_1, τ_2) always mean bitopological spaces in which no separation axioms are assumed unless explicitly stated. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [5] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ. The intersection of all ω-closed sets containing A is called the ω-closure [5] of A and is denoted by $\omega \operatorname{Cl}(A)$. The family of all ω-open sets of X is denoted by $\omega(\tau)$.

Definition 2.1. *[1] Let (X, τ_1, τ_2) be a bitopological space and let $A \subset X$. Then*

1. *A is said to be u-ω-open in (X, τ_1, τ_2) if $A \in \omega(\tau_1) \cup \omega(\tau_2)$,*
2. *A is said to be u-ω-closed in (X, τ_1, τ_2) if $X - A$ is u-ω-open in (X, τ_1, τ_2).*

The family of all u-ω-open sets in (X, τ_1, τ_2) is denoted by $\omega(\tau_1, \tau_2)$, and the family of all u-ω-closed sets in (X, τ_1, τ_2) is denoted by $\omega c(\tau_1, \tau_2)$.

Definition 2.2.
1. *The u-ω-closure of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2)-$\omega \operatorname{Cl}(A)$ and defined as follows: $(\tau_1, \tau_2)\text{-}\omega \operatorname{Cl}(A) = \omega \operatorname{Cl}_{\tau_1}(A) \cap \omega \operatorname{Cl}_{\tau_2}(A)$.*
2. *The u-ω-interior of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2)-$\omega \operatorname{Int}(A)$ and defined as follows: $(\tau_1, \tau_2)\text{-}\omega \operatorname{Int}(A) = \omega \operatorname{Int}_{\tau_1}(A) \cup \omega \operatorname{Int}_{\tau_2}(A)$.*

3. Weak forms bitopological ω-open sets

In this section, we study some fundamental properties of u-ω-minimal closed sets and u-ω-maximal open sets.

Definition 3.1. *A proper nonempty u-ω-closed subset F of (X, τ_1, τ_2) is said to be a minimal u-ω-closed set if any u-ω-closed set contained in F is $\emptyset$ or F.*

Definition 3.2. *A proper nonempty u-ω-open U of (X, τ_1, τ_2) is said to be a maximal u-ω-open set if any u-ω-open set containing U is either X or U.*

Example 3.1. *Let $X = \mathbb{R}$, $\tau_1 = \tau_2 = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$. Then $A = \mathbb{Q}$ is a u-ω-closed set and $B = \mathbb{R} \setminus \mathbb{Q}$ is a u-ω-open set. Observe that the set A is not a minimal u-ω-closed set and the set B is not a maximal u-ω-open. In the same form, any unitary set of the set A is a minimal u-ω-closed and the set $D = \mathbb{R} \setminus E$, were E is a unitary set of A is a maximal u-ω-open.*

Remark 3.1. *The collection of all minimal u-ω-closed sets of X is denoted by $\omega^{-}c(\tau_1,\tau_2)$ and the collection of all maximal u-ω-open sets of X is denoted by $\omega^{+}(\tau_1,\tau_2)$.*

Theorem 3.1. *A proper nonempty subset $U \in \omega^{+}(\tau_1,\tau_2)$ if and only if $X\backslash U \in \omega^{-}c(\tau_1,\tau_2)$.*

Proof. Let $U \in \omega^{+}(\tau_1,\tau_2)$. Suppose $X\backslash U \notin \omega^{-}c(\tau_1,\tau_2)$. Then there exists $V \in \omega c(\tau_1,\tau_2)$ and $V \neq X\backslash U$ such that $\emptyset \neq V \subset X\backslash U$. That is $U \subset X\backslash V$ and $X\backslash V \in \omega(\tau_1,\tau_2)$, a contradiction for $U \in \omega^{-}c(\tau_1,\tau_2)$. Conversely, let $X\backslash U \in \omega^{-}c(\tau_1,\tau_2)$. Suppose $U \notin \omega^{+}(\tau_1,\tau_2)$. Then there exists $E \in \omega(\tau_1,\tau_2)$ and $E \neq U$ such that $U \subset E \neq X$. That is $\emptyset \neq X\backslash E \subset X\backslash U$ and $X\backslash E \in \omega c(\tau_1,\tau_2)$, a contradiction for $X\backslash U \in \omega^{-}c(\tau_1,\tau_2)$. Therefore, $U \in \omega^{+}(\tau_1,\tau_2)$. ☐

Lemma 3.1.
(1) *If $U \in \omega^{-}c(\tau_1,\tau_2)$ and $V \in \omega c(\tau_1,\tau_2)$, then $U \cap V = \emptyset$ or $U \subset V$.*
(2) *If $U, V \in \omega^{-}c(\tau_1,\tau_2)$, then $U \cap V = \emptyset$ or $U = V$.*

Proof. (1). If $U \cap V = \emptyset$, then there is nothing to prove. If $U \cap V \neq \emptyset$, then $U \cap V \subset U$. Since $U \in \omega^{-}c(\tau_1,\tau_2)$, $U \cap V = U$. Hence $U \subset V$.
(2). If $U \cap V \neq \emptyset$, then $U \subset V$ and $V \subset U$ by (1). Hence $U = V$. ☐

Theorem 3.2. *Let $U \in \omega^{-}c(\tau_1,\tau_2)$. If $x \in U$, then $U \subset W$ for some $W \in \omega c(\tau_1,\tau_2,x)$.*

Proof. Let $x \in U$ and $W \in \omega c(\tau_1,\tau_2,x)$. Then $U \cap W = \emptyset$. By Lemma 3.1 (1), $U \subset W$. ☐

Theorem 3.3. *If $U \in \omega^{-}c(\tau_1,\tau_2)$, then $U = \cap\{W : W \in \omega c(\tau_1,\tau_2,x)\}$.*

Proof. By Theorem 3.2 and $U \in \omega c(\tau_1,\tau_2,x)$, we have $U \subset \cap\{W : W \in \omega c(\tau_1,\tau_2,x)\}$. Next let, $x \in \cap\{W : W \in \omega c(\tau_1,\tau_2,x)\}$. Then $x \in W$ for all $W \in \omega c(\tau_1,\tau_2)$. As $U \in \omega c(\tau_1,\tau_2)$, $x \in U$; hence $\cap\{W : W \in \omega c(\tau_1,\tau_2,x)\} = U$. ☐

Theorem 3.4. *Let U be a nonempty u-ω-closed subset of (X,τ_1,τ_2). Then the following statements are equivalent:*

(1) *$U \in \omega^{-}c(\tau_1,\tau_2)$.*
(2) *$U \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$ for any nonempty subset S of U.*
(3) *(τ_1,τ_2)-$\omega\,\mathrm{Cl}(U) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$ for any nonempty subset S of U.*

