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# Certain Bounds for a Subclasses of Analytic Functions of Reciprocal Order

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Abstract. In this article we obtain the best possible estimates, Fekete Szegö inequality and bounds of second Hankel determinant for the function belonging to the new subclass of reciprocal order.

## 1. Introduction

Let $\mathcal{A}$ denote the family of functions $f$ normalized by

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i, \quad z \in \mathcal{E} \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$ . We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are univalent in $\mathcal{E}$.

Let $0 \le \lambda < 1$ the function $f \in \mathcal{A}$ is called starlike function of order $\lambda$ and convex function of order $\lambda$ are denoted by $\mathcal{S}^*(\lambda)$ and $\mathcal{C}(\lambda)$ respectively, if

$$\mathcal{R}\left(\frac{zf'(z)}{f(z)}\right) > \lambda, \quad z \in \mathcal{E}$$

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and

$$\mathcal{R}\left(1+\frac{zf''(z)}{f'(z)}\right)>\lambda, \quad z\in\mathcal{E}.$$

The class of all such starlike and convex functions were introduced by Robertson [14]. and we let

$$\mathcal{S}^*(0)\equiv\mathcal{S} \qquad \textit{and} \qquad \mathcal{C}(0)\equiv\mathcal{C}.$$

Now, let $\mathcal{S}_*(\lambda)$ and $\mathcal{C}_*(\lambda)$ be the classes of starlike and convex functions of reciprocal order $\lambda$ for $0\leq\lambda<1$ respectively if

$$\mathcal{R}\left(\frac{f(z)}{zf'(z)}\right)>\lambda, \quad z\in\mathcal{E}. \tag{1.2}$$

and

$$\mathcal{R}\left(\frac{f'(z)}{zf''(z)+f'(z)}\right)>\lambda, \quad z\in\mathcal{E}. \tag{1.3}$$

By seeing literature, it is the fact that

$$\mathcal{R}[p(z)]>0\Rightarrow\mathcal{R}\left(\frac{1}{p(z)}\right)=\mathcal{R}\left(\frac{p(z)}{|p(z)|^2}\right)>0.$$

In 2008, Nunokawa *et al.* [11] proved that every starlike function of reciprocal order $\lambda$ is starlike and hence univalent in $\mathcal{E}$.

When $0<\lambda<1/2$, for $f(z)\in\mathcal{S}_*(\lambda)$ if and only if

$$\left|\frac{zf'(z)}{f(z)}-\frac{1}{2\lambda}\right|<\frac{1}{2\lambda}.$$

**Example 1.1.** *[15] Let $f\in\mathcal{A}$ and $0\leq\lambda<1$ satisfy the inequality*

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\lambda$$

*then*

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-\frac{2-\lambda}{2}\right| &\leq \left|\frac{zf'(z)}{f(z)}-1\right|+\frac{\lambda}{2}.\\ &<1-\lambda+\frac{\lambda}{2}=\frac{2-\lambda}{2}\end{aligned}$$

*and therefore such functions are starlike of reciprocal order $\frac{1}{2-\lambda}$.*

Miller, Mocanu and others studied extensively (see [6, 8–10]) the class of $\alpha$-convex functions $M_\alpha$, $\alpha$ be real, defined as

$$M_\alpha=\left\{f\in\mathcal{A}: \quad \mathcal{R}\left((1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>0, \quad z\in\mathcal{E}\right\}.$$

Taking note of these results, we now define the following subclass of reciprocal order,

**Definition 1.1.** *For $0 \leq \alpha < 1$ and $b \in \mathbb{C} \setminus \{0\}$, a function $f \in \mathcal{A}$ is in the class $\widetilde{\mathfrak{M}(\alpha)}$ satisfying the inequality*

$$Re\left\{1+\frac{1}{b}\left((1-\alpha)\frac{f(z)}{zf'(z)}+\alpha\frac{f'(z)}{zf''(z)+f'(z)}-1\right)\right\}>0 \quad z\in\mathcal{E}. \tag{1.4}$$

The Hankel determinant $H_q(n)$ of Taylor's coefficients of function $f \in \mathcal{A}$ of the form(1.1),is defined by

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}, \quad (n,q\in\mathbb{N}=1,2,3...). \tag{1.5}$$

The Hankel determinant is useful,in showing that a function of bounded characteristic in $\mathcal{E}$, *i.e.,* a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [1]. Pommerenke [12] proved that the Hankel determinants of univalent functions satisfy $|H_q(n)| < \mathcal{K}n^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, where $\beta > 1/4000$ and $\mathcal{K}$ depends only on q. Later Hayman [4] proved that $|H_q(n)| < An^{1/2}$ (A is an absolute constant) for a really mean univalent functions.

