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Bifurcation Analysis and Chaos Control for Prey-Predator Model With Allee Effect

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Abstract. The main purpose of this work is to discuss the dynamics of a predator-prey dynamical system with Allee effect. The conformable fractional derivative is applied to convert the fractional derivatives which appear in the governing model into ordinary derivatives. We use the piecewise-constant approximation method to discritize the considered model. We also investigate the occurrence of positive equilibrium points. Moreover, we analyse the stability of the equilibrium point using some stability theorems. This work also explores a Neimark-Sacker bifurcation and a period-doubling bifurcation using the theory of bifurcations. The distance between the obtained equilibrium point and some closed curves is examined for various values for the considered bifurcation parameter. The chaos control is nicely analysed using the hybrid control approach. Furthermore, we present the maximum Lyapunov exponents for different values for the bifurcation parameter. Numerical simulations are utilized to ensure that the obtained theoretical results are correct. The used techniques can be applied to deal with predator-prey models in various versions.

1. Introduction

In some communities, individual organisms interact with each other in a variety of ways. This interaction may be advantageous to both individuals or advantageous to one organism at the detriment of the other. An antagonistic interaction is one in which one organism benefits at the expense of the other. The antagonistic interactions can be clearly observed in parasitism, herbivory, and predation. The predation can be defined as one organism kills and eats another. Predation often occurs between

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two populations or more. The influences of the predation can highly affect both populations. The density of predators increases if the density of prey is high while the number of the predators decreases if the number of prey is low. In animals' world, lions hunt zebras, wolves and hyenas hunt deer, shrews hunt worms and insects, foxes hunt rabbits, tigers hunt buffaloes, etc. Not all predators are animals. Some plants can be predators. For instance, the Venus flytrap and the pitcher plants consume some insects. It is worth to mention that the communication between predators and prey plays a significant role in maintaining the ecological equilibrium among various species of organisms. In particular, in the absence of predators, competition on food resources between some prey species may extinct other species. Moreover, predators cannot remain alive without prey. More information about biological phenomena can be obtained in refs. [1–6].

Since some biological processes are naturally complicated, mathematics can simply explain the future behavior of some of these processes. Differential or difference equations are commonly used in population dynamics to explain how populations behave and interact over time. For example, Lotka-Volterra system, that was independently developed by Lotka (1925) and Volterra (1926), describes predator-prey (parasite-host or herbivore-plant) interactions. The predator-prey system contains a pair of differential equations in which the predators and prey affect each other. Many scholars have investigated this system. Some researchers have perfectly developed this model to give a more realistic explanation about predator and prey populations. For instance, Leslie [7, 8] developed a well-known Leslie predator-prey dynamical system in which the carrying capacity of the predator population is proportional to the density of prey population. Pal et al. [9] used the classical predator-prey model to investigate the dynamics of the predation when prey populations are frightened by predators. More specifically, predators sometimes cooperate during hunting which frighten prey populations. Kumar and Kharbanda [10] analyzed the occurrence and stability conditions of the fixed points, the boundedness of the solutions, and the bifurcation of the predator-prey model in the presence of defense. Furthermore, Zhou et al. [11] discussed the existence of positive fixed points, the stability, and the Hopf bifurcations of a predator–prey system with Holling-II type functional response. Akhtar et al. [12] used a non-standard finite difference equations to approximate the solutions of a continuous-time Leslie prey-predator system. They also discussed the stability of the equilibrium points and Neimark-Sacker bifurcation of the model. The hybrid control technique was also applied in [12] to control the chaos and the bifurcation. Deng et al. [13] investigated the positive and negative effect of cannibalism on the stability of the Lotka–Volterra system.

One of the most fascinating ecological phenomena is the Allee effect which was discovered by Allee [14]. The Allee effect is described as a positive correlation between average individual fitness and population size over some finite interval [15]. The Allee effect can be resulted from different environmental conditions such as genetic inbreeding, difficulties in obtaining mating partners at low populations, etc. Several scientists have recently shown a strong interest in studying the influence of including the

Allee effect in the predator-prey dynamics. For example, Sen et al. [16] discussed a ratio-dependent predator–prey equations with a strong Allee effect and a weak Allee effect. Further, the authors compared some dynamical properties of the considered system with and without the Allee effect and presented a considerable difference in the dynamics of these models [16]. The same system was also investigated in [17] by Cheng and Cao. Cheng and Cao found that the predator and prey populations cannot coexist if the predator growth rate is smaller than its death rate. However, if the predator growth rate exceeds the rate of death, the model has two interior fixed points [17]. Vinoth et al. [18] investigated the influence of the fear and the Allee effect on the existence of positive fixed points of a Leslie–Gower ratio-dependent predator–prey system. The authors also used the Jacobian matrix to examine the local attractivity of all positive equilibrium points. Furthermore, Ahmed et al. [19] explored the topological classification of the equilibrium points of a discrete-time predator-prey model with Allee effect.

Motivated by the above mentioned studies, this work aims to explore the qualitative behavior of the following predator-prey system with Allee effect:

$$\begin{cases} \mathcal{T}^{\beta}x(t) = x(t)(\alpha - \gamma y(t)), \\ \mathcal{T}^{\beta}y(t) = y(t)\left(\sigma - \frac{\varepsilon y(t)}{x(t)}\left(\frac{y(t)}{\eta + y(t)}\right)\right), \end{cases} \tag{1.1}$$

where $0 < \beta < 1$ is the fractional-order parameter. The variables $x(t) > 0$ and $y(t) > 0$ are considered to be the population densities of prey and predator at time t, respectively. Moreover, α and σ are the intrinsic growth rates of prey and predator, respectively. The parameter γ represents the strength of competition among individuals of prey population. The parameter ε denotes the food quantity the prey provides and is converted to predator birth. The term $\left(\frac{y(t)}{\eta+y(t)}\right)$ is the Allee effect while $\eta > 0$ represents the constant of Allee effect. Moreover, $\mathcal{T}^{\beta}$ is the fractional derivative of the conformable-type. We first use the piecewise-constant approximation method to discritize system (1.1) and then we obtain the corresponding difference equations. This paper analyses the presence of fixed points of the considered problem. The stability of these points is found. We utilize some bifurcation theorems to explore the existence of period-doubling and Neimark–Sacker bifurcations at the equilibrium points. The hybrid control approach is used to discuss the chaos control. Finally, this work shows some numerical investigations of the theoretical results under some suitable parameters.

This article is written as follows. In Section 2, we present some important concepts related to this paper. Section 3 discritizes system (1.1) while Section 4 presents the existence of the equilibrium point and its local stability. In Section 5, we analyse the period-doubling and Neimark–Sacker bifurcations. chaos control is discussed in Section 6. In addition, some numerical computations for the obtained theoretical results are given in Section 7 while Section 8 concludes this article.

2. Preliminaries

This part introduces the most important terminologies used in this paper.

Definition 2.1 ([20]). *The Neimark sacker bifurcation occurs when an equilibrium point changes stability by a pair of complex eigenvalues with unit modulus in a discrete time model. This type of bifurcation can be subcritical or supercritical in an unstable or stable limit cycle.*

Definition 2.2 ([20]). *The period doubling bifurcation (flip bifurcation) takes place when a small change in the bifurcation parameter gives a new structure with twice the period of the original system. Mathematically, we can define a flip bifurcation as follows. Let $G : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a one-parameter family of C^3 maps satisfying the following conditions:*

$$\begin{aligned} G(0,0) &= 0, \\ \left(\frac{\partial G}{\partial x}\right)_{h=0,x=0} &= -1, \\ \left(\frac{\partial^2 G}{\partial x^2}\right)_{h=0,x=0} &< 0, \\ \left(\frac{\partial^3 G}{\partial x^3}\right)_{h=0,x=0} &< 0. \end{aligned}$$

Then, there are intervals $(h_1, 0)$, $(0, h_2)$ and $\epsilon > 0$ where

- *If $h \in (0, h_2)$, then $G_h(x)$ has one stable orbit of period two for $x \in (-\epsilon, \epsilon)$ and one unstable equilibrium point.*
- *If $h \in (h_1, 0)$, then $G_h(x)$ has a unique stable equilibrium point for $x \in (-\epsilon, \epsilon)$.*

Note that this bifurcation is referred to as a flip bifurcation.

Definition 2.3 ([21]). *Allee effect is described as an ecological phenomena in which a positive connection between any component of individual fitness and either numbers or density of conspecifics.*

It can be noted that the Allee effect was first observed by an American ecologist Warder Clyde Allee (1931) [14]. This effect may occur due to a several reasons such as genetic inbreeding and difficulties in finding mates.