Proof. (1) ⇒ (2): Let $x \in U$; $U \in \omega^{-}c(\tau_1,\tau_2)$ and $S(\neq \emptyset) \subset U$. By Theorem 3.2, for any $W \in \omega c(\tau_1,\tau_2,x)$, $S \subset U \subset W$ gives $S \subset W$. Now $S = S \cap U \subset S \cap W$. Since $S \neq \emptyset$, $S \cap W \neq \emptyset$. Since $W \in \omega c(\tau_1,\tau_2,x)$, by Theorem 3.2, $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$. That is, $x \in U \Rightarrow x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$. Hence $U \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$ for any nonempty subset S of U.
(2) ⇒ (3): Let S be a nonempty subset of U. Then (τ_1,τ_2)-$\omega\,\mathrm{Cl}(S) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(U)$. By (2), (τ_1,τ_2)-$\omega\,\mathrm{Cl}(U) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}((\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$. That is, (τ_1,τ_2)-$\omega\,\mathrm{Cl}(U) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$. We have (τ_1,τ_2)-$\omega\,\mathrm{Cl}(U) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(S)$ for any nonempty subset S of U.
(3) ⇒ (1): Suppose $U \notin \omega^{-}c(\tau_1,\tau_2)$. Then there exists $V \in \omega c(\tau_1,\tau_2)$ such that $V \subset U$ and

$V \neq U$. Now, there exists $a \in U$ such that $a \notin V$. That is, (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\{a\}) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(X\backslash V) = X\backslash V$, as $X\backslash V \in \omega c(\tau_1,\tau_2)$. Then (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\{a\}) \neq (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(U)$, a contradiction for (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\{a\}) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(U)$ for any $\{a\}(\neq \emptyset) \subset U$. Therefore, $U \in \omega^- c(\tau_1,\tau_2)$. ☐

Theorem 3.5. *If V is a nonempty finite u-ω-closed subset of (X,τ_1,τ_2), then there exists at least one (finite) $U \in \omega^- c(\tau_1,\tau_2)$ such that $U \subset V$.*

Proof. If $V \in \omega^- c(\tau_1,\tau_2)$, we may set $U = V$. If $V \notin \omega^- c(\tau_1,\tau_2)$, then there exists (finite) $V_1 \in \omega c(\tau_1,\tau_2)$ such that $\emptyset \neq V_1 \subset V$. If $V_1 \in \omega^- c(\tau_1,\tau_2)$, we may set $U = V_1$. If $V_1 \notin \omega^- c(\tau_1,\tau_2)$, then there exists (finite) $V_2 \in \omega c(\tau_1,\tau_2)$ such that $\emptyset \neq V_2 \subset V_1$. Continuing this process, we have a sequence of u-ω-closed sets $V \supset V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_k \supset \cdots$. Since V is a finite set, this process repeats only finitely many time and finally we get a minimal u-ω-closed set $U = V_n$ for some positive integer n. ☐

Theorem 3.6. *Let $U, U_\alpha \in \omega^- c(\tau_1,\tau_2)$ for any element $\alpha \in \Delta$. If $U \subset \bigcup_{\alpha\in\Delta} U_\alpha$, then there exists $\alpha \in \Delta$ such that $U = U_\alpha$.*

Proof. Let $U \subset \bigcup_{\alpha\in\Delta} U_\alpha$. Then $U \cap (\bigcup_{\alpha\in\Delta} U_\alpha) = U$. That is $\bigcup_{\alpha\in\Delta} (U \cap U_\alpha) = U$. Also by Lemma 3.1 (2), $U \cap U_\alpha = \emptyset$ or $U = U_\alpha$ for any $\alpha \in \Delta$. Then there exists $\alpha \in \Delta$ such that $U = U_\alpha$. ☐

Theorem 3.7. *Let $U, U_\alpha \in \omega^- c(\tau_1,\tau_2)$ for any $\alpha \in \Delta$. If $U \neq U_\alpha$ for any $\alpha \in \Delta$, then $(\bigcup_{\alpha\in\Delta} U_\alpha) \cap U = \emptyset$.*

Proof. Suppose that $(\bigcup_{\alpha\in\Delta} U_\alpha) \cap U \neq \emptyset$. Then there exists $\alpha \in \Delta$ such that $U \cap U_\alpha \neq \emptyset$. By Lemma 3.1 (2), we have $U = U_\alpha$, which contradicts the fact that $U \neq U_\alpha$ for any $\alpha \in \Delta$. Hence $(\bigcup_{\alpha\in\Delta} U_\alpha) \cap U = \emptyset$. ☐

Lemma 3.2. *For the subsets A and B of X, we have the following:*

(1) *If $A \in \omega^+(\tau_1,\tau_2)$ and $B \in \omega(\tau_1,\tau_2)$, then $A \cup B = X$ or $B \subset A$.*
(2) *Let $A, B \in \omega^+(\tau_1,\tau_2)$, then $A \cup B = X$ or $A = B$.*

Proof. (1). If $A \cup B = X$, then there is nothing to prove. If $A \cup B \neq X$, then $A \cup B \in \omega(\tau_1,\tau_2)$ such that $A \subset A \cup B$. Then $A \cup B = A$. Hence $B \subset A$.
(2). If $A \cup B \neq X$, then $A \cup B \in \omega(\tau_1,\tau_2)$ such that A, $B \subset A \cup B$, that is, $A \cup B = A$ and $A \cup B = B$. Hence $A = B$. ☐

Theorem 3.8. *Let $F \in \omega^+(\tau_1,\tau_2)$. If $x \in F$, then $S \subset F$ for some $S \in \omega(\tau_1,\tau_2,x)$.*

Proof. Similar to the proof of Theorem 3.2. ☐

Theorem 3.9. *Let $A, B, C \in \omega^+(\tau_1,\tau_2)$ such that $A \neq B$. If $A \cap B \subset C$, then either $A = C$ or $B = C$.*

Proof. If $A = C$, then there is nothing to prove. If $A \neq C$, then we have to prove $B = C$. Now $B \cap C = B \cap (C \cap X) = B \cap (C \cap (A \cup B)$ (by Theorem 3.2 (2)) $= B \cap ((C \cap A) \cup (C \cap B)) = (B \cap C \cap A) \cup (B \cap C) = (A \cap B) \cup (C \cap B) = (A \cup C) \cap B = X \cap B = B$ (Since $A, C \in \omega^+(\tau_1, \tau_2)$ by Theorem 3.2 (2), $A \cup C = X$). That is, $B \cap C = B \Rightarrow B \subset C$. Since $B, C \in \omega^+(\tau_1, \tau_2)$, $B = C$. Hence $B = C$. □