A classical theorem of Fekete-Szegö [2] considered the second Hankel determinant $|H_2(1)| = |a_3 - a_2^2|$ for univalent functions. They made an early study for the estimate of well known Fekete-Szegö functional $|a_3 - \mu a_2^2|$ when $\mu$ is real. Janteng [5] investigated the sharp upper bound for second Hankel determinant $|H_2(2)| = |a_2a_4 - a_3^2|$ for univalent functions whose derivative has positive real part.

In this present paper, we estimate the coefficient bounds, classical Fekete-Szegö function and second Hankel determinant for the function $f \in \mathcal{A}$ belonging to the new defined sublass.

For understanding the concept of this article, it is now neccessary to review the following fundamental lemmas.

## 2. Preliminary Results

**Lemma 2.1.** *[13] Let $P$ denote the class of functions*

$$p(z) = 1 + c_1z + c_2^2 + \cdots \tag{2.1}$$

*which are regular in $\mathcal{E}$ and satisfy $\mathcal{R}\,[p(z)] > 0$, $z \in \mathcal{E}$*

$$|c_n| \leq 2, \quad n \geq 1.$$

**Lemma 2.2.** *[3] If $p(z)$ is of the form (2.1) with positive real part then,*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.2}$$

*and*

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2 c_1(4 - c_1^2) + 2z(1 - |x|^2)(4 - c_1^2) \tag{2.3}$$

*for some $x, z$ such that $|x| \leq 1$ and $|z| \leq 1$.*

**Lemma 2.3.** *[7] If $p(z)$ is of the form (2.1) with positive real part then the following inequality holds,*

$$|c_2 - \nu c_1^2| \leq 2max\{1, |2\nu - 1|\}, \quad where \quad \nu \in \mathbb{C}.$$

## 3. Main Results

The most important aspect of this study is that we are simultaneously examining the characteristics of the defined class of analytic functions with respect to the reciprocal order. Taking the aforementioned importance of Hankel determinant into consideration, we now proving the following result.

**Theorem 3.1.** *If $f \in \mathcal{A}$ is in the class $\widetilde{\mathfrak{M}(\alpha)}$,then*

$$|a_2| \leq \frac{2|b|}{1+\alpha},$$

$$|a_3| \leq \frac{|b|}{(1+2\alpha)}\left[1 + \frac{4|b|(1+3\alpha)}{(1+\alpha)^2}\right]$$

*and for any real number $\mu$ we have,*

$$|a_3 - \mu a_2^2| \leq \frac{|b|}{(1+2\alpha)}\, max\left\{1, \left|\frac{4b}{(1+\alpha)^2}[(1+3\alpha) - \mu(1+2\alpha)] - 1\right|\right\}. \tag{3.1}$$

*Proof.* From definition (1.1) we have,

$$1 + \frac{1}{b}\left[(1-\alpha)\frac{f(z)}{zf'(z)} + \alpha\frac{f'(z)}{zf''(z)+f'(z)} - 1\right] = p(z), \quad for \quad 0 \leq \alpha < 1 \tag{3.2}$$

and $p(z)$ is analytic in $\mathcal{E}$ with $p(0) = 1$ and $Re[p(z)] > 0$.

Simple calculation yields,

$$\begin{aligned} &1 - \frac{1}{b}(1+\alpha)a_2 z + \frac{1}{b}[2(1+3\alpha)a_2^2 - 2(1+2\alpha)a_3]z^2 \\ &- \frac{1}{b}[4(1+7\alpha)a_2^3 - 7(1+5\alpha)a_2a_3 + 3(1+3\alpha)a_4]z^3 \\ &+ \frac{1}{b}[10(1+7\alpha)a_2a_4 - 4(1+4\alpha)a_5 - 20(1+11\alpha)a_2^2a_3 \\ &+ 6(1+8\alpha)a_3^2 + 8(1+15\alpha)a_2^4]z^4 + \cdots = 1 + c_1z + c_2z^2 + c_3z^3 \ldots. \end{aligned} \tag{3.3}$$

Now equating the coefficients on both sides, we obtain

$$a_2 = -\frac{bc_1}{1+\alpha} \tag{3.4}$$

and

$$a_3 = -\frac{b}{2(1+2\alpha)}\left[c_2 - \frac{2b(1+3\alpha)}{(1+\alpha)^2}c_1^2\right]. \tag{3.5}$$

For any real $\mu$, we can derive

$$|a_3 - \mu a_2^2| \leq \frac{|b|}{2(1+2\alpha)} \left| c_2 - \frac{2b}{(1+\alpha)^2} \left[(1+3\alpha) - \mu(1+2\alpha)\right] c_1^2 \right|$$

where

$$\mu = \frac{2b}{(1+\alpha)^2} \left[(1+3\alpha) - \mu(1+2\alpha)\right].$$

□

By lemma 2.3 we obtain our desired results.