Definition 2.4 ([22, 23]). *Let the discrete dynamical system*

$$x_{k+1} = G(x_k) = G^{k+1}(x_0).$$

Here, $x \in \mathbf{R}^n$. Consider a minor change Δx_0 in the initial values x_0, the sensitivity to initial conditions can be measured as

$$||\Delta x_k|| \approx ||\Delta x_0|| e^{k\lambda}.$$

Here, λ is the maximum Lyapunov exponent (MLE), which can be calculated by

$$L = \lim_{k \to \infty} \frac{1}{k} \ln \frac{||\Delta x_k||}{||\Delta x_0||}.$$

For $n = 1$ (one dimensional system), the Lyapunov exponents is defined as

$$L = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k-1} \ln \mid G'(x_i) \mid .$$

Definition 2.5 ([24, 25]). *Let $G : (0, \infty) \to R$ be a function. Then, the conformable fractional derivative of order $0 < \beta \leq 1$ of f at $t > 0$ is defined by*

$$\mathcal{T}_h^{\beta} G(t) = \lim_{\epsilon\to 0} \frac{G(t + \epsilon(t-h)^{1-\beta}) - G(t)}{\epsilon}, \quad 0 < \beta \leq 1. \tag{2.1}$$

Here, $\mathcal{T}_h^{\beta}$ is the conformable-type fractional derivative. The discretization parameter is $h > 0$. It was presented in [26], the derivative of Eq. (2.1) is given by

$$\mathcal{T}_h^{\beta} G(t) = (t-h)^{1-\beta} G'(t). \tag{2.2}$$

3. Discretization Process

The discretization process of system (1.1) using the piecewise-constant approximation method [27, 28] is shown in this section. Applying this technique, we obtain

$$\begin{cases} \frac{\mathcal{T}^{\beta} x(t)}{x(t)} = (\alpha - \gamma y(\left[\frac{t}{h}\right] h)), \\ \frac{\mathcal{T}^{\beta} y(t)}{y(t)} = \left(\sigma - \frac{\varepsilon y(\left[\frac{t}{h}\right] h)}{x(\left[\frac{t}{h}\right] h)} \left(\frac{y(\left[\frac{t}{h}\right] h)}{\eta + y(\left[\frac{t}{h}\right] h)}\right)\right), \end{cases} \tag{3.1}$$

where $\left[\frac{t}{h}\right]$ represents the integer part of $t \in [nh, (n+1)h)$, $n = 0, 1, \cdots$, and $h > 0$ is a discretization parameter. Applying the rule of the conformable fractional derivative (Definition 2.5) on the first equation of system (3.1), we end up with the following differential equation:

$$\frac{dx(t)}{x(t)dt} = (\alpha - \gamma y(nh))(t - nh)^{\beta - 1}, \tag{3.2}$$

which can be solved over the interval $[nh, t)$ as follows:

$$\ln(x(t)) - \ln(x(nh)) = (\alpha - \gamma y(nh)) \frac{(t - nh)^{\beta}}{\beta}. \tag{3.3}$$

Let $t \to (n+1)h$ and replace $y(nh)$ and $x(nh)$ by y_n and x_n, respectively, then Eq. (3.3) becomes

$$x_{n+1} = x_n e^{(\alpha - \gamma y_n)\frac{h^{\beta}}{\beta}}. \tag{3.4}$$

Similarly, one can solve the second equation of system (3.1) and end up with

$$y_{n+1} = y_n e^{\left(\sigma - \frac{\varepsilon y_n^2}{x_n(\eta + y_n)}\right)\frac{h^{\beta}}{\beta}}. \tag{3.5}$$

As a result, we have

$$\begin{cases} x_{n+1} = x_n e^{(\alpha - \gamma y_n)\frac{h^{\beta}}{\beta}}, \\ y_{n+1} = y_n e^{\left(\sigma - \frac{\varepsilon y_n^2}{x_n(\eta + y_n)}\right)\frac{h^{\beta}}{\beta}}. \end{cases} \tag{3.6}$$

4. Local stability of equilibrium point

The existence of an equilibrium point of system (3.6) is explored in this part. We then analyse the stability of the equilibrium points. There exists only one equilibrium point for system (3.6) which is $E = (\frac{\varepsilon\alpha^2}{\gamma\sigma(\alpha+\gamma\eta)}, \frac{\alpha}{\gamma})$.

Lemma 4.1 ([27]). *Assume that (x, y) is an equilibrium point of model (3.6) with multipliers (eigenvalues of Jacobien matrix) ω_1 and ω_2. Then,*

1. *The equilibrium point (x, y) is a sink (locally asymptotic stable) if $|\omega_1| < 1$ and $|\omega_2| < 1$.*
2. *The equilibrium point (x, y) is a source if $|\omega_1| > 1$ and $|\omega_2| > 1$.*
3. *The equilibrium point (x, y) is a saddle if $|\omega_1| < 1$ and $|\omega_2| > 1$, or if $|\omega_1| > 1$ and $|\omega_2| < 1$.*
4. *The equilibrium point (x, y) is a non-hyperbolic if $|\omega_1| = 1$ or $|\omega_2| = 1$.*

Lemma 4.2 ([27, 29–32]). *Assume that the polynomial $K(\omega) = \omega^2 - p\omega + q$, where $P(1) > 0$, and ω_1 and ω_2 are the two roots of $K(\omega) = 0$. Then,*

1. *$|\omega_1| < 1$ and $|\omega_2| < 1$ if and only if $K(-1) > 0$ and $K(0) < 1$.*
2. *$|\omega_1| > 1$ and $|\omega_2| > 1$ if and only if $K(-1) > 0$ and $K(0) > 1$.*
3. *$|\omega_1| < 1$ and $|\omega_2| > 1$ (or $|\omega_1| > 1$ and $|\omega_2| < 1$) if and only if $K(-1) < 0$.*
4. *$\omega_1 = -1$ and $|\omega_2| \neq 1$ if and only if $K(-1) = 0$ and $p \neq 0, 2$.*
5. *ω_1 and ω_2 are complex numbers and $|\omega_1| = |\omega_2| = 1$ if and only if $|p| < 2$ and $K(0) = 1$.*

Now, we consider the stability of system (3.6). The Jacobian matrix of Eqs. (3.6) is shown as follows:

$$\mathcal{J}((x, y)) = \begin{pmatrix} e^{(\alpha-\gamma y_n)\frac{h^\beta}{\beta}} & -\dfrac{\gamma x h^\beta}{\beta} e^{(\alpha-\gamma y_n)\frac{h^\beta}{\beta}} \\ \dfrac{\varepsilon y^2 h^\beta}{\beta x^2(\eta+y_n)} e^{\left(\sigma - \frac{\varepsilon y_n^2}{x_n(\eta+y_n)}\right)\frac{h^\beta}{\beta}} & \left(1 - \dfrac{\varepsilon y^2 h^\beta(2\eta+y)}{\beta x(\eta+y)}\right) e^{\left(c - \frac{\varepsilon y_n^2}{x_n(\eta+y_n)}\right)\frac{h^\beta}{\beta}} \end{pmatrix}. \tag{4.1}$$

Hence,

$$\mathcal{J}(E) = \begin{pmatrix} 1 & -\dfrac{\varepsilon\alpha^2 h^\beta}{\beta\sigma(\alpha+\gamma\eta)} \\ \dfrac{\sigma^2 h^\beta(\alpha+\gamma\eta)}{\beta\alpha\varepsilon} & 1 - \dfrac{\sigma h^\beta(2\gamma\varepsilon+\alpha)}{\beta(\alpha+\gamma\varepsilon)} \end{pmatrix}. \tag{4.2}$$

The characteristic polynomial of the Jacobian matrix can be expressed by

$$K(\omega) = \omega^2 - p\omega + q, \tag{4.3}$$

where p is the trace and q is the determinant of the Jacobian matrix $\mathcal{J}(E)$, which are written as

$$p = 2 - \frac{h^\beta C(2\gamma\eta+\alpha)}{\beta(\alpha+\gamma\eta)},$$

$$q = 1 - \frac{h^\beta\sigma(2\gamma\eta+A)}{\beta(\alpha+\gamma\eta)} + \frac{\alpha\sigma h^{2\beta}}{\beta^2}.$$

Simple calculations on Eq. (4.3) lead to

$$\begin{aligned} K(0) &= 1 - \frac{h^{\beta}\sigma(2\gamma\eta+\alpha)}{\beta(\alpha+\gamma\eta)} + \frac{\alpha\sigma h^{2\beta}}{\beta^2}, \quad K(1) = \frac{\alpha\sigma h^{2\beta}}{\beta^2} > 0, \\ K(-1) &= 4 - \frac{2h^{\beta}}{\beta}\left(\frac{\sigma(2\gamma\eta+\alpha)}{(\alpha+\gamma\eta)}\right) + \frac{\alpha\sigma h^{2\beta}}{\beta^2}. \end{aligned}$$

Using Lemmas 4.1 and 4.2, we end up with the following result.