Theorem 3.10. *If $A, B, C \in \omega^+(\tau_1, \tau_2)$ which are different from each other, then $(A \cap B) \not\subseteq (A \cap C)$.*

Proof. Let $A \cap B \subset A \cap C$. Then $(A \cap B) \cup (C \cap B) \subset (A \cap C) \cup (C \cap B)$. That is, $(A \cup C) \cap B \subset C \cap (A \cup B)$. By Theorem 3.2 (2), $A \cup C = X = A \cup B$. Hence $X \cap B \subset C \cap X \Rightarrow B \subset C$. Thus from the definition of maximal u-ω-open set, we have $B = C$, a contradiction to the fact that A, B and C are different to each other. Therefore, $(A \cap B) \not\subseteq (A \cap C)$. □

Theorem 3.11. *If $F \in \omega^+(\tau_1, \tau_2)$ and $x \in F$, then $F = \cup\{S : S \in \omega(\tau_1, \tau_2, x)$ such that $F \cup S \neq X\}$.*

Proof. Similar to the proof of Theorem 3.3. □

We call a set cofinite if its complement is finite.

Theorem 3.12. *If F is a proper nonempty cofinite u-ω-open set, then there exists (cofinite) $E \in \omega^+(\tau_1, \tau_2)$ such that $F \subset E$.*

Proof. If $F \in \omega^+(\tau_1, \tau_2)$, we may set $E = F$. If $F \notin \omega^+(\tau_1, \tau_2)$, then there exists (cofinite) $F_1 \in \omega(\tau_1, \tau_2)$ such that $F \subset F_1 \neq X$. If $F_1 \in \omega^+(\tau_1, \tau_2)$, we may set $E = F_1$. If $F_1 \notin \omega^+(\tau_1, \tau_2)$, then there exists (cofinite) $F_2 \in \omega(\tau_1, \tau_2)$ such that $F \subset F_1 \subset F_2 (\neq X)$. Continuing this process, we have a sequence of u-ω-open sets such that $F \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots$. Since F is cofinite, this process repeats only finitely many times and finally we get a maximal u-ω-open set $E = F$. □

Theorem 3.13. *For a bitopological space (X, τ_1, τ_2), we have the following:*

1. *If $A \in \omega^+(\tau_1, \tau_2)$ and $x \in X \backslash A$, then $X \backslash A \subset B$ for any $B \in \omega(\tau_1, \tau_2, x)$.*
2. *If $A \in \omega^+(\tau_1, \tau_2)$, then either of the following (i) or (ii) holds:*
    - *(i) For each $x \in X \backslash A$ and each $B \in \omega(\tau_1, \tau_2, x)$, $B = X$.*
    - *(ii) There exists $B \in \omega(\tau_1, \tau_2)$ such that $X \backslash A \subset B$.*
3. *If $A \in \omega^+(\tau_1, \tau_2)$, then either of the following (i) or (ii) holds:*
    - *(i) For each $x \in X \backslash A$ and each $B \in \omega(\tau_1, \tau_2, x)$, $X \backslash A \subset B$.*
    - *(ii) There exists $B \in \omega(\tau_1, \tau_2)$ such that $X \backslash A = B$.*

Proof. (1). Since $x \in X \backslash A$, $B \not\subseteq A$ for any $B \in \omega(\tau_1, \tau_2, x)$. Then by Theorem 3.2 (1), $A \cup B = X \Rightarrow X \backslash A \subset B$.

(2). If (i) holds, we are done. Let (i) do not hold. Then there exist $x \in X \backslash A$ and $B \in \omega(\tau_1, \tau_2, x)$ such that $B \subset X$. Then by Theorem 3.2 (1), $A \cup B = X$ or $B \subset A$. But $B \not\subseteq A \Rightarrow A \cup B = X \Rightarrow X \backslash A \subset B$.

(3). If (ii) holds, we are done. Let (ii) do not hold. Then (by (i)) for each $x \in X\backslash A$ and each $B \in \omega(\tau_1, \tau_2, x)$, $X\backslash A \subset B$. Hence by assumption $X\backslash A \subset B$. □

Theorem 3.14. *If* $A \in \omega^+(\tau_1, \tau_2)$*, then either* (τ_1, τ_2)*-*$\omega\,\mathrm{Cl}(A) = X$ *or* (τ_1, τ_2)*-*$\omega\,\mathrm{Cl}(A) = A$*.*

Proof. Since $A \in \omega^+(\tau_1, \tau_2)$, only the following cases (i) and (ii) occur by Theorem 3.13 (3).

(i). For each $x \in X$ and $x \in X\backslash A$ and each $B \in \omega(\tau_1, \tau_2, x)$, we have $X\backslash A \subset B$. Let $x \in X\backslash A$ and $B \in \omega(\tau_1, \tau_2, x)$. Since $X\backslash A \neq B$, $B \cap A \neq \emptyset$ and $X\backslash A \subset (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(A)$. Since $X = A \cup (X\backslash A) \subset A \cup (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(A) = (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(A) \subset X$, $X = (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(A)$.

(ii). There exists $B \in \omega(\tau_1, \tau_2)$ such that $X\backslash A = B(\neq X)$. Since $X\backslash A = B$, $A \in \omega c(\tau_1, \tau_2) \Rightarrow (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(A) = A$.

□

Theorem 3.15. *If* $A \in \omega^+(\tau_1, \tau_2)$*, then either* (τ_1, τ_2)*-*$\omega\,\mathrm{Int}(X\backslash A) = X\backslash A$ *or* (τ_1, τ_2)*-*$\omega\,\mathrm{Int}(X\backslash A) = \emptyset$*.*

Proof. By Theorem 3.14, we have (τ_1, τ_2)-$\omega\,\mathrm{Cl}(A) = A$ or (τ_1, τ_2)-$\omega\,\mathrm{Cl}(A) = X$. That is, (τ_1, τ_2)-$\omega\,\mathrm{Int}(X\backslash A) = X\backslash A$ or (τ_1, τ_2)-$\omega\,\mathrm{Int}(X\backslash A)$
$= \emptyset$. □

Theorem 3.16. *If* $A \in \omega^+(\tau_1, \tau_2)$ *and* $\emptyset \neq B \subset X\backslash A$*, then* (τ_1, τ_2)*-*$\omega\,\mathrm{Cl}(B) = X\backslash A$*.*

