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Note on Type L-Functions of Euler Product of Second Degree

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Abstract. In this paper we are concerned with a special case of Euler functions. We shall study the Euler product of degree two (Type of L-Euler functions) and give some results. More precisely, we shall deal with some Dirichlet series associated with a class of arithmetic $\{a_n\}$ under the condition that $a_p a_{p^k} = a_{p^{k+1}} + p^{\alpha} a_{p^{k-1}}$, provided p is prime, $k \geqslant 1$, and α is a fixed complex number. We will demonstrate that there is an Euler's product for the Dirichlet series $\sum_n a_n/n^s$. This result is important in analysis, especially in analytic number theory.

1. Introduction

We start this note by recalling the definition of Riemann Zeta Function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s \in \mathbb{C}, \quad \mathrm{Re}\, s > 1. \tag{1.1}$$

We shall write in accordance with standard notation in analytic number theory $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. So, for example, $\{s : \sigma > 1\}$ is the set of all s which have real part greater than one.

Next, it is known that the Euler's identity or Euler product is given by

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}, \quad \forall s \in \mathbb{C}. \tag{1.2}$$

Here in the product in the right hand side of (1.2), p runs over all primes, and on other hand (1.2) in particular contains this infinite product is convergent when $\sigma > 1$. Hence we have: $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\sigma > 1$ (see [11]). For more details about (1.1) and (1.2) see, for instance, the references [1–12].

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In this paper, we are interested in studying type of L-functions that appear from several sources in number theory, such as modular forms, elliptic curves, Galois representations which is Euler product of degree 2 or L-function of Euler product of degree 2. For further elaboration, please refer to the cited sources [13–19].

In fact, we shall deal with some Dirichlet series associated with a class of arithmetic functions $\{a_n\}$ satisfying the following condition: $a_p a_{p^k} = a_{p^{k+1}} + p^{\alpha} a_{p^{k-1}}$ with p is prime, $k \geqslant 1$ and α is a fixed complex number. We shall show that the Dirichlet series $\sum_n a_n/n^s$ has Euler's product.

Our main results are

Theorem 1.1. *Let $A \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ be given constants, and let $\{a_n\}$ be a sequence of complex numbers satisfying $|a_n| = O(n^A)$ for all $n \in \mathbb{Z}^+$. Assume that the sequence $\{a_n\}$ is multiplicative, not identically zero, and that $a_p a_{p^k} = a_{p^{k+1}} + p^{\alpha} a_{p^{k-1}}$ holds for every prime p and every $k \geqslant 1$. Then we have*

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a_p p^{-s} + p^{\alpha} p^{-2s})^{-1} \quad (\sigma > A+1).$$

Using Theorem 1.1, we can prove the next result.

Theorem 1.2. *Let $\alpha \in \mathbb{C}$ be a given constant and let $\{a_n\}$ be a sequence of complex numbers. Then the following two assertions are equivalent:*

(i) The sequence $\{a_n\}$ is multiplicative, not identically zero, satisfies $|a_n| \ll n^A$ for all $n \in \mathbb{Z}^+$ and some constant $A \in \mathbb{R}$; and for every prime p and every $k \geqslant 1$ we have

$$a_p a_{p^k} = a_{p^{k+1}} + p^{\alpha} a_{p^{k-1}}.$$

(ii) There is some $B \in \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a_p p^{-s} + p^{\alpha} p^{-2s})^{-1}$$

holds for all $s \in \mathbb{C}$ with $\sigma > B$ (in particular both the sum and the product converge when $\sigma > B$).