Lemma 4.3. *Assume that*

$$\begin{aligned} h_- &= \left[\frac{\beta\left(\sigma(2\gamma\eta+\alpha) - \sqrt{\sigma^2(2\gamma\eta+\alpha)^2 - 4\alpha\sigma(\alpha+\gamma\eta)^2}\right)}{\alpha\sigma(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}}, \\ h_+ &= \left[\frac{\beta\left(\sigma(2\gamma\eta+\alpha) + \sqrt{\sigma^2(2\gamma\eta+\alpha)^2 - 4\alpha\sigma(\alpha+\gamma\eta)^2}\right)}{\alpha\sigma(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}}. \end{aligned}$$

Then, for the point E of system (3.6), the following results are true.

(1) *If one of the following requirements is satisfied, then E is sin:*
  i- $\sigma^2(2\gamma\eta+\alpha)^2 < 4\alpha\sigma(\alpha+\gamma\eta)^2$ *and* $0 < h < h_1 = \left[\frac{\beta(2\gamma\eta+\alpha)}{\alpha(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}}$.
  ii- $\sigma^2(2\gamma\eta+\alpha)^2 \geq 4\alpha\sigma(\alpha+\gamma\eta)^2$ *and* $0 < h < h_-$.

(2) *If one of the following requirements is satisfied, then E is source:*
  i- $\sigma^2(2\gamma\eta+\alpha)^2 < 4\alpha\sigma(\alpha+\gamma\eta)^2$ *and* $h > h_1$.
  ii- $\sigma^2(2\gamma\eta+\alpha)^2 \geq 4\alpha\sigma(\alpha+\gamma\eta)^2$ *and* $h > h_-$.

(3) *The equilibrium point E is unstable saddle when*

$$\sigma^2(2\gamma\eta+\alpha)^2 \geq 4\alpha\sigma(\alpha+\gamma\eta)^2, \text{ and } h_- < h < h_+.$$

(4) *The equilibrium point E is non-hyperbolic and the roots of Eq. (4.3) are $\omega_1 = -1$ and $|\omega_2| \neq 1$ if*

$$\sigma^2(2\gamma\eta+\alpha)^2 \geq 4\alpha\sigma(\alpha+\gamma\eta)^2, \ h = h_{\mp} \text{ and } h \neq \left[\frac{4\beta(\alpha+\gamma\eta)}{\sigma(2\gamma\eta+\alpha)}\right]^{\frac{1}{\beta}}.$$

(5) *The equilibrium point E is non-hyperbolic and the roots of Eq. (4.3) are complex numbers with modulus one if*

$$\sigma^2(2\gamma\eta+\alpha)^2 < 4\alpha\sigma(\alpha+\gamma\eta)^2, \ h = h_1 = \left[\frac{\beta(2\gamma\eta+\alpha)}{\alpha(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}}.$$

Based on conditions 4 and 5 presented in Lemma 4.2, we can present the next Lemma.

Lemma 4.4. *The point E of system (3.6) loses its stability in two cases.*

(1) *The point E loses its stability via a period-doubling bifurcation if*

$$\sigma^2(2\gamma\eta+\alpha)^2 \geq 4\alpha\sigma(\alpha+\gamma\eta)^2, \ h = h_{\mp} = \left[\frac{\beta\left(\sigma(2\gamma\eta+\alpha) \mp \sqrt{\sigma^2(2\gamma\eta+\alpha)^2 - 4\alpha\sigma(\alpha+\gamma\eta)^2}\right)}{\alpha\sigma(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}}.$$

And

$$h \neq \left[\frac{4\beta(\alpha+\gamma\eta)}{\sigma(2\gamma\eta+\alpha)}\right]^{\frac{1}{\beta}}.$$

(2) *The point E loses its stability via a Neimark-Sacker bifurcation if*

$$\sigma^2(2\gamma\eta+\alpha)^2 < 4\alpha\sigma(\alpha+\gamma\eta)^2, \ \textit{and} \ h = h_1 = \left[\frac{\beta(2\gamma\eta+\alpha)}{\alpha(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}}.$$

5. Bifurcation Analysis

This section explores the existence and the stability examination of period-doubling and Neimark–Sacker bifurcations at the equilibrium point E. It is worth noting that the discretization parameter h is considered as the bifurcation parameter.

Lemma 5.1 ([33–35]). *Assume that $U_{k+1} = F_m(U_k)$ is a n-dimensional discrete dynamical system where $m \in R$ is a bifurcation parameter. Let U^* be an equilibrium point of F_m and suppose that the characteristic equation of the Jacobian matrix $J(U^*) = (b_{ij})_{n\times n}$ of n-dimensional map $F_m(U_k)$ is expressed as*

$$K_m(\omega) = \omega^n + b_1\omega^{n-1} + \cdots + b_{n-1}\omega + b_n. \tag{5.1}$$

Here, $b_i = b_i(m, u)$, $i = 1, 2, 3, \cdots, n$ and u is a control parameter. Suppose that $\Delta_0^{\pm}(m, u) = 1$, $\Delta_1^{\pm}(m, u), \cdots, \Delta_n^{\pm}(m, u)$ are a sequence of the determinants described by

$$\Delta_i^{\pm}(m, u) = \det(N_1 \pm N_2), \quad i = 1, 2, \cdots, n, \tag{5.2}$$

where

$$N_1 = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{i-1} \\ 0 & 1 & b_1 & \cdots & b_{i-2} \\ 0 & 0 & 1 & \cdots & b_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \ N_2 = \begin{pmatrix} b_{n-i+1} & b_{n-i+2} & \cdots & b_{n-1} & b_n \\ b_{n-i+2} & b_{n-i+3} & \cdots & b_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_n & \cdots & 0 & 0 \\ b_n & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.3}$$

Furthermore, suppose that the following statements hold.

- **H1-** *Eigenvalue assignment $\Delta_{n-1}^{-}(m_0, u) = 0$, $\Delta_{n-1}^{+}(m_0, u) > 0$, $K_{m_0}(1) > 0$, $(-1)^n K_{m_0}(-1) > 0$, $\Delta_i^{\pm}(m, u) > 0$, for $i = n-3, n-5, \cdots, 2$ (or 1), when n is odd (or even), respectively.*
- **H2-** *Transversality condition: $\left[\frac{d\Delta_{n-1}^{-}(m,u)}{dm}\right]_{m=m_0} \neq 0$.*
- **H3-** *Non-resonance condition: $\cos(2\pi/j) \neq \phi$, or resonance condition $\cos(2\pi/j) = \phi$, where $j = 3, 4, 5, \cdots$, and $\phi = 1 - 0.5P_{m_0}(1)\Delta_{n-3}^{-}(m_0, u)/\Delta_{n-2}^{+}(m_0, u)$. Then, a Neimark-Sacker bifurcation occurs at m_0.*

Lemma 5.2 ([33, 35]). *Let*

$$U_{k+1} = F_m(U_k),$$

be an n-dimensional system where $U_k \in R^n$ and $m \in R$ denotes the bifurcation parameter. In addition, suppose that the constraints (5.1)-(5.3) of Lemma 5.1 are satisfied and suppose that the following conditions are true:

- **P1-** *Eigenvalue criterion:* $K_{m_0}(-1) = 0$, $\Delta^{\pm}_{n-1}(m_0, u) > 0$, $K_{m_0}(1) > 0$, $\Delta^{\pm}_{i}(m_0, u) > 0$, $i = n-2, n-4, \cdots, 1$ *(or 1), when n is even (or odd), respectively.*
- **P2-** *Transversality criterion:* $\dfrac{\sum_{i=1}^{n}(-1)^{n-i} b_i'}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)b_{i-1}} \neq 0$, *where b_i' represents the derivative of $b(m)$ at $m = m_0$. Then, a period-doubling bifurcation exists at critical value m_0.*

Theorem 5.1. *Model (3.6) goes through a Neimark-Sacker bifurcation at the unique positive equilibrium point E, if the following constraints hold:*

$$
\begin{aligned}
1 - q &= 0,\\
1 + q &> 0,\\
1 - p + q &> 0,\\
1 + p + q &> 0.
\end{aligned}
$$

Thus, there exists a Neimark-Sacker bifurcation occurs at h if $\alpha, \gamma, \sigma, \varepsilon, \eta, \beta$, and h modify in a neighborhood of the set

$$
\mathcal{N}_{SB} = \left\{ (\alpha, \gamma, \sigma, \varepsilon, \eta, \beta, h) \in \mathrm{R}^7 \,\middle|\, \beta \in (0, 1],\ \sigma^2(2\gamma\eta + \alpha)^2 < 4\alpha\sigma(\alpha + \gamma\eta)^2,\ \ h = h_1 = \left[\frac{\beta(2\gamma\eta + \alpha)}{\alpha(\alpha + \gamma\eta)}\right]^{\frac{1}{\beta}} \right\}.
$$