Proof. Since $\emptyset \neq B \subset X\backslash A$, $W \cap B \neq \emptyset$ for any element $x \in X\backslash A$ and any $W \in \omega(\tau_1, \tau_2, x)$, by Theorem 3.13 (1). Thus, $X\backslash A \subset (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(B)$. Since $X\backslash A \in \omega c(\tau_1, \tau_2)$ and $B \subset X\backslash A$, we have (τ_1, τ_2)-$\omega\,\mathrm{Cl}(B) \subset X\backslash A$. □

Corollary 3.1. *If* $A \in \omega^+(\tau_1, \tau_2)$ *and* $A \subset B$*, then* (τ_1, τ_2)*-*$\omega\,\mathrm{Cl}(B) = X$*.*

Proof. The proof follows from Theorem 3.14. □

Theorem 3.17. *If* $A \in \omega^+(\tau_1, \tau_2)$ *and* $X\backslash A$ *have at least two elements, then* (τ_1, τ_2)*-*$\omega\,\mathrm{Cl}(X\backslash\{a\}) = X$ *for any* $a \in X\backslash A$*.*

Proof. As $A \subset X\backslash\{a\}$, we have, by Corollary 3.1, (τ_1, τ_2)-$\omega\,\mathrm{Cl}(X\backslash\{a\}) = X$. □

Theorem 3.18. *If* $A \in \omega^+(\tau_1, \tau_2)$ *and* $\emptyset \neq G \subset X$ *with* $A \subset G$*, then* (τ_1, τ_2)*-*$\omega\,\mathrm{Int}(G) = A$*.*

Proof. If $G = A$, then (τ_1, τ_2)-$\omega\,\mathrm{Int}(G) = (\tau_1, \tau_2)$-$\omega\,\mathrm{Int}(A) = A$. If $G \neq A$, then $A \subset G$. Thus $A \subset (\tau_1, \tau_2)$-$\omega\,\mathrm{Int}(G)$. Since $A \in \omega^+(\tau_1, \tau_2)$, (τ_1, τ_2)-$\omega\,\mathrm{Int}(G) \subset A$. Hence (τ_1, τ_2)-$\omega\,\mathrm{Int}(G) = A$. □

Theorem 3.19. *If* $A \in \omega^+(\tau_1, \tau_2)$ *and* $F \subset X\backslash A$*, then* $X\backslash(\tau_1, \tau_2)$*-*$\omega\,\mathrm{Cl}(F) = A$*.*

Proof. Since $A \subset X\backslash F \subset X$, by our assumption and by Theorem **??**, $X\backslash(\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(F) = A$. □

4. Basic properties of u-ω-radical

In this section, we study some fundamental properties of radical of maximal u-ω-open sets. We establish a very useful decomposition theorem for a maximal u-ω-open sets.

Definition 4.1. *Let* $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\} \in \omega^+(\tau_1, \tau_2)$. *Then* $\cap\mathcal{U} = \bigcap_{\alpha\in\Delta} U_\alpha$ *is called the u-ω-radical of* $\mathcal{U}$.

Theorem 4.1. *Suppose that* $|\Delta| \geq 2$. *Let* $U_\alpha \in \omega^+(\tau_1, \tau_2)$ *for any* $\alpha \in \Delta$ *and* $U_\alpha \neq U_\beta$, *for any* $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$. *If* $\beta \in \Delta$, *then the following hold:*

1. $X \backslash \bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha \subseteq U_\beta$.
2. $\bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha \neq \emptyset$.

Proof. (1). By Lemma 3.2 (2), we have $X \backslash U_\beta \subseteq U_\alpha$ for any $\alpha \in \Delta$ with $\alpha \neq \beta$. Then $X\backslash U_\beta \subseteq \bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha$. Therefore, $X \backslash \bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha \subseteq U_\beta$.
(2). If $\bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha = \emptyset$. By (1), we have $X = U_\beta$, a contradiction to our supposition that $U_\alpha \in \omega^+(\tau_1, \tau_2)$. Therefore, $\bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha \neq \emptyset$. ☐

Corollary 4.1. *Let* $U_\alpha \in \omega^+(\tau_1, \tau_2)$ *for any* $\alpha \in \Delta$ *and* $U_\alpha \neq U_\beta$ *for any* $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$. *If* $|\Delta| \geq 2$, *then* $U_\alpha \cap U_\beta \neq \emptyset$ *for any* $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$.

Proof. The proof follows from Theorem 4.1 (2). ☐

Theorem 4.2. *Let* $U_\alpha \in \omega^+(\tau_1, \tau_2)$ *for any* $\alpha \in \Delta$ *and* $U_\alpha \neq U_\beta$ *for any* $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$. *Assume that* $|\Delta| \geq 2$. *If* $\beta \in \Delta$, *then* $\bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha \nsubseteq U_\beta \nsubseteq \bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha$.

Proof. If $\bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha \subseteq U_\beta$. Then by Theorem 4.1 (2), we have $X = (X \backslash \bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha) \cup (\bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha) \subseteq U_\alpha$, a contradiction. If $U_\beta \subseteq \bigcap_{\alpha\in\Delta\backslash\{\beta\}} U_\alpha$, then $U_\beta \subseteq U_\alpha$ and $U_\beta = U_\alpha$ for any element $\alpha \in (\Delta\backslash\{\beta\})$. This contradicts our assumption that $U_\beta \neq U_\alpha$ when $\alpha \neq \beta$. ☐

Corollary 4.2. *Let* $U_\alpha \in \omega^+(\tau_1, \tau_2)$ *for any* $\alpha \in \Delta$ *and* $U_\alpha \neq U_\beta$ *for any* $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$. *If* $\emptyset \neq \Delta^\star \subseteq \Delta$, *then* $\bigcap_{\alpha\in\Delta\backslash\Delta^\star} U_\alpha \nsubseteq \bigcap_{\iota\in\Delta^\star} U_\iota \nsubseteq \bigcap_{\alpha\in\Delta\backslash\Delta^\star} U_\alpha$.

Proof. Let $\iota \in \Delta^\star$. By Theorem 4.2, $\bigcap_{\alpha\in\Delta\backslash\Delta^\star} U_\alpha = \bigcap_{\alpha\in\Delta\backslash(\Delta^\star\cup\{\iota\})} U_\alpha \nsubseteq U_i$. Then $\bigcap_{\alpha\in\Delta\backslash\Delta^\star} U_\alpha \nsubseteq \bigcap_{\iota\in\Delta^\star} U_\iota$. On the other hand, since $\bigcap_{\iota\in\Delta^\star} U_\iota = \bigcap_{\iota\in\Delta\backslash(\Delta\backslash\delta^\star)} U_\iota \nsubseteq \bigcap_{\alpha\in\Delta\backslash\Delta^\star} U_\alpha$, we have $\bigcap_{\iota\in\Delta^\star} U_\iota \nsubseteq \bigcap_{\alpha\in\Delta\backslash\Delta^\star} U_\alpha$. ☐

Theorem 4.3. *Let* $U_\alpha \in \omega^+(\tau_1, \tau_2)$ *for any* $\alpha \in \Delta$ *and* $U_\alpha \neq U_\beta$ *for any* $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$. *If* $\emptyset \neq \Delta^\star \subseteq \Delta$, *then* $\bigcap_{\alpha\in\Delta} U_\alpha \nsubseteq \bigcap_{\iota\in\Delta^\star} U_\iota$.