2. Review Some Basic Facts about Convergence of Dirichlet Series

A series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ referred to as a Dirichlet series. Let $a_1, a_2, \ldots \in \mathbb{C}$ and assume that the $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is convergent if $s = s_0 = \sigma_0 + it_0$, and consider $H > 0$ be an arbitrary constant. Then the series $\alpha(s)$ is uniformly convergent in the sector

$$\mathbb{S} = \{s = \sigma + it : \sigma \geqslant \sigma_0,\ |t - t_0| \leqslant H(\sigma - \sigma_0)\}.$$

By assuming a large value for H, we can observe that the series $\alpha(s)$ converges for all s in the halfplane $\sigma > \sigma_0$, indicating that the halfplane is the domain of convergence. To be more specific, we have

Theorem 2.1. ([10], Corollary 3.7) *Any Dirichlet series* $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *has an abscissa of convergence* $\sigma_c \in \mathbb{R} \cup \{\pm\infty\}$ *with the property that* $\alpha(s)$ *converges for all* s *with* $\sigma > \sigma_c$*, and for no* s *with* $\sigma < \sigma_c$*. Furthermore* $\alpha(s)$ *converges uniformly in any compact subset of* $\{s : \sigma > \sigma_c\}$*.*

In extreme circumstances, a Dirichlet series may converge anywhere ($\sigma_c = +\infty$) or everywhere ($\sigma_c = -\infty$) in the plane.

The series may converge everywhere along the line $\sigma_c + it$, at some places on the line but not all of them, or not at all when the abscissa of convergence is finite.

We define the abscissa of absolute convergence and contrast it with conditional convergence, σ_a, of a Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ as the infimum of those σ for which $\sum_{n=1}^{\infty} |a_n| n^{-s} < \infty$ (By Theorem 2.1, σ_a equals the abscissa of convergence of the Dirichlet series $\sum_{n=1}^{\infty} |a_n| n^{-s}$).

Since $\left|a_n n^{-s}\right| = |a_n| n^{-\sigma}$ we immediately see that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent for all s with $\sigma > \sigma_a$, but not for any s with $\sigma < \sigma_a$. Hence if $\sigma_c < \sigma_a$ then $\sum_{n=1}^{\infty} a_n n^{-s}$ is conditionally convergent for all s with $\sigma_c < \sigma < \sigma_a$. In fact one can prove that

Theorem 2.2. ([11], Proposition 3.9) *For every Dirichlet series we have* $\sigma_c \leqslant \sigma_a \leqslant \sigma_c + 1$*.*

The fact that a Dirichlet series' coefficients can only be derived from the resulting function is crucial:

Theorem 2.3. ([11], Proposition 3.10) *If* $\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}$ *for all* s *with* $\sigma > \sigma_0$ *then* $a_n = b_n$ *for all* $n \geqslant 1$*.*

For more details abut convergence of Dirichlet series the reader can see, for example, ([11], pp. 36-45) and ([12], pp. 137-164).

3. Basic Result

We borrow the following fact from ([9], Theorem 5) (also see [11], Proposition 2.7) which is originally is discovered by Euler in 1737, is sometimes called the analytic version of the fundamental theorem of arithmetic.

Theorem 3.1. *Let* $f : \mathbb{Z}^+ \to \mathbb{C}$ *be a multiplicative function which is not identically zero. Then*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} (1 + f(p) + f(p^2) + \cdots), \tag{3.1}$$

provided that the series on the left is absolutely convergent, in which case the product is also absolutely convergent.

If f *is multiplicative without restrictions, then also*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \frac{1}{1 - f(p)}, \tag{3.2}$$

In each case of (3.1) and (3.2), the product is called the Euler product of the series. We shall utilize this important result in our work more than one time.

4. Proof of Theorem 1.1

For any s with $\sigma > A+1$ the Dirichlet series $\Sigma_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent, since $\Sigma_{n=1}^{\infty} \left|a_n n^{-s}\right| \ll \sum_{n=1}^{\infty} n^{A-\sigma} < \infty$. Hence for every such s we have, Theorem 3.1:

$$\sum_{n=1}^{\infty} a_n n^s = \prod_p (1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots). \tag{4.1}$$