Proof. Using Lemmas 4.4, and 5.1, and from the evaluation of Eq. (4.3) at E, we have

$$
\begin{aligned}
&\Delta^{\mp}_0(h) = 1 > 0,\\
&\Delta^{-}_1(h) = 1 - q = 0,\\
&\Delta^{+}_1(h) = 1 + q > 0,\\
&K(1) = 1 - p + q > 0,\\
&(-1)^2 K(-1) = 1 + p + q > 0,
\end{aligned}
$$

if and only if

$$
h = h_1 = \left[\frac{\beta(2\gamma\eta + \alpha)}{\alpha(\alpha + \gamma\eta)}\right]^{\frac{1}{\beta}} \ \textit{and}\ \sigma^2(2\gamma\eta + \alpha)^2 < 4\alpha\sigma(\alpha + \gamma\eta)^2.
$$

Moreover, the transversality condition is

$$
\left[\frac{d\Delta^{-}_1(h)}{dh}\right]_{h=h_1} = \left[\frac{d(1-q)}{dh}\right]_{h=h_1} = -\left(\frac{\sigma(2\gamma\eta + \alpha)}{\alpha + \gamma\eta}\right)\left[\frac{\beta(2\gamma\eta + \alpha)}{\alpha(\alpha + \gamma\eta)}\right]^{\frac{\beta-1}{\beta}} \neq 0.
$$

Then, the Neimark-Sacker bifurcation occurs at h_1. Hence, the proof is done. $\square$

Theorem 5.2. *Model (3.6) goes through a period-doubling bifurcation at the unique positive point E, if the following conditions hold:*

$$1+q>0,$$

$$1+p+q=0,$$

$$1-p+q>0.$$

Thus, the period-doubling bifurcation takes place at h if the parameters $(\alpha,\gamma,\sigma,\varepsilon,\eta,\beta,h)$ vary in a neighborhood of the set

$$\mathcal{P}_{DB}^{-}=\left\{\begin{array}{l} (\alpha,\gamma,\sigma,\varepsilon,\eta,\beta,h)\in \mathrm{R}^{7} \left| \sigma^{2}(2\gamma\eta+\alpha)^{2}\geq 4\alpha\sigma(\alpha+\gamma\eta)^{2},\ \ h\neq\left[\dfrac{4\beta(\alpha+\gamma\eta)}{\sigma(2\gamma\eta+\alpha)}\right]^{\frac{1}{\beta}}\right. \\ h=h_{-}=\left[\dfrac{\beta\left(\sigma(2\gamma\eta+\alpha)-\sqrt{\sigma^{2}(2\gamma\eta+\alpha)^{2}-4\alpha\sigma(\alpha+\gamma\eta)^{2}}\right)}{\alpha\sigma(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}},\ \beta\in(0,1] \end{array}\right\}.$$

Or,

$$\mathcal{P}_{DB}^{+}=\left\{\begin{array}{l} (\alpha,\gamma,\sigma,\varepsilon,\eta,\beta,h)\in \mathrm{R}^{7} \left| \sigma^{2}(2\gamma\eta+\alpha)^{2}\geq 4\alpha\sigma(\alpha+\gamma\eta)^{2},\ \ h\neq\left[\dfrac{4\beta(\alpha+\gamma\eta)}{\sigma(2\gamma\eta+\alpha)}\right]^{\frac{1}{\beta}}\right. \\ h=h_{-}=\left[\dfrac{\beta\left(\sigma(2\gamma\eta+\alpha)-\sqrt{\sigma^{2}(2\gamma\eta+\alpha)^{2}+4\alpha\sigma(\alpha+\gamma\eta)^{2}}\right)}{\alpha\sigma(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}},\ \beta\in(0,1] \end{array}\right\}.$$

Proof. Using Lemmas 4.4, and 5.2, and from the evaluation of Eq. (4.3) of system (3.6) at E, we have

$$\Delta_{0}^{\mp}(h)=1>0,$$

$$\Delta_{1}^{+}(h)=1+q>0,$$

$$(-1)^{2}K(-1)=1+p+q=0,$$

$$K(1)=1-p+q>0,$$

if and only if

$$h=h_{\mp}=\left[\frac{\beta\left(\sigma(2\gamma\eta+\alpha)-\sqrt{\sigma^{2}(2\gamma\eta+\alpha)^{2}\mp 4\alpha\sigma(\alpha+\gamma\eta)^{2}}\right)}{\alpha\sigma(\alpha+\gamma\eta)}\right]^{\frac{1}{\beta}}\ \ \textit{and}\ \sigma^{2}(2\gamma\eta+\alpha)^{2}\geq 4\alpha\sigma(\alpha+\gamma\eta)^{2}.$$

In addition, the transversality condition is

$$\frac{p'+q'}{p+2}=\frac{2\sqrt{\sigma^{2}(2\gamma\eta+\alpha)^{2}-4\alpha\sigma(\alpha+\gamma\eta)^{2}}}{(\alpha+\gamma\eta)}\left[\frac{\beta\left(\sigma(2\gamma\eta+\alpha)\mp\sqrt{\sigma^{2}(2\gamma\eta+\alpha)^{2}-4\alpha\sigma(\alpha+\gamma\eta)^{2}}\right)}{\alpha\sigma(\alpha+\gamma\eta)}\right]^{\frac{\beta-1}{\beta}}\neq 0,$$

with $p'=\frac{dp}{dh}\Big|_{h=h_{\mp}}$ and $q'=\frac{dq}{dh}\Big|_{h=h_{\mp}}$.

Then, the period-doubling bifurcation arises at h. Thus, the proof is done. ☐

6. Chaos Control

In this part, we discuss the chaos control of model (3.6) using the hybrid control approach. The chaos control technique is to improve the stability of a given model when an unexpected perturbations occur. In particular, the chaos control is the stabilization, by means of small system perturbations, of one of unstable periodic orbits. In order to apply the hybrid control approach on model (3.6), we write the corresponding control model in the following from:

$$\begin{cases} x_{n+1} = \theta x_n e^{(\alpha - \gamma y_n)\frac{h^\beta}{\beta}} + (1-\theta)x_n, \\ y_{n+1} = \theta y_n e^{\left(\sigma - \frac{\varepsilon y_n^2}{x_n(\eta + y_n)}\right)\frac{h^\beta}{\beta}} + (1-\theta)y_n. \end{cases} \tag{6.1}$$

Here, $0 < \theta < 1$ is a control parameter for the hybrid control technique. The Jacobian matrix of system (6.1) is given by

$$\mathcal{J}(E) = \begin{pmatrix} 1 & -\dfrac{\varepsilon\theta\alpha^2 h^\beta}{\beta\sigma(\alpha+\gamma\eta)} \\ \dfrac{\theta\sigma^2 h^\beta(\alpha+\gamma\eta)}{\beta\varepsilon\alpha} & 1 - \dfrac{\theta\sigma h^\beta(2\gamma\eta+\alpha)}{\beta(\alpha+\gamma\eta)} \end{pmatrix}, \tag{6.2}$$

whose characteristic equation is

$$\omega^2 - p\omega + q = 0, \tag{6.3}$$

where

$$p = 2 - \frac{\theta h^\beta \sigma(2\gamma\eta+\alpha)}{\beta(\alpha+\gamma\eta)},$$
$$q = 1 - \frac{\theta h^\beta \sigma(2\gamma\eta+\alpha)}{\beta(\alpha+\gamma\eta)} + \frac{\alpha\sigma\theta^2 h^{2\beta}}{\beta^2}.$$

Lemma 6.1. *The point $E = (\frac{\varepsilon\alpha^2}{\gamma\sigma(\alpha+\gamma\eta)}, \frac{\alpha}{\gamma})$ of system (6.1) is locally asymptotically stable, if the following inequality satisfies*

$$\left|2 - \frac{\theta h^\beta \sigma(2\gamma\eta+\alpha)}{\beta(\alpha+\gamma\eta)}\right| < 2 - \frac{\theta h^\beta \sigma(2\gamma\eta+\alpha)}{\beta(\alpha+\gamma\eta)} + \frac{\alpha\sigma\theta^2 h^{2\beta}}{\beta^2} < 2. \tag{6.4}$$

7. Numerical Computations and Discussion

This part shows the numerical simulation of the stability, the maximum Lyaponov exponents, the Neimark-Sacker and the period-doubling bifurcations, and the phase portrait of the governing system. The comparison of our work with other studies is explained as follows. The author in [36] only considered the stability of two species with Allee effect. Isık [37] examined the stability and the occurrence of a Neimark-Sacker bifurcation of a discrete-time predator-prey model with Allee effect on prey. Kangalgil [38] studied the Neimark-Sacker bifurcation of a discrete-time prey–predator model with Allee effect on the prey population. The considered derivatives in these models are ordinary.