Proof. By Corollary 4.2, we have $\bigcap\limits_{\alpha\in\Delta} U_\alpha = (\bigcap\limits_{\alpha\in\Delta\setminus\Delta^\star} U_\alpha)\cap(\bigcap\limits_{\iota\in\Delta} U_\iota) \nsubseteq \bigcap\limits_{\iota\in\Delta} U_\iota$. ☐

The following theorem shows the useful decomposition theorem for a maximal u-ω-open sets.

Theorem 4.4. *Let $|\Delta| \geq 2$. If $U_\alpha \in \omega^+(\tau_1,\tau_2)$ for any $\alpha\in\Delta$ and $U_\alpha \neq U_\beta$ for any $\alpha,\beta\in\Delta$ with $\alpha\neq\beta$. Then for any $\beta\in\Delta$, $U_\beta = (\bigcap\limits_{\alpha\in\Delta} U_\alpha)\cup(X\setminus \bigcap\limits_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha)$.*

Proof. Let $\beta\in\Delta$. By Theorem 4.1 (1), we have

$$\begin{aligned}
(\bigcap_{\alpha\in\Delta} U_\alpha)\cup(X\setminus \bigcap_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha) &= ((\bigcap_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha)\cap U_\beta)\cup(X\setminus \bigcap_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha)\\
&= ((\bigcap_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha)\cup(X\setminus \bigcap_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha))\\
&\quad \cap(U_\beta\cup(X\setminus \bigcap_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha))\\
&= U_\beta\cup(X\setminus \bigcap_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha)\\
&= U_\beta.
\end{aligned}$$

Therefore, $U_\beta = (\bigcap\limits_{\alpha\in\Delta} U_\alpha)\cup(X\setminus \bigcap\limits_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha)$. ☐

Theorem 4.5. *Let Δ be a finite set and $U_\alpha\in\omega^+(\tau_1,\tau_2)$ for any $\alpha\in\Delta$ and $U_\alpha\neq U_\beta$ for any $\alpha,\beta\in\Delta$ with $\alpha\neq\beta$. If $\bigcap\limits_{\alpha\in\Delta} U_\alpha \in \omega c(\tau_1,\tau_2)$, then $U_\alpha\in\omega c(\tau_1,\tau_2)$ for any $\alpha\in\Delta$.*

Proof. By Theorem 4.4, we have $U_\beta = (\bigcap\limits_{\alpha\in\Delta} U_\alpha)\cup(X\setminus \bigcap\limits_{\alpha\in\Delta\setminus\{\beta\}} U_\alpha) = (\bigcap\limits_{\alpha\in\Delta} U_\alpha)\cup(\bigcup\limits_{\alpha\in\Delta\setminus\{\beta\}}(X\setminus U_\alpha))$. Since Δ is finite, $\bigcup\limits_{\alpha\in\Delta\setminus\{\beta\}} X\setminus U_\alpha \in \omega c(\tau_1,\tau_2)$. Hence $U_\alpha\in\omega c(\tau_1,\tau_2)$. ☐

Theorem 4.6. *Assume that $|\Delta|\geq 2$. Let $U_\alpha\in\omega^+(\tau_1,\tau_2)$ for any $\alpha\in\Delta$ and $U_\alpha\neq U_\beta$ for any $\alpha,\beta\in\Delta$ with $\alpha\neq\beta$. If $\bigcap\limits_{\alpha\in\Delta} U_\alpha=\emptyset$, then $\{U_\alpha : \alpha\in\Delta\}\in\omega^+(\tau_1,\tau_2)$.*

Proof. If there exists $U_\upsilon\in\omega^+(\tau_1,\tau_2)$, which is not equal to U_α for any $\alpha\in\Delta$, then $\emptyset = \bigcap\limits_{\alpha\in\Delta} U_\alpha = \bigcap\limits_{\alpha\in(\Delta\cup\{\upsilon\})\setminus\{\upsilon\}} U_\alpha$. By Theorem 4.1 (2), $\bigcap\limits_{\alpha\in(\Delta\cup\{\upsilon\})\setminus\{\upsilon\}} U_\alpha\neq\emptyset$, a contradiction. ☐

Proposition 4.1. *If (τ_1,τ_2)-ω $\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta} U_\alpha)=X$, then (τ_1,τ_2)-ω $\mathrm{Cl}(U_\alpha)=X$ for any $\alpha\in\Delta$.*

Proof. We see that $X=(\tau_1,\tau_2)$-ω $\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta} U_\alpha)\subseteq(\tau_1,\tau_2)$-$\omega$ $\mathrm{Cl}(U_\alpha)$. Then (τ_1,τ_2)-ω $\mathrm{Cl}(U_\alpha)=X$ for any $\alpha\in\Delta$. ☐

Theorem 4.7. *Let Δ be a finite set and $U_\alpha\in\omega^+(\tau_1,\tau_2)$ for any $\alpha\in\Delta$. If (τ_1,τ_2)-ω $\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta} U_\alpha)\neq X$, then there exists $\alpha\in\Delta$ such that (τ_1,τ_2)-ω $\mathrm{Cl}(U_\alpha)=U_\alpha$.*

Proof. Suppose that (τ_1,τ_2)-$\omega\,\mathrm{Cl}(U_\alpha)=X$ for any $\alpha\in\Delta$. Let $\beta\in\Delta$. Then $\bigcap\limits_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha\in\omega(\tau_1,\tau_2)$. Also

$$\begin{aligned}(\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(\bigcap_{\alpha\in\Delta}U_\alpha) &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}((\bigcap_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha)\cap U_\beta)\\ &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(\bigcap_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha)\cap(\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(U_\beta)\\ &\supseteq \bigcap_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha\cap(\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(U_\beta)\\ &= \bigcap_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha\cap X\\ &= \bigcap_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha.\end{aligned}$$