**Corollary 3.1.** *If $\alpha = 0$ and $b = 1-\lambda$, then the class $\widetilde{\mathfrak{M}(\alpha)}$ reduced to $\mathcal{S}_*(\lambda)$ and we will get*

$$|a_2| \leq 2(1-\lambda) \quad \textit{and} \quad |a_3| \leq (1-\lambda)(5-4\lambda).$$

**Corollary 3.2.** *If $\alpha = 1$ and $b = 1-\lambda$, then the class $\widetilde{\mathfrak{M}(\alpha)}$ reduced to $\mathcal{C}_*(\lambda)$ and we will get*

$$|a_2| \leq (1-\lambda) \quad \textit{and} \quad |a_3| \leq \frac{(1-\lambda)}{3}(5-4\lambda).$$

**Remark 3.1.** *[16] If $\alpha = 0$ and $b = 1$, then the class $\widetilde{\mathfrak{M}(\alpha)}$ reduced to $\widehat{\mathcal{ST}}$ and we will get*

$$a_2 = -c_1 \quad \textit{and} \quad a_3 = -\frac{1}{2}(c_2 - 2c_1^2).$$

**Remark 3.2.** *[16] If $\alpha = 1$ and $b = 1$, then the class $\widetilde{\mathfrak{M}(\alpha)}$ reduced to $\widehat{\mathcal{CV}}$ and we will get*

$$a_2 = -\frac{c_1}{2} \quad \textit{and} \quad a_3 = -\frac{1}{6}(c_2 - 2c_1^2).$$

**Theorem 3.2.** *If $f \in \mathcal{A}$ is in the class $\widetilde{\mathfrak{M}(\alpha)}$,then*

$$|a_2 a_4 - a_3^2| \leq \frac{4}{3(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \left(\eta_1 + \frac{\eta_2}{2} + \frac{\eta_3}{4} + \frac{\eta_4}{4}\right)$$

*Proof.* Let $f \in \widetilde{\mathfrak{M}(\alpha)}$, from (3.3) we have

$$\begin{aligned} a_4 =& \frac{-b}{3(1+3\alpha)} \left[ c_3 - \frac{7b(1+5\alpha)}{2(1+\alpha)(1+2\alpha)} c_1 c_2 \right. \\ & \left. + \left( \frac{7(1+3\alpha)(1+5\alpha)}{(1+\alpha)^3(1+2\alpha)} - \frac{4(1+7\alpha)}{(1+\alpha)^3} \right) b^2 c_1^3 \right]. \end{aligned} \tag{3.6}$$

From equation (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} |a_2 a_4 - a_3^2| =& \frac{b^2}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \left\{ 4(1+\alpha)^3(1+2\alpha)^2 c_1 c_3 \right. \\ & - 2b(1+\alpha)^2[7(1+2\alpha)(1+5\alpha) - 6(1+3\alpha)^2] c_1^2 c_2 \\ & + 4b^2[7(1+2\alpha)(1+3\alpha)(1+5\alpha) - 4(1+2\alpha)^2(1+7\alpha) - 3(1+3\alpha)^3] c_1^4 \\ & \left. -3(1+\alpha)(1+3\alpha) c_2^2 \right\}. \end{aligned}$$

Replacing the values of $c_2$ and $c_3$ from Lemma 2.2 and assume that $c_1 = c \in [0, 2]$, then applying triangle inequality with $|x| = \rho$, we obtain,

$$
\begin{aligned}
|a_2a_4 - a_3^2| \leq & \frac{|b|^2}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \Big\{ \Big[(4\alpha^5 + 16\alpha^4 + 26\alpha^3 + 7\alpha^2 + 19\alpha + 1) \\
& + |b|(1+\alpha)^2(16\alpha^2 + 13\alpha + 1) + |b|^2(17\alpha^3 + 8\alpha^2 - \alpha) + \frac{3}{4}(3\alpha^2 + 4\alpha + 1)\Big] c^4 \\
& + \Big[4(4\alpha^5 + 16\alpha^4 + 25\alpha^3 + 19\alpha^2 + 7\alpha + 1) + 2|b|(1+\alpha)^2) \\
& \left((16\alpha^2 + 13\alpha + 1) + \frac{3}{2}(3\alpha^2 + 4\alpha + 1)\right)\Big] (4 - c^2)c^2\rho \\
& + \left[c^2(4\alpha^5 + 16\alpha^4 + 25\alpha^3 + 19\alpha^2 + 7\alpha + 1) + \frac{3}{4}(3\alpha^2 + 4\alpha + 1)(4 - c^2)\right] \rho^2(4 - c^2) \\
& + 2(1+2\alpha)^2 c(4 - c^2)(1 - \rho^2)\Big\} = F(c, \rho).
\end{aligned}
$$