However, we have considered the existence of a coexistence equilibrium point, the stability, the period-doubling bifurcation and the Neimark-Sacker bifurcation of a predator-prey model will Allee effect in which the derivatives are fractional. Fractional derivatives are generalization of the ordinary derivatives. The most significant outcomes in this work are covered in the following examples.

Example 7.1. *Here, we show the behavior of system* (3.6) *when it encounters a supercritical Neimark-Sacker bifurcation. In particular, we consider system* (3.6) *when* $\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$, $h \in [0, 0.19]$ *and the initial conditions* $P_1 = (0.24, 0.73)$ *and* $P_2 = (0.248, 0.71)$*. When Taking* h *as a bifurcation parameter, we observe that at* $h_1 = 0.1443$*, the positive equilibrium point* E *loses stability and system* (3.6) *undergoes the Neimark-Sacker bifurcation. The Jacobian matrix calculated at* E *is shown as*

$$J(E) = \begin{pmatrix} 1.0000 & -0.5678 \\ 6.4871 & -2.6835 \end{pmatrix},$$

with

$$K(\omega) = \omega^2 + 1.6835\omega + 1. \tag{7.1}$$

Hence, the roots of Eq. (7.1) *are* $\omega_{1,2} = -0.8418 \mp 0.5399i$ *where* $|\omega_{1,2}| = 1$*. Also, we have*

$$\begin{aligned}
&\Delta_0^{\mp}(h) = 1 > 0,\\
&\Delta_1^{-}(h) = 1 - q = 1 - 1 = 0,\\
&\Delta_1^{+}(h) = 1 + q = 2 > 0,\\
&K(1) = 1 - p + q = 3.6835 > 0,\\
&(-1)^2 K(-1) = 1 + p + q = 0.3165 > 0.
\end{aligned}$$

Note that the requirements of the Neimark-Sacker bifurcation shown in Theorem 5.1 are satisfied. In Fig. 1, the bifurcation diagrams (for x_n *and* y_n*) and maximum Lyaponov exponents (MLE) are given with the above parameters. In particular, Fig. 1a and Fig. 1b represent that the fixed point* E *is stable when* $0 < h < 0.1443$*. At* $h = 0.1443$*, the point* E *loses its stability to become unstable when* $0.1443 < h < 0.19$*. Fig. 3 and Fig. 4 present some phase portraits for system* (3.6) *and the evolution of* x_n *under the values of the bifurcation parameter* $h = 0.143, 0.1441, 0.14429, 0.1443, 0.14445$ *and* 0.1446*. For* $0 < h < 0.1443$*, the equilibrium point is stable and all orbits tends to* E *(see Fig. 3). If* $0.1443 \leq h < 0.19$*, we obtain an attracting closed invariant curve* Γ_s *encircling the equilibrium point. Here, the stability of the point* E *is lost because all trajectories asymptotically approaches the closed invariant curve* Γ_s *(see Figs. 4). Ultimately, chaotic attractors are occurred for* $h = 0.189$*, as presented in Diagram 5. Biologically, the Allee effect improves the dynamics of the considered system and balances the density of populations. In Fig. 2b, we illustrate how the distance* $D(E, \Gamma_s)$ *varies between the equilibrium point* E *and the closed curve* Γ_s*. We notice that the distance vanishes when* $0 < h < 0.1443$ *at which the point* E *is asymptotically stable. At* $h = 0.1443$*, the Neimark-Sacker* *bifurcation exits. Therefore, the curve Γ_s, which encloses the point E, emerges. It is worth noting that when the value of h increases inside $[0.1443, 0.17]$ the distance increases. The increment in the distance is plotted in Fig. 2a for diverse values of h. We can observe in this figure that the radius increases if h increases.*

Example 7.2. *This example discusses the behavior of model (3.6) when $\alpha = 1.52$, $\gamma = 4.81$, $\sigma = 2.39$, $\eta = 2.15$, $\varepsilon = 3.5$, $\beta = 0.55$, $h \in [0, 0.30]$ and the initial condition $P_0 = (0.059, 0.316)$. When the bifurcation parameter $h_- = 0.1280$, the positive equilibrium point E loses stability and the model (3.6) undergoes the period-doubling bifurcation. The Jacobian matrix calculated at E can be expressed by*

$$J(E) = \begin{pmatrix} 1.0000 & -0.1674 \\ 7.4748 & -1.6257 \end{pmatrix}.$$

Hence, the characteristic equation is given by

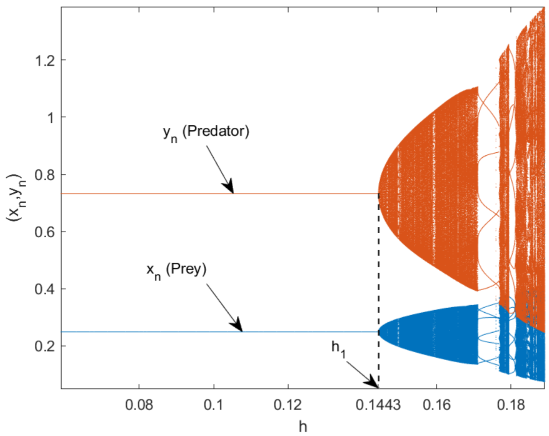
$$K(\omega) = \omega^2 + 0.6257\omega - 0.3743, \tag{7.2}$$

whose roots are $\omega_1 = -1$, $\omega_2 = 0.3743$ where $|\omega_2| \neq 1$. We also have

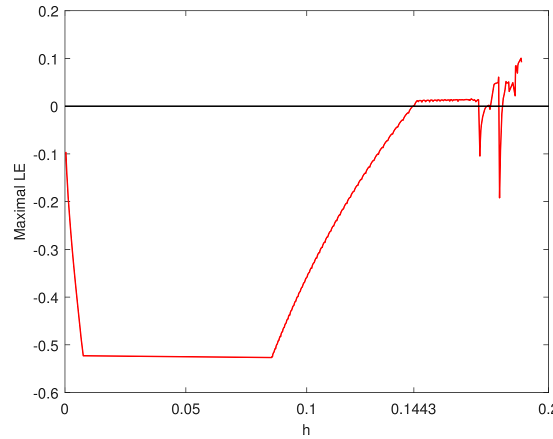
$$\begin{aligned} &\Delta_0^{\mp}(h) = 1 > 0, \\ &\Delta_1^{+}(h) = 1 + q = 0.6257 > 0, \\ &(-1)^2 K(-1) = 1 + p + q = 0, \\ &K(1) = 1 - p + q = 1.2514 > 0, \end{aligned}$$

Note that the requirements of the Neimark-Sacker bifurcation shown in Theorem 5.2 are satisfied. From bifurcation diagrams of x_n and y_n (shown in Fig. 6a and Fig. 6b, respectively), we notice that the positive point E of model (3.6) is stable for $0 < h < 0.1280$ (see Fig. 7a) while this point loses its stability through a period doubling bifurcation if $h \geq 0.1280$. Further, there is a period doubling cascade in orbits of periods-2,4,8 (see Fig. 7b, Fig. 7c, and Fig. 7d) and chaotic set for different values of h. The maximum Lyapunov exponents associated with Figs. 6a and 6b are clearly plotted in Fig. 6c. This certainly confirms the existence of the chaotic behavior and period orbits in the parametric space. In Fig. 6c, it is observed that some "Maximal LE" values are positive and some of them are negative. As a result, there exists a stable equilibrium point or stable period orbits in the chaotic region. We can deduce that the presence of Allee effect in the considered model contributes in improving the stability and balancing the populations.

Example 7.3. *To assess the performance of the hybrid control approach in improving chaotic (unstable) system dynamics (3.6), we take the same parameter values as given in Example 7.1 ($\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$) with $h = 0.16$. Then, it shows that the equilibrium point E of system (3.6) is unstable (see Fig. 8a and Fig. 8c). However, this equilibrium point is stable for* *the control system* (6.1) *if* $0 < \theta < 0.9497$ *(see Fig. 8b and Fig. 8d). This is certainly confirmed by the bifurcation figures of* x_n *and* y_n *as illustrated in Fig. 8e and Fig. 8f, respectively.*

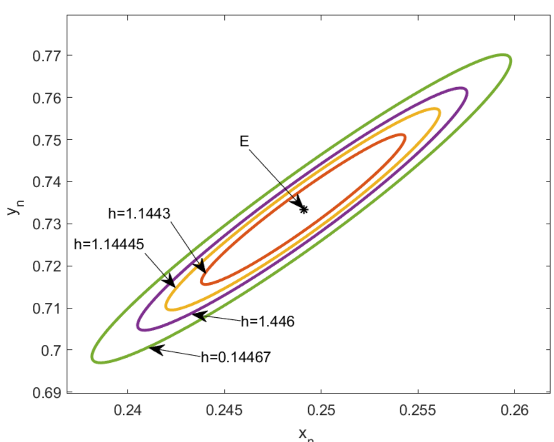


(a) Bifurcation diagram for x_n and y_n.