So (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha)\subseteq(\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}((\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha))=(\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)$. On the other hand, $\bigcap\limits_{\alpha\in\Delta}U_\alpha\subseteq\bigcap\limits_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha$. It follows that (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)=(\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta\setminus\{\beta\}}U_\alpha)$. Then by induction on the element of Δ, (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)\neq X$. Then there exists $\alpha\in\Delta$ such that (τ_1,τ_2)-$\omega\,\mathrm{Cl}(U_\alpha)=U_\alpha$. □

Theorem 4.8. *Let* Δ *be finite and* $U_\alpha\in\omega^+(\tau_1,\tau_2)$ *for each* $\alpha\in\Delta$. *Let* $\Delta^\star\subseteq\Delta$ *such that*

$$(\tau_1,\tau_2)-\omega\,\mathrm{Cl}(U_\alpha)=\begin{cases}U_\alpha & \text{for any }\alpha\in\Delta^\star,\\ X & \text{for any }\alpha\in\Delta\setminus\Delta^\star.\end{cases}$$

Then (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)=\bigcap\limits_{\alpha\in\Delta}U_\alpha(=X,$ *if* $\Delta^\star=\emptyset)$.

Proof. If $\Delta=\emptyset$, then we have the result by Theorem 4.7. Otherwise, $\Delta\neq\emptyset$, and

$$\begin{aligned}(\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(\bigcap_{\alpha\in\Delta}U_\alpha) &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}((\bigcap_{\alpha\in\Delta^\star}U_\alpha)\bigcap(\bigcap_{\alpha\in\Delta\setminus\Delta^\star}U_\alpha))\\ &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(\bigcap_{\alpha\in\Delta^\star}U_\alpha)\cap(\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(\bigcap_{\alpha\in\Delta\setminus\Delta^\star}U_\alpha)\\ &\supseteq (\bigcap_{\alpha\in\Delta^\star}U_\alpha)\cap(\tau_1,\tau_2)\text{-}\omega\,\mathrm{Cl}(\bigcap_{\alpha\in\Delta\setminus\Delta^\star}U_\alpha)\\ &= \bigcap_{\alpha\in\Delta^\star}U_\alpha\cap X\\ &= \bigcap_{\alpha\in\Delta^\star}U_\alpha.\end{aligned}$$

By Theorem 4.7 and the fact that $\bigcap\limits_{\alpha\in\Delta^\star}U_\alpha\in\omega(\tau_1,\tau_2)$. Hence (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)=(\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}((\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha))\supseteq(\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta^\star}U_\alpha)$. On the other hand, we see that $\bigcap\limits_{\alpha\in\Delta}U_\alpha\subseteq\bigcap\limits_{\alpha\in\Delta^\star}U_\alpha$, and hence (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)\subseteq(\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta^\star}U_\alpha)$. It follows that (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)=(\tau_1,\tau_2)$-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta^\star}U_\alpha)$. The *u*-$\omega$-radical $\bigcap\limits_{\alpha\in\alpha}U_\alpha\in\omega c(\tau_1,\tau_2)$ since $U_\alpha\in\omega c(\tau_1,\tau_2)$ for any $\alpha\in\Delta^\star$ by our assumption. Therefore, (τ_1,τ_2)-$\omega\,\mathrm{Cl}(\bigcap\limits_{\alpha\in\Delta}U_\alpha)=\bigcap\limits_{\alpha\in\Delta^\star}U_\alpha$. □

5. Bitopological u-ω-paraopen / u-ω-paraclosed sets

In this section, a new class of sets called paraopen sets and paraclosed sets in bitopological spaces are introduced and studied. Some properties of the new concepts have been studied.

Definition 5.1. *An u-ω-open set A is said to be u-ω-paraopen if it is neither u-ω-minimal open nor u-ω-maximal open. The collection of all u-ω-paraopen sets of X is denoted by $\omega po(\tau_1, \tau_2)$.*

Definition 5.2. *An u-ω-closed A is said to be an u-ω-paraclosed set if its complement is an u-ω-paraopen set. The collection of all u-ω-paraclosed sets of X is denoted by $\omega pc(\tau_1, \tau_2)$.*

Example 5.1. *From Example 3.1, the set $B = \mathbb{R} \setminus \mathbb{Q}$ is u-ω-paraopen, and the set $A = \mathbb{Q}$ is a u-ω-paraclosed.*

Now we describe some properties of the u-ω-paraopen and u-ω-paraclosed sets.

Proposition 5.1. *If $A \in \omega po(\tau_1, \tau_2)$ such that $A \neq \emptyset$, then there exists $B \in \omega^-(\tau_1, \tau_2)$ such that $B \subset A$.*

Proof. It is evident that $B \subset A$, by the definition of u-ω-minimal open set. □

Proposition 5.2. *If $A \in \omega po(\tau_1, \tau_2)$ such that $A \neq \emptyset$, then there exists $B \in \omega^+(\tau_1, \tau_2)$ such that $A \subset B$.*

Proof. It is apparent that $A \subset B$ by the definition of u-ω-maximal open set. □

Proposition 5.3. *If $A \in \omega po(\tau_1, \tau_2)$ and $B \in \omega^-(\tau_1, \tau_2)$, then $A \cap B = \emptyset$ or $B \subset A$.*

Proof. Since $A \in \omega po(\tau_1, \tau_2)$ and $B \in \omega^-(\tau_1, \tau_2)$, $A \cap B = \emptyset$ or $A \cap B \neq \emptyset$. If $A \cap B = \emptyset$, then there is nothing to prove. If $A \cap B \neq \emptyset$, then $A \cap B \in \omega(\tau_1, \tau_2)$ and $A \cap B \subseteq B$. Since $B \in \omega^-(\tau_1, \tau_2)$, $A \cap B = B$ which implies $B \subset A$. □

Proposition 5.4. *If $A \in \omega po(\tau_1, \tau_2)$ and $B \in \omega^+(\tau_1, \tau_2)$, then $A \cap B = \emptyset$ or $A \subset B$.*

Proof. Since $A \in \omega po(\tau_1, \tau_2)$ and $B \in \omega^+(\tau_1, \tau_2)$, $A \cap B = \emptyset$ or $A \cap B \neq \emptyset$. If $A \cap B = \emptyset$, then there is nothing to prove. If $A \cap B \neq \emptyset$, then $A \cap B \in \omega(\tau_1, \tau_2)$ and $B \subseteq A \cap B$. Since $B \in \omega^+(\tau_1, \tau_2)$, $A \cap B = B$, which implies $A \subset B$. □

Proposition 5.5. *If $A_1, A_2 \in \omega po(\tau_1, \tau_2)$, then $A_1 \cap A_2 \in \omega po(\tau_1, \tau_2) \cup \omega^-(\tau_1, \tau_2)$.*