Here for each prime p the sum $1 + a_p p^s + a_{p^2} p^{-2s} + \cdots$ is absolutely convergent, and hence we may multiply termwise with $(1 - a_p p^{-s} + p^{\alpha} p^{-2s})$ to get

$$\begin{aligned}
&(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots)(1 - a_p p^{-s} + p^{\alpha} p^{-2s}) \\
&= 1 - a_p p^{-s} + a_p p^{-2s} + \sum_{k=2}^{\infty} (a_{p^k} - a_p a_{p^{k-1}} + p^{\alpha} a_{p^{k-2}}) p^{-ks} \\
&= 1 + 0 + \sum_{k=2}^{\infty} 0 p^{-ks} = 1,
\end{aligned}$$

where we used our assumption about $\{a_n\}$. Hence for each prime p (and our fixed s with $\sigma > A+1$) we have

$$1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots = \frac{1}{1 - a_p p^{-s} + p^{\alpha} p^{-2s}}, \tag{4.2}$$

and hence from (4.1) we get

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \frac{1}{1 - a_p p^{-s} + p^{\alpha} p^{-2s}}.$$

5. Proof of Theorem 1.2

To prove Theorem 1.2, we need the help of the following well known lemma.

Lemma 5.1. *Let $f : \mathbb{Z}^+ \to \mathbb{C}$ be a multiplicative function which is not identically zero and assume that the product*

$$\prod_p (1 + |f(p)| + \left|f(p^2)\right| + \cdots)$$

is absolutely convergent (in particular we assume that each sum $1+|f(p)|+\left|f(p^2)\right|+\cdots$ is convergent). Then also the sum $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, and hence by Theorem 3.1 we have

$$\sum_{n=1}^{\infty} f(n) = \prod_p (1 + |f(p)| + \left|f(p^2)\right| + \cdots).$$

Proof. It follows from the assumption in the lemma that

$$X = \prod_p (1 + |f(p)| + \left|f(p^2)\right| + \cdots)$$

is a finite real number ($X \geqslant 1$). We will prove the absolute convergence of $\Sigma_{n=1}^{\infty} |f(n)|$ by proving $\Sigma_{n=1}^{\infty} |f(n)| \leqslant X$. To prove this it suffices to prove that $\Sigma_{n=1}^{N} |f(n)| \leqslant X$ for each $N \in \mathbb{Z}^+$.

Let N be given, and let $p_1, p_2, \ldots, p_M$ be all prime numbers $\leqslant N$. Then

$$\prod_{k=1}^{M} (1 + |f(p_k)| + \left|f(p_k^2)\right| + \cdots) \leqslant \prod_{p} (1 + |f(p)| + \left|f(p^2)\right| +) \leqslant X.$$

On the other hand, by Cauchy's theorem, the finite product $\Pi_{k=1}^{M}(1 + |f(p_k)| + \left|f(p_k^2)\right| + \cdots)$ may be multiplied out as

$$\prod_{k=1}^{M} (1 + |f(p_k)| + \left|f(p_k^2)\right| + \cdots) = \sum_{v_1}^{\infty} \sum_{v_2}^{\infty} \cdots \sum_{v_M}^{\infty} \left|f(p_1^{v_1} p_2^{v_2} \cdots p_M^{v_M})\right|.$$

In the last sum, $p_1^{v_1} p_2^{v_2} \cdots p_M^{v_M}$ runs through exactly those positive integers which only have prime factors $p_1 p_2, \ldots, p_M$ (or a subset of these) in their prime factorization. In particular $p_1^{v_1} p_2^{v_2} \cdots p_M^{v_M}$ visits all the numbers $1, 2, \ldots, N$. Hence

$$\sum_{n=1}^{N} |f(n)| \leqslant \prod_{k=1}^{M} (1 + |f(p_k)| + \left|f(p_k^2)\right| + \cdots) \leqslant X.$$

This concludes the proof.