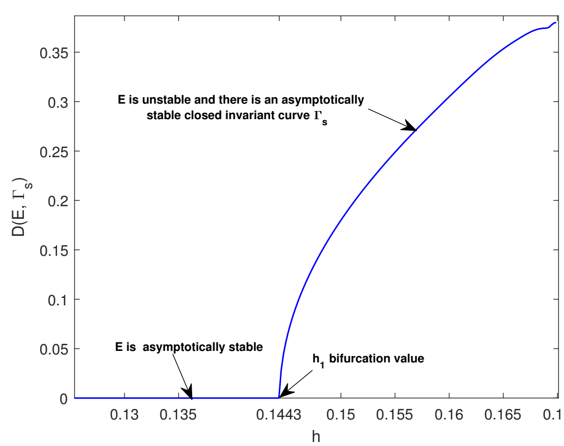


(b) Maximum Lyaponov exponents (MLE).

Figure 1. The Neimark-Sacker Bifurcation figure is represented in the left figure while the right figure illustrates the maximum Lyapunov exponent of model (3.6). These graphs are plotted under the values $\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$ and $h \in [0, 0.19]$.

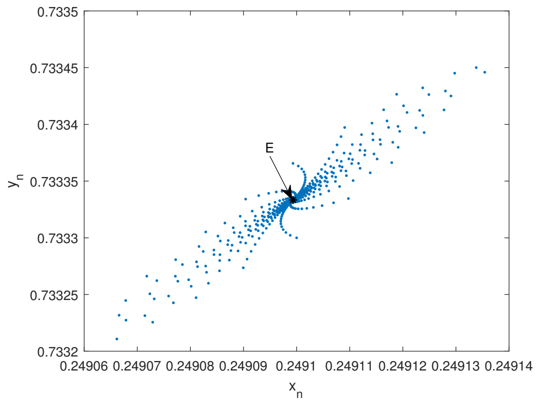


(a) The closed invariant curves Γ_s for $h = h_1 = 0.1443$, $h = 0.14445$, $h = 0.1446$, $h = 0.14467$.

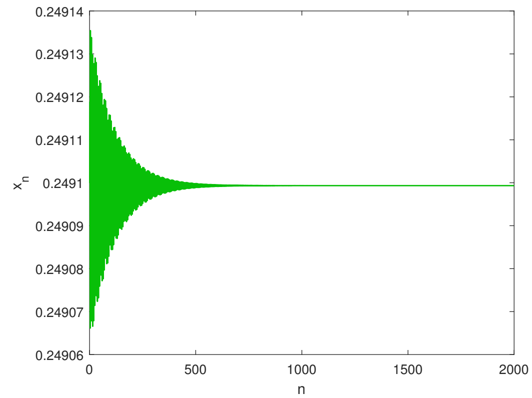


(b) The distance between the equilibrium point E and the closed invariant curves Γ_s for $h \in [0.13, 0.17]$.

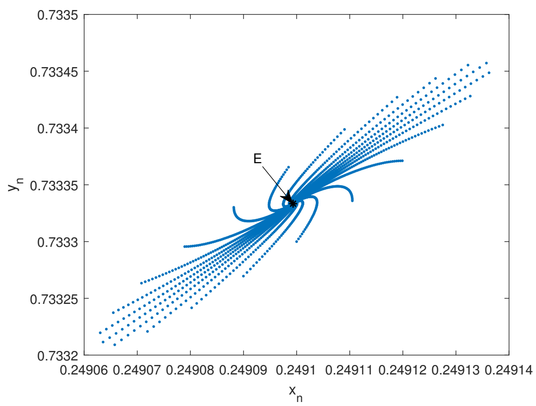
Figure 2. Figure (a) illustrates the closed invariant curves Γ_s while figure (b) presents the distance between the equilibrium point and the closed invariant curve Γ_s.



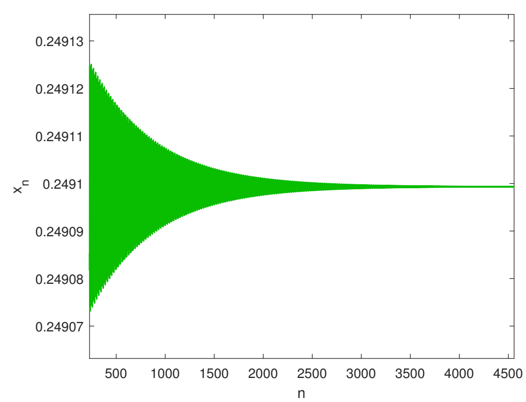
(a) Phase portrait of model (3.6) when $h = 0.143$.



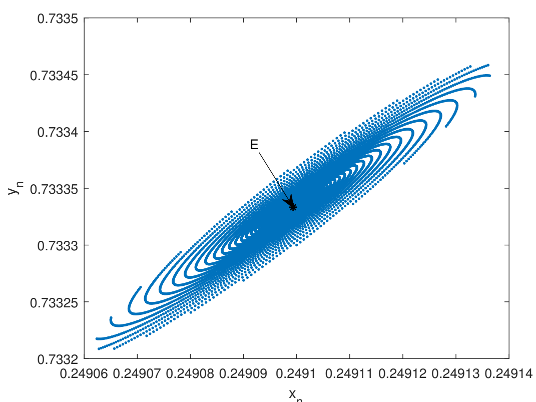
(b) Time evolution of x_n when $h = 0.143$.



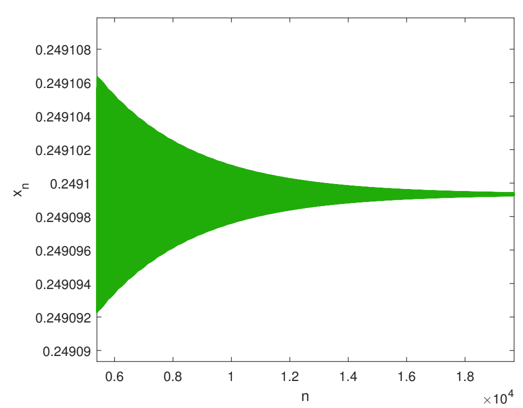
(c) Phase portrait of model (3.6) when $h = 0.1441$.



(d) Time evolution of x_n when $h = 0.1441$.

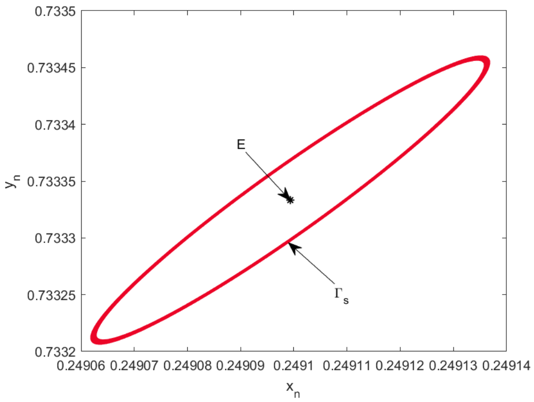


(e) Phase portrait of model (3.6) when $h = 0.14429$.

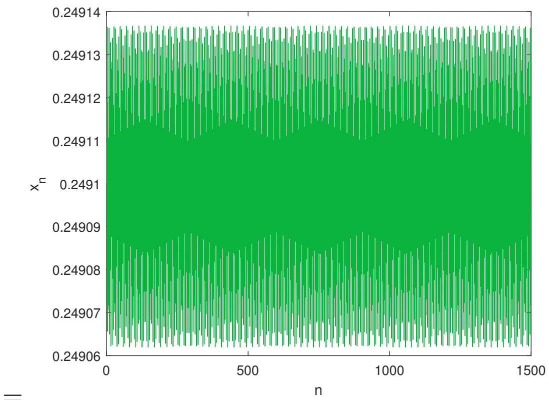


(f) Time evolution of x_n when $h = 0.14429$.