Proof. Let $A_1 \cap A_2 \in \omega po(\tau_1, \tau_2)$. If $A_1 \cap A_2 \in \omega po(\tau_1, \tau_2)$, then there is nothing to prove. If $A_1 \cap A_2 \notin \omega po(\tau_1, \tau_2)$, then $A_1 \cap A_2 \in \omega po(\tau_1, \tau_2) \cup \omega^+(\tau_1, \tau_2)$. If $A_1 \cap A_2 \in \omega^+(\tau_1, \tau_2)$, then there is nothing to prove. So, let $A_1 \cap A_2 \in \omega^+(\tau_1, \tau_2)$. Now $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$, a contradiction, since $A_1, A_2 \in \omega po(\tau_1, \tau_2)$. Hence $A_1 \cap A_2 \notin \omega^+(\tau_1, \tau_2)$. Hence $A_1 \cap A_2 \in \omega^-(\tau_1, \tau_2)$. □

Proposition 5.6. *If* $A_1, A_2 \in \omega po(\tau_1, \tau_2)$*, then* $A_1 \cup A_2 \in \omega po(\tau_1, \tau_2) \cup \omega^+(\tau_1, \tau_2)$*.*

Proof. Let $A_1, A_2 \in \omega po(\tau_1, \tau_2)$. If $A_1 \cup A_2 \in \omega po(\tau_1, \tau_2)$, then there is nothing to prove. If $A_1 \cup A_2 \notin \omega po(\tau_1, \tau_2)$, then $A_1 \cup A_2 \in \omega^-(\tau_1, \tau_2) \cup \omega^+(\tau_1, \tau_2)$. If $A_1 \cup A_2 \in \omega^+(\tau_1, \tau_2)$, then there is nothing to prove. So, let $A_1 \cup A_2 \in \omega^-(\tau_1, \tau_2)$. Now $A_1 \subseteq A_1 \cup A_2$ and $A_2 \subseteq A_1 \cup A_2$, a contradiction to the fact that $A_1, A_2 \in \omega po(\tau_1, \tau_2)$. Hence $A_1 \cup A_2 \notin \omega^-(\tau_1, \tau_2)$. Hence $A_1 \cup A_2 \in \omega^+(\tau_1, \tau_2)$. ☐

Proposition 5.7. *For a subset A of X,* $A \in \omega pc(\tau_1, \tau_2) \Leftrightarrow A \notin \omega^+c(\tau_1, \tau_2) \cap \omega^-c(\tau_1, \tau_2)$*.*

Proof. It is apparent from the facts that the complement of an u-ω-minimal open set is a u-ω-maximal closed set and the complement of an u-ω-maximal open set is a u-ω-minimal closed set. ☐

Proposition 5.8. *If* $\emptyset \neq A \in \omega pc(\tau_1, \tau_2)$*, then there exists* $B \in \omega^-c(\tau_1, \tau_2)$ *such that* $B \subset A$*.*

Proof. It is evident that $B \subset A$, by the definition of u-ω-minimal closed set. ☐

Proposition 5.9. *If* $A \in \omega pc(\tau_1, \tau_2)$*, then there exists* $B \in \omega^+c(\tau_1, \tau_2)$ *such that* $A \subset B$*.*

Proof. It is apparent that $A \subset B$, by the definition of u-ω-maximal closed set. ☐

Proposition 5.10. *If* $\emptyset \neq C \in \omega pc(\tau_1, \tau_2)$*, then there exist* $A, B(\neq C) \in \omega c(\tau_1, \tau_2)$ *such that* $A \subset C \subset B$*.*

Proof. Follows from the respective Definition. ☐

Proposition 5.11. *If* $A \in \omega pc(\tau_1, \tau_2)$ *and* $B \in \omega^-c(\tau_1, \tau_2)$*, then* $A \cap B = \emptyset$ *or* $B \subset A$*.*

Proof. Let $A \in \omega pc(\tau_1, \tau_2)$ and $B \in \omega^-c(\tau_1, \tau_2)$. Then $X \backslash A \in \omega po(\tau_1, \tau_2)$ and $X \backslash B \in \omega^+(\tau_1, \tau_2)$. Then $X \backslash A \cup X \backslash B = X$ or $B \subset A$. Hence $A \cap B = \emptyset$ or $B \subset A$. ☐

Proposition 5.12. *If* $A \in \omega pc(\tau_1, \tau_2)$ *and* $B \in \omega^+c(\tau_1, \tau_2)$*, then* $A \cup B = X$ *or* $A \subset B$*.*

Proof. Since $A \in \omega pc(\tau_1, \tau_2)$ and $B \in \omega^+c(\tau_1, \tau_2)$, $X \backslash A \in \omega po(\tau_1, \tau_2)$ and $X \backslash B \in \omega^-(\tau_1, \tau_2)$. Then $(X \backslash A) \cap (X \backslash B) = \emptyset$ or $X \backslash B \subset X \backslash A$, which implies that $X \backslash (A \cup B) = \emptyset$ or $A \subset B$. Hence $A \cup B = X$ or $A \subset B$. ☐

Proposition 5.13. *If* $A_1, A_2 \in \omega pc(\tau_1, \tau_2)$*, then* $A_1 \cap A_2 \in \omega pc(\tau_1, \tau_2) \cup \omega^-c(\tau_1, \tau_2)$*.*

Proof. Let $A_1, A_2 \in \omega pc(\tau_1, \tau_2)$. If $A_1 \cap A_2 \in \omega pc(\tau_1, \tau_2)$, then there is nothing to prove. If $A_1 \cap A_2 \notin \omega pc(\tau_1, \tau_2)$, then $A_1 \cap A_2 \in \omega^-c(\tau_1, \tau_2) \cup \omega^+c(\tau_1, \tau_2)$. If $A_1 \cap A_2 \in \omega^-c(\tau_1, \tau_2)$, then there is nothing to prove. Suppose $A_1 \cap A_2 \in \omega^+(\tau_1, \tau_2)$. Now $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$ which is a contradiction to the fact that $A_1, A_2 \in \omega pc(\tau_1, \tau_2)$. Hence $A_1 \cap A_2 \notin \omega^+(\tau_1, \tau_2)$. Hence $A_1 \cap A_2 \in \omega^-c(\tau_1, \tau_2)$. ☐

Proposition 5.14. *If* $A_1, A_2 \in \omega pc(\tau_1, \tau_2)$*, then* $A_1 \cup A_2 \in \omega pc(\tau_1, \tau_2) \cup \omega^+c(\tau_1, \tau_2)$*.*

Proof. Similar to the Proposition 5.13. ☐

6. Bitopological u-ω-mean open / u-ω-closed sets

In this section, we introduce and study the concept of u-ω-mean open sets and u-ω-mean closed sets in bitopological spaces.