Proof of Theorem 1.2. We have already proved that (i) $\Longrightarrow$ (ii) (any $B \geqslant A + 1$ works), cf. Theorem 1.1. We now prove (ii) $\Longrightarrow$ (i). Assume that (ii) holds. Then $\sum_{n=1}^{\infty} a_n n^{-s}$ has abscissa of convergence $\leqslant B$ and hence as in the proof of Theorem 2.2 ([11, Proposition 3.9]) we know that $|a_n| \ll n^{B+\varepsilon}$ for any fixed $|a_n| \leqslant Cn^{B+1}$ and all $n \in \mathbb{Z}^+$; in particular there is a constant $C > 1$ such that $|a_n| \leqslant Cn^{B+1}$ for all $n \in \mathbb{Z}^+$. Now fix a real constant B_0 so large that $2^{B_0 - B - 1} > 2C$, (thus $B_0 > B + 1$) and $2B_0 - 2 > \operatorname{Re}\alpha$. Then for all $n \geqslant 2$ we have $Cn^{B+1} < \frac{1}{2} n^{B_0}$, since $n^{B_0 + B - 1} \geqslant 2^{B_0 + B - 1} > 2C$; hence

$$|a_n| \leqslant \frac{1}{2} n^{B_0}, \quad \forall n \geqslant 2 \tag{5.1}$$

Now let us define a new sequence $b_1, b_2, b_3, \ldots$ by the following recipe: Set b_1; for each prime p set $b_p := a_p$, and define $b_{p^2}, b_{p^3}, \ldots$ recursively by $b_{p^{k+1}} = b_p b_{p^k} - p^\alpha b_{p^{k-1}}$ for $k = 1, 2, \ldots,$ and finally define b_n for composite n in the unique way which makes b_n multiplicative. We intend to prove that $b_n = a_n$ for all n, this will clearly complete the proof that (i) holds. For each prime p, note that the recursion formula gives for b_{p^2}, using (5.1):

$$\left|b_{p^2}\right| = \left|b_p^2 - p^\alpha b_1\right| \leqslant \left|b_p^2\right| + p^{\operatorname{Re}\alpha} |b_1| \leqslant \frac{1}{4} p^{2B_0} + p^{\operatorname{Re}\alpha}.$$

Using $2B_0 - 2 > \operatorname{Re}\alpha$ we see that this is

$$\left|b_{p^2}\right| \leqslant \frac{1}{4} p^{2B_0} + p^{2B_0 - 2} \leqslant \frac{1}{4} p^{2B_0} + \frac{1}{4} p^{2B_0} = \frac{1}{2} p^{2B_0}.$$

Similarly one proves by induction that (Similarly we prove by induction that)

$$\left|b_{p^k}\right| \leqslant \frac{1}{2} p^{B_0 k}, \quad \forall k \geqslant 1. \tag{5.2}$$

Note that already done for $k = 1, 2, \ldots$. Now take $k \geqslant 3$ and assume that the inequality is true for $k-1$ and $k-2$. By the recursion formula we have

$$\begin{aligned}\left|b_{p^k}\right| &= \left|b_p b_{p^{k-1}} - p^{\alpha} b_{p^{k-2}}\right| \leqslant \frac{1}{4} p^{B_0 + B_0(k-1)} + \frac{1}{2} p^{\operatorname{Re}\alpha} p^{B_0(k-2)} \\ &< \frac{1}{4} p^{B_0 k} + \frac{1}{2} p^{2B_0 - 2} p^{B_0(k-2)} \leqslant \frac{1}{2} p^{B_0 k}.\end{aligned}$$

It follows from (5.2) that

$$1 + \left|b_p p^{-s}\right| + \left|b_{p^2} p^{-2s}\right| + \left|b_{p^3} p^{-3s}\right| + \cdots < \infty$$

for all s with $\sigma > B_0$, and in fact if $\sigma > B_0 + 1$, then

$$\sum_{k=1}^{\infty} \left|b_{p^k} p^{-ks}\right| \leqslant \frac{1}{2} \sum_{k=1}^{\infty} p^{(B_0 - \sigma)k} = \frac{1}{2} \frac{p^{B_0 - \sigma}}{1 - p^{B_0 - \sigma}} < \frac{1}{2} \frac{p^{B_0 - \sigma}}{1 - 2^{-1}} = p^{B_0 - \sigma},$$

and thus for $\sigma > B_0 + 1$ the product

$$\prod_p \left(1 + \left|b_p p^{-s}\right| + \left|b_{p^2} p^{-2s}\right| + \left|b_{p^3} p^{-3s}\right| + \ldots\right)$$