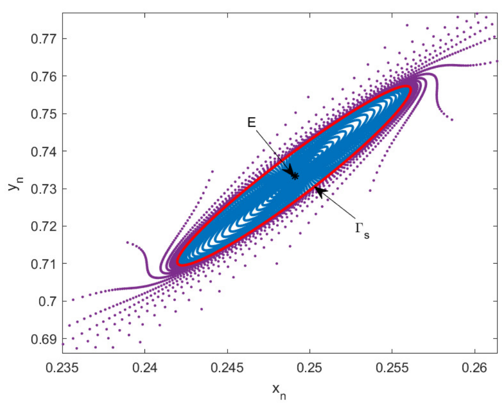
Figure 3. Figures (a), (c) and (e) demonstrate the phase portraits of model (3.6) for various values of the parameter h while the diagrams (b), (d) and (f) show time evolution of x_n under the values $\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$, $h \in [0, 0.443]$.



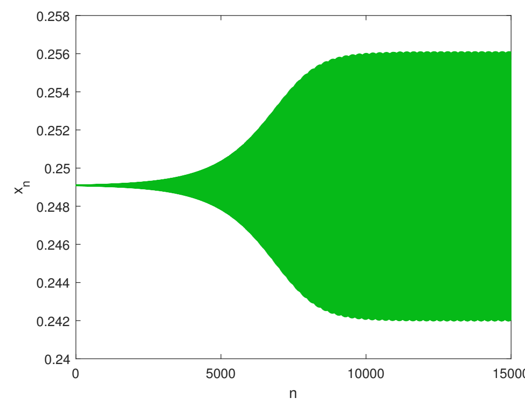
(a) Phase portrait of model (3.6) when $h = h_1 = 0.1443$.



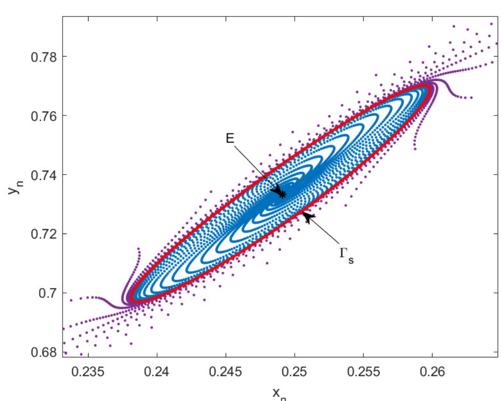
(b) Time evolution of x_n when $h = h_1 = 0.1443$.



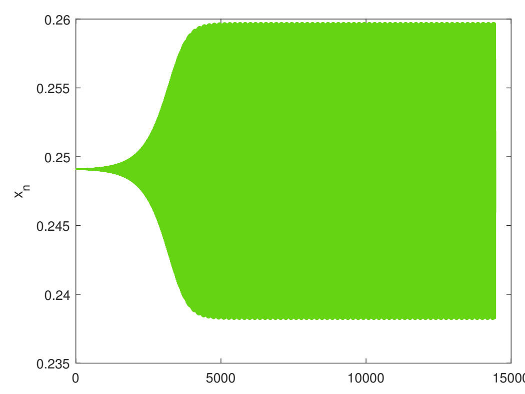
(c) Phase portrait of model (3.6) when $h = 0.14445$.



(d) Time evolution of x_n when $h = 0.14445$.

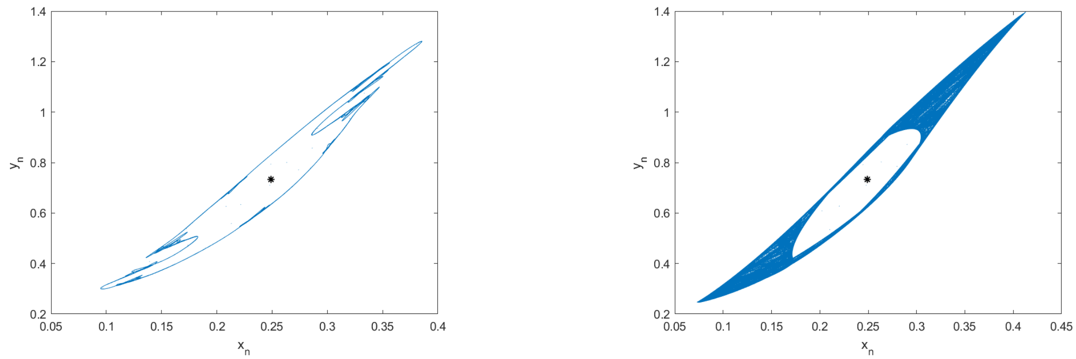


(e) Phase portrait of model (3.6) when $h = 0.1446$.



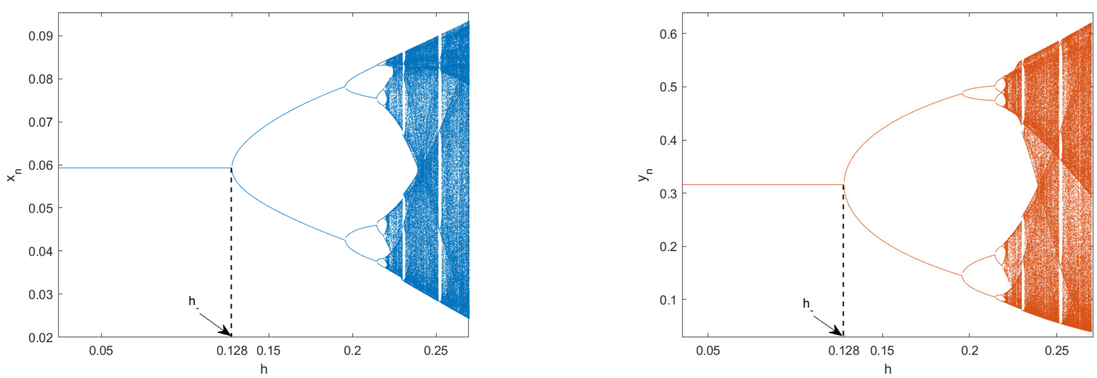
(f) Time evolution of x_n when $h = 0.1446$.

Figure 4. Figures (a), (c) and (e) demonstrate the phase portraits of model (3.6) for various values of the parameter h while the diagrams (b), (d) and (f) show time evolution of x_n under the values $\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$, $h \in [0.443, 0.19]$.

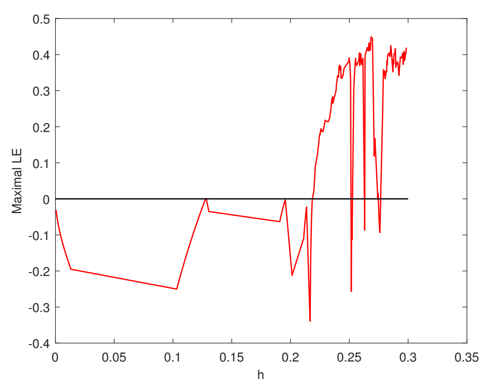


(a) Phase portrait of system (3.6) when $h = 0.182$. (b) Phase portrait of system (3.6) when $h = 0.189$.

Figure 5. These graphs show the strange attractors under the values $\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$, $h = 0.182$ (left) and $h = 0.189$ (right).

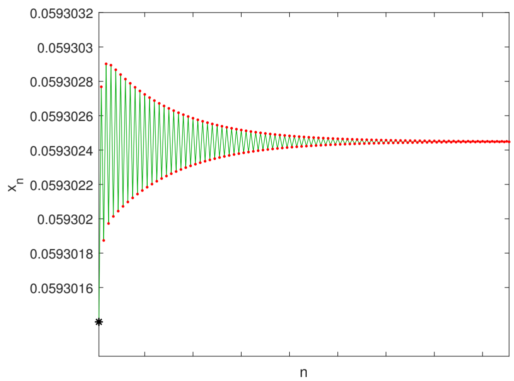


(a) Bifurcation sketch for x_n with $h \in [0.02, 0.29]$. (b) Bifurcation sketch for y_n with $h \in [0, 0.3]$.

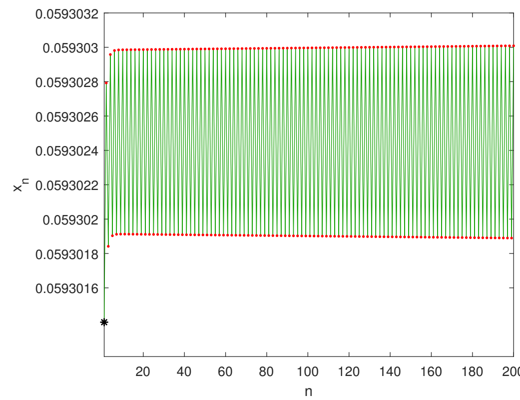


(c) Maximum Lyapunov exponent (MLE) with $h \in [0, 0.29]$.