Definition 6.1. *An u-ω-open set A is said to be u-ω-mean open if there exist two distinct proper u-ω-open sets $A_1, A_2(\neq A)$ such that $A_1 \subset A \subset A_2$. The collection of all u-ω-mean open sets of X is denoted by $\overline{\omega}(\tau_1, \tau_2)$.*

Definition 6.2. *An u-ω-closed set A is said to be u-ω-mean closed if there exist two distinct proper u-ω-closed sets $A_1, A_2(\neq A)$ such that $A_1 \subset A \subset A_2$. The collection of all u-ω-mean closed sets of X is denoted by $\overline{\omega}c(\tau_1, \tau_2)$.*

Example 6.1. *From Example 3.1, the set $B = \mathbb{R} \setminus \mathbb{Q}$ is u-ω-mean open, and the set $A = \mathbb{Q}$ is a u-ω-mean closed. For the first case, take $B_1 = B \cup \{\alpha_0, \alpha_0 \in \mathbb{Q}\}$ and $B_2 = B \setminus \{\beta_0, \beta_0 \in \mathbb{R} \setminus \mathbb{Q}\}$.*

Theorem 6.1. *An u-ω-open set of X is u-ω-mean open if and only if its complement is u-ω-mean closed.*

Proof. Let $B \in \overline{\omega}(\tau_1, \tau_2)$. Then $A_1(\neq \emptyset), B, A_2(\neq B), X \in \omega(\tau_1, \tau_2)$ such that $A_1 \subset B \subset A_2$ and so $X \backslash A_2 \subset X \backslash B \subset X \backslash A_1$. Since $X \backslash A_2 \neq \emptyset, X \backslash B$ and $X \backslash A_1 \neq X \backslash B, X$; hence $X \backslash B \in \overline{\omega}c(\tau_1, \tau_2)$. Conversely, let $B \in \omega(\tau_1, \tau_2)$ such that $X \backslash B \in \overline{\omega}c(\tau_1, \tau_2)$. Hence there exist $C_1 \neq \emptyset, X \backslash B, C_2 \neq X \backslash B, X \in \omega c(\tau_1, \tau_2)$ such that $C_1 \subset X \backslash B \subset C_2$. Then $X \backslash C_2 \subset B \subset X \backslash C_1$. Since $X \backslash C_2 \neq \emptyset, B$ and $X \backslash C_1 \neq B, X$ and hence $B \in \overline{\omega}(\tau_1, \tau_2)$. □

Theorem 6.2. (1) *A proper u-ω-paraopen set is an u-ω-mean open set and vice-versa.*

(2) *A proper u-ω-paraclosed set is an u-ω-mean closed set and vice-versa.*

Proof. (1). If $\alpha \in \omega po(\tau_1, \tau_2)$ such that $\alpha \notin \{\emptyset, X\}$, then $\alpha \in \overline{\omega}(\tau_1, \tau_2)$. Conversely, let $B \in \overline{\omega}(\tau_1, \tau_2)$. Then there exist $B_1(\neq B), B_2(\neq B) \in \omega(\tau_1, \tau_2)$ such that $B_1 \subset B \subset B_2$ and $B_1, B_2 \notin \{\emptyset, X\}$. Since $B_1 \neq \emptyset, B$ and $B_2 \neq X, B$, $B \notin \omega^-(\tau_1, \tau_2) \cap \omega^+(\tau_1, \tau_2)$. As $B \neq \emptyset, X$, $B \in \omega po(\tau_1, \tau_2)$.
(2). Similar to (1). □

Theorem 6.3. (1) *If $C_1, C_2 \in \omega^+(\tau_1, \tau_2)$ with $C_1 \neq C_2$ and $\omega \in \overline{\omega}(\tau_1, \tau_2)$, then $C_1 \cap C_2 \neq \emptyset$.*

(2) *If $C_1, C_2 \in \omega^-(\tau_1, \tau_2)$ with $C_1 \neq C_2$ and $\omega \in \overline{\omega}(\tau_1, \tau_2)$, then $C_1 \cup C_2 \neq X$.*

(3) *If $C_1, C_2 \in \omega^+c(\tau_1, \tau_2)$ with $C_1 \neq C_2$ and $\omega \in \overline{\omega}c(\tau_1, \tau_2)$, then $C_1 \cap C_2 \neq \emptyset$.*

(4) *If $C_1, C_2 \in \omega^+(\tau_1, \tau_2)$ with $C_1 \neq C_2$ and $\omega \in \overline{\omega}(\tau_1, \tau_2)$, then $C_1 \cup C_2 \neq X$.*

Proof. (1). Let $C_1, C_2 \in \omega^+(\tau_1, \tau_2)$ with $C_1 \neq C_2$ and $A \in \overline{\omega}(\tau_1, \tau_2)$. Then $C_1 \cup C_2 = X$. $A \in \overline{\omega}(\tau_1, \tau_2)$ implies $A \notin \omega^+(\tau_1, \tau_2) \cap \omega^-(\tau_1, \tau_2)$. Then $A \neq C_1, C_2$. Also $A \neq X$. Then $A \nsubseteq C_1$ or $A \cup C_1 = X$ and $A \nsubseteq C_2$ or $A \cup C_2 = X$. Then we have the following cases: (i). $A \nsubseteq C_1$ and $A \nsubseteq C_2$, (ii). $A \nsubseteq C_1$ and $A \cup C_2 = X$, (iii). $A \cup C_1 = X$ and $A \nsubseteq C_2$ and (iv). $A \cup C_1 = X$ and $A \cup C_2 = X$.

Case (i): Obviously, $C_1 \cap C_2 \neq \emptyset$ if $A \nsubseteq C_1$ and $A \nsubseteq C_2$. Case (ii): If $A \cap C_2 \neq \emptyset$, then $C_1 \cap C_2 \neq \emptyset$. Now suppose $A \cap C_2 \neq \emptyset$. As $A \nsubseteq C_1$, then there exists $x \in C_1$ such that $x \notin C_2$. Since $A \cup C_2 = X$, $x \in C_2$. So $C_1 \cap C_2 \neq \emptyset$. Case (iii): Similar to Case (ii). Case (iv): $A \cup C_1 = X$ and $A \cup C_2 = X$ imply that $A \cup (C_1 \cap C_2) = X$ which in turn imply that $A \neq X$, we have $C_1 \cap C_2 \neq \emptyset$.

(2). Similar to (1).

(3). Follows from (1).

(4). Follows from (2). □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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