We assume that the upper bound occurs at the interior point of the rectangle $[0, 2] \times [0, 1]$. Differentiating the above equation with respect to $\rho$,we get

$$
\begin{aligned}
\frac{\partial F}{\partial \rho} = & \frac{|b|^2}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \Big\{ \Big[4(4\alpha^5 + 16\alpha^4 + 25\alpha^3 + 19\alpha^2 + 7\alpha + 1) \\
& + 2|b|(1+\alpha)^2 \left((16\alpha^2 + 13\alpha + 1) + \frac{3}{2}(3\alpha^2 + 4\alpha + 1)\right)\Big] (4 - c^2)c^2 \\
& + \Big[2c^2(4\alpha^5 + 16\alpha^4 + 25\alpha^3 + 19\alpha^2 + 7\alpha + 1) \\
& + \frac{3}{4}(3\alpha^2 + 4\alpha + 1)(4 - c^2)\Big] \rho(4 - c^2) \\
& - 4(1+2\alpha)^2 c(4 - c^2)\rho\Big\}.
\end{aligned}
$$

For $0 < \rho < 1$ and fixed $c \in [0, 2]$, it can be easily seen that $\frac{\partial F}{\partial \rho} > 0$. This shows that $F(c, \rho)$ is an increasing function of $\rho$.

Therefore, $\max F(c, \rho) = F(c, 1) = G(c)$

$$
\begin{aligned}
G(c) = & \frac{|b|^2}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \Big\{ \Big[(4\alpha^5 + 16\alpha^4 + 26\alpha^3 + 7\alpha^2 + 19\alpha + 1) \\
& + |b|(1+\alpha)^2(16\alpha^2 + 13\alpha + 1) + |b|^2(17\alpha^3 + 8\alpha^2 - \alpha) + \frac{3}{4}(3\alpha^2 + 4\alpha + 1)\Big] c^4 \\
& + \Big[4(4\alpha^5 + 16\alpha^4 + 25\alpha^3 + 19\alpha^2 + 7\alpha + 1) + 2|b|(1+\alpha)^2) \qquad . \\
& \left((16\alpha^2 + 13\alpha + 1) + \frac{3}{2}(3\alpha^2 + 4\alpha + 1)\right)\Big] (4 - c^2)c^2 \\
& + \left[c^2(4\alpha^5 + 16\alpha^4 + 25\alpha^3 + 19\alpha^2 + 7\alpha + 1) + \frac{3}{4}(3\alpha^2 + 4\alpha + 1)(4 - c^2)\right] (4 - c^2)
\end{aligned}
$$

Now differentiating with respect to c,

$$\begin{aligned}G'(c) =&\frac{|b|^2}{12(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}\Big\{\big[4(4\alpha^5+16\alpha^4+26\alpha^3+7\alpha^2+19\alpha+1)\\&+|b|(1+\alpha)^2(16\alpha^2+13\alpha+1)+|b|^2(17\alpha^3+8\alpha^2-\alpha)+\frac{3}{4}(3\alpha^2+4\alpha+1)\Big]c^3\\&+\big[4(4\alpha^5+16\alpha^4+25\alpha^3+19\alpha^2+7\alpha+1)+2|b|(1+\alpha)^2)\\&\left((16\alpha^2+13\alpha+1)+\frac{3}{2}(3\alpha^2+4\alpha+1)\right)\Big](8c-c^3)\\&+\left[(4\alpha^5+16\alpha^4+25\alpha^3+19\alpha^2+7\alpha+1)(8c-c^3)-\frac{3}{4}(3\alpha^2+4\alpha+1)(4c-c^3)\right].\end{aligned}$$

Since $\dfrac{\partial G}{\partial \rho}>0$ for $c\in[0,2]$, G(c) has maximum value at c=2. Hence,

$$\begin{aligned}|a_2a_4-a_3^2|\leq&\frac{4}{3(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}\big[(4\alpha^5+16\alpha^4+26\alpha^3+7\alpha^2+19\alpha+1)\\&+|b|(1+\alpha)^2(16\alpha^2+13\alpha+1)+|b|^2(17\alpha^3+8\alpha^2-\alpha)+\frac{3}{4}(3\alpha^2+4\alpha+1)\Big]\times|b|^2.\end{aligned}$$

□

**Remark 3.3.** *If $\alpha=1$ and $b=1$, then the results of Theorem 3.2 reduced to the results of Vamshee Krishna and Ramreddy [16].*

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