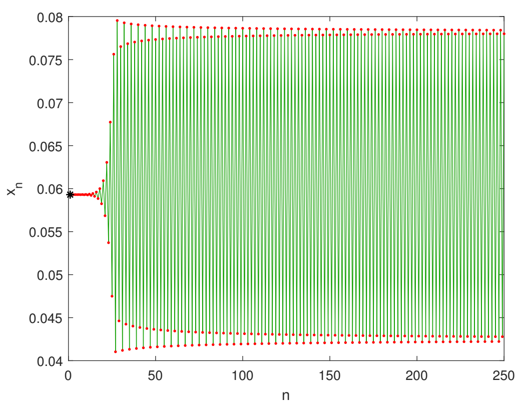
Figure 6. Figures (a) and (b) present the period doubling bifurcation of model (3.6) while figure (c) illustrates the maximum Lyapunov exponent for model (3.6) when $\alpha = 1.52$, $\gamma = 4.81$, $\sigma = 2.39$, $\eta = 2.15$, $\varepsilon = 3.5$, $\beta = 0.55$, and $h \in [0, 0.3]$.



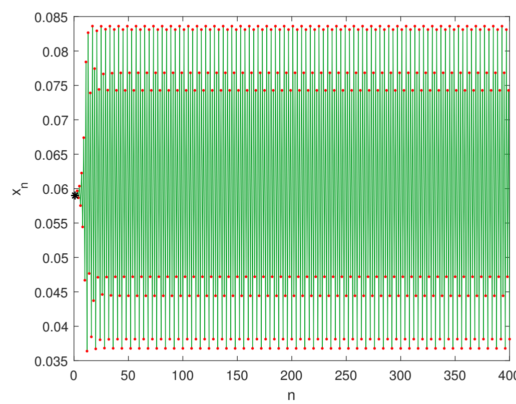
(a) Stability of model (3.6) when $h = 0.124$.



(b) Time evolution of x_n with period-2 when $h = 0.12801$.

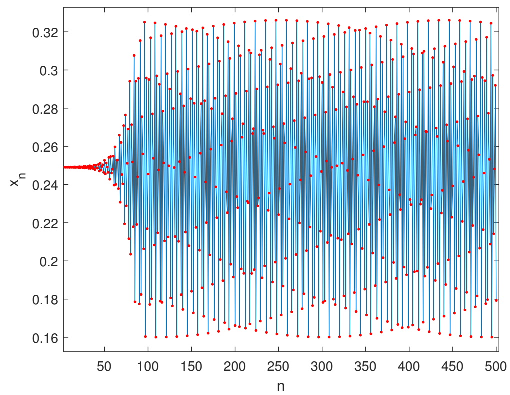


(c) Time evolution of x_n with period-4 when $h = 0.1955$.

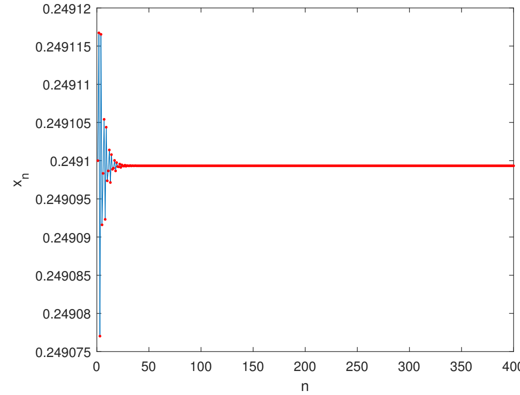


(d) Time evolution of x_n with period-8 when $h = 0.2145$.

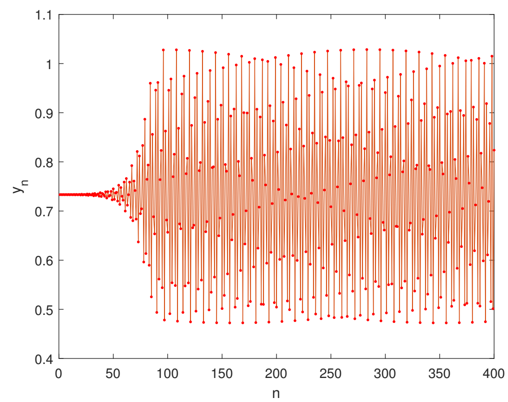
Figure 7. The time evolution of x_n for different values of the parameter h.



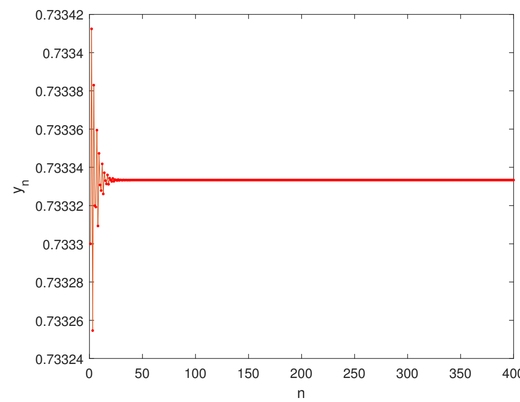
(a) Time evolution of x_n in model (3.6) when $h = 0.16$.



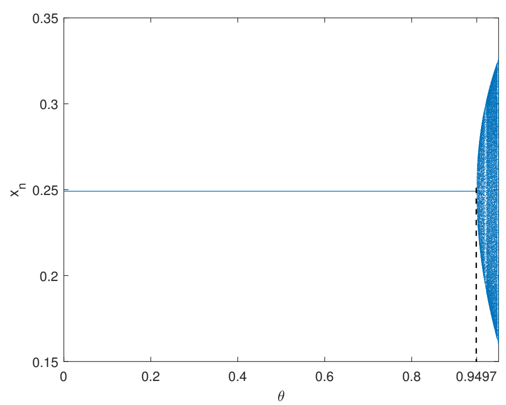
(b) Time evolution of x_n in model (6.1) when $h = 0.16$.



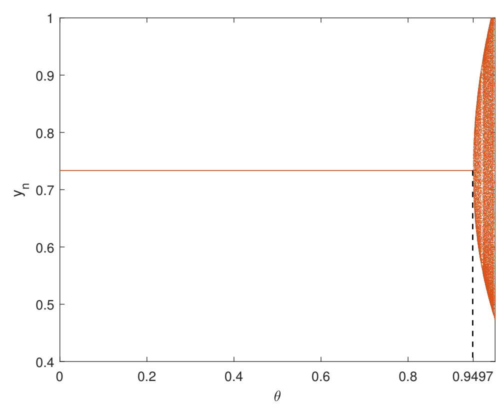
(c) Time evolution of y_n in model (3.6) when $h = 0.16$.



(d) Time evolution of y_n in model (6.1) when $h = 0.16$.



(e) Bifurcation diagram of x_n in model (6.1).



(f) Bifurcation diagram of y_n in model (6.1).

Figure 8. The time evolution and bifurcation diagrams of models (3.6) and (6.1) under the values $\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$, $h = 0.16$ and $\theta \in [0, 1]$ (Figs. 8b and 8d are plotted when $\theta = 0.82$).

8. Conclusion

This article has analysed the qualitative behaviors of model (1.1). This system has been discretized and converted into difference equations. The stability of the unique positive point E is extensively discussed in Lemma 4.3. In this Lemma, we have presented some conditions under which the point E is locally asymptotically stable, unstable source, unstable saddle, or non-hyperbolic. Lemma 4.4 shows that the stability of the point E is lost via a period-doubling bifurcation and via a Neimark-Sacker bifurcation. System (3.6) goes through a Neimark-Sacker bifurcation at the point E under some requirements given in Theorem 5.1. Numerically, system (3.6) undergoes a Neimark-Sacker bifurcation if we use the parameters values $\alpha = 2.2$, $\gamma = 3$, $\sigma = 2.9$, $\varepsilon = 3$, $\eta = 1.5$, $\beta = 0.5$, $h_1 = 0.1443$ and the initial conditions $P_1 = (0.24, 0.73)$ and $P_2 = (0.248, 0.71)$, as shown in Fig. 1. Furthermore, model (3.6) goes through a period-doubling bifurcation at the point E under some conditions given in Theorem 5.2. This can be clearly seen in Example 7.2, Fig. 6a and Fig. 6b when we assume that the parameter values are $\alpha = 1.52$, $\gamma = 4.81$, $\sigma = 2.39$, $\eta = 2.15$, $\varepsilon = 3.5$, $\beta = 0.55$, $h_- = 0.1280$ and the initial condition $P_0 = (0.059, 0.316)$. The chaos control of system (3.6) has been successfully explored using the hybrid control method. We have noticed that the point E of model (6.1) is locally asymptotically stable if condition (6.4) is satisfied. Biologically, the presence of the Allee effect in the proposed system improves the stability and balances the populations. One of the open problems which will be discussed later is the investigation of rank-one strange attractors to the considered system using Torus-breakdown theory. We can conclude that the approaches employed are applicable to other nonlinear dynamical systems.

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