is absolutely convergent. Hence by Lemma 5.1 we have

$$\prod_p \left(1 + b_p p^{-s} + b_{p^2} p^{-2s} + b_{p^3} p^{-3s} + \ldots\right) = \sum_{n=1}^{\infty} b_n n^{-s}$$

for all s with $\sigma > B_0 + 1$ (the right hand side also being absolutely convergent for these s). But on the other hand, because of the recursion formula for b_{p^k}, we have for each prime p (by the same computation as in the proof to Theorem 1.1):

$$1 + b_p p^{-s} + b_{p^2} p^{-2s} + b_{p^3} p^{-3s} + \ldots = \frac{1}{1 - b_p p^{-s} + p^{\alpha} p^{-2s}} = \frac{1}{1 - a_p p^{-s} + p^{\alpha} p^{-2s}}.$$

Hence, using now our assumption that (ii) holds, we get

$$\sum_{n=1}^{\infty} b_n n^{-s} = \prod_p \left(1 + b_p p^{-s} + b_{p^2} p^{-2s} + b_{p^3} p^{-3s} + \ldots\right) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{\alpha} p^{-2s}} = \sum_{n=1}^{\infty} a_n n^{-s}$$

for all s with $\sigma > B_0 + 1$. Hence by Theorem 2.3 we have $a_n = b_n$ for all $n \in \mathbb{Z}^+$.

Remark 3.1 From the relations in (i) above one can derive the following general multiplication formula:

$$a_m a_n = \sum_{d \mid (m,n)} d^{\alpha} a_{mn/d^2}, \quad \forall m, n \in \mathbb{Z}^+.$$

Remark 3.2 Note that for any $\alpha \in \mathbb{C}$, the sequence $a_n = \sigma_{\alpha}(n)$ satisfies conditions (i) $\Longleftrightarrow$ (ii) above, by Remark 3.3 below. However, as already mentioned, similar types of L-functions also arise from more advanced sources, e.g. modular forms, elliptic curves and Galois representations.

Remark 3.3 We have used the following result.

Theorem 5.1. *Let $d(n)$ be the number of divisors of n, for each $n \in \mathbb{Z}^+$. Then for $\sigma > 1$, we have $\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta(s)^2$. More generalization for any $\alpha \in \mathbb{C}$ if we set $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, then $\sum_{n=1}^{\infty} \sigma_\alpha(n)n^{-s} = \zeta(s)\zeta(s-\alpha)$ when $\sigma > \max(1, 1 + \operatorname{Re}\alpha)$.*

Proof. Note that $d(n) = \sigma_0(n)$; hence the first part of the theorem follows as a special case of the second one.

When $\sigma > \max(1, 1 + \operatorname{Re}\alpha)$ we have $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ and $\zeta(s-\alpha) = \sum_{m=1}^{\infty} m^{\alpha-s}$, with both sums being absolutely convergent. Hence we may multiply the two sums termwise to get an absolutely convergent double sum:

$$\zeta(s)\zeta(s-\alpha) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} k^{-s}m^{\alpha-s} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m^{\alpha}(km)^{-s}.$$

Here substitute $n = km$; then we get

$$\zeta(s)\zeta(s-\alpha) = \sum_{n=1}^{\infty}\left(\sum_{m|n} m^{\alpha}\right) n^{-s} = \sum_{n=1}^{\infty} \sigma_\alpha(n)n^{-s}.$$

□

6. Conclusion

Using some well-known and classical results of basic analytic number theory, we have proved that the Dirichlet series $\sum_n a_n n^{-s}$ has a Euler's product (Theorem 1.1). In dead, we have motivated the study comes from number theory, modular functions, elliptic curves etc. We have considered the following condition: $a_p a_{p^k} = a_{p^{k+1}} + p^\alpha a_{p^{k-1}}$ with p is prime, $k \geqslant 1$ and α is a fixed complex number. Then Theorem 1.1 and 1.2 holds for sequences satisfying this condition.

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