Connectedness in Single-Valued Neutrosophic Soft Grill Topological Spaces

Yaser Saber$^{1,2,*}$

$^1$Department of Business Administration, College of Science and Human Studies, Hotat Sudair, Majmaah University, Majmaah, 11952, Saudi Arabia

$^2$Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut, 71524, Egypt

*Corresponding author: y.sber@mu.edu.sa

Abstract. The incentive of this article is to continue discovering more interesting results and concepts related to the single-valued neutrosophic soft topological spaces. The concept of the single-valued neutrosophic soft operator $\phi$ created from a single-valued neutrosophic soft grill $(K^\sigma, K^\tau, K^\delta)$ and a single-valued neutrosophic soft topological space $(B, \tilde{\sigma}, \tilde{\tau}, \tilde{\delta})$ is presented. Connectedness of single-valued neutrosophic soft topological spaces with single-valued neutrosophic soft grills is given. Moreover, the concept of $\gamma$-connectedness associated with a single-valued neutrosophic soft operator $\gamma$ is extended on the set $B$.

1. Introduction and Preliminaries

In real life, there are many mathematical tools that are precise, deterministic, and crisp-like for that of computing, reasoning, and formal modeling in character. On the other hand, others are not, such as the problems in engineering, social science, economics, environment and medical science, etc. The inadequacy of the classical parameterization tool in general may be considered to be the reason for these difficulties. For this and to avoid the above difficulties, Molodtsov (1999) [14] created the concept of soft set theory as a new mathematical tool for dealing with uncertainties and vagueness. The soft set theory was applied in several directions, such as game theory, theory of measurement, Riemann integration, smoothness of functions, and Perron integration by Molodtsov (2001) [15]. Practical application of soft sets in decision-making problems has been also given by Maji et al. (2002) [13].
Maji et al. (2001) [12], have also introduced the concept of fuzzy soft set which is a more generalized concept and a combination of fuzzy set (Zadeh 1965) [30] and soft set (Molodtsov 1999) [14] and also studied some of its properties. Later, some researchers studied the concept of fuzzy soft sets (Acharjee and Tripathy [4]; Ahmad and Kharal (2009) [5]; Kharal and Ahmad (2009) [11], Tanay and Kandemir (2011) [26]; Aygünoglu et al. (2014) [8]; Çetkin et al. (2014) [9]; Abbas et al. (2016, 2018) [1,2]; Gunduz and Bayramov (2013) [10]).

Smarandache [24] initiated the neutrosophic set as a generalization of an intuitionistic fuzzy set. Salama et al [23] set up the notion of neutrosophic crisp set. Correspondingly, Salama and Alblowi [22], introduced neutrosophic topology as they claimed a number of its characteristics. The single-valued neutrosophic set concept was given by Wang et al [27]. The concept of fuzzy ideal topological spaces, single-valued neutrosophic ideal, single-valued neutrosophic ideal open local function, connectedness in single-valued neutrosophic topological spaces \((\mathcal{L}, \tilde{\sigma}, \tilde{\tau}, \tilde{\delta})\) and compactness in single-valued neutrosophic ideal topological spaces and studied the basic notions by following Šostak’s [25] fuzzy topological spaces were obtained by Saber et al [3,6,7,16–21,31,32].

This article aims to explore and define the properties and characterizations of the single-valued neutrosophic soft operator \(\Theta\) in single-valued neutrosophic soft grill topological spaces. Also, an \(r\)-single-valued neutrosophic soft grill connectedness which has relations with an \(r\)-single-valued neutrosophic soft connectedness and some basic definitions and theorems about it have been given and investigated. Moreover, the \(r\)-single-valued neutrosophic soft \(\aleph\)-connectedness and \(r\)-fuzzy soft \(\aleph\)-disconnectedness related to a single-valued neutrosophic soft operator \(\aleph\) on the set \(B\) is introduced.

Throughout this work, \(B\) denotes the initial universe, \(\xi^B\) is the collection of all single-valued neutrosophic sets (simply, svns) on \(B\) (where, \(\xi = [0, 1]\), \(\xi_0 = (0, 1]\) and \(\xi_1 = [0, 1)\)) and \(E\) is the set of each parameters on \(B\).

All characterizations and concepts of svns are originate in Smarandache [24], Wang et al. [27], Yang et al. [28], Ye et al. [29].

\(\mathcal{H}_z\) is a single-valued neutrosophic soft set [17] (simply, svnfs) on \(B\) where, \(\mathcal{H}_z : E \rightarrow \xi^B\); i.e., \(\mathcal{H}_z \cong \mathcal{H}(\bar{e})\) is a svns on \(B\), for all \(\bar{e} \in z\) and \(\mathcal{H}(\bar{e}) = (0, 1, 1)\), if \(\bar{e} \not\in \ell\).

The svns \(\mathcal{H}(\bar{e})\) is termed as an element of the svnfs \(\mathcal{H}_z\). Thus, a svnfs \(\mathcal{H}_E\) on \(B\) it can be defined as:

\[
(\mathcal{H}, E) = \{(\bar{e}, \mathcal{H}(\bar{e})) \mid \bar{e} \in E, \mathcal{H}(\bar{e}) \in \xi^B\}
\]

\[
= \{(\bar{e}, (\sigma_{\mathcal{H}}(\bar{e}), \tau_{\mathcal{H}}(\bar{e}), \delta_{\mathcal{H}}(\bar{e}))) \mid \bar{e} \in E, \mathcal{H}(\bar{e}) \in \xi^B\},
\]

where \(\sigma_{\mathcal{H}} : E \rightarrow \xi\) (\(\sigma_{\mathcal{H}}\) is termed as a membership function), \(\tau_{\mathcal{H}} : E \rightarrow \xi\) (\(\tau_{\mathcal{H}}\) is termed as indeterminacy function), and \(\delta_{\mathcal{H}} : E \rightarrow \xi\) (\(\delta_{\mathcal{H}}\) is termed as a non-membership function) of svnf set. \((\overline{B, E})\) refers to the collection of all svnfs on \(B\) and is termed svnfs-universe.

A svnfs \(\mathcal{H}_z\) on \(B\) is termed as a null svnfs (simply, \(\phi\)), if \(\sigma_{\mathcal{H}}(\bar{e}) = 0, \tau_{\mathcal{H}}(\bar{e}) = 1\) and \(\delta_{\mathcal{H}}(\bar{e}) = 1\), for any \(\bar{e} \in E\).
A svnf set $\mathcal{H}_E$ on $\mathcal{B}$ is termed as an absolute svnf set (simply, $\mathcal{E}$), if $\sigma_{\mathcal{H}}(\tilde{e}) = 1, \tau_{\mathcal{H}}(\tilde{e}) = 0$ and $\delta_{\mathcal{H}}(\tilde{e}) = 0$, for any $\tilde{e} \in E$.

A svnf set $\mathcal{H}_E$ on $\mathcal{B}$ is termed as an t-absolute svnf set (simply, $\mathcal{E}^t$), if $\sigma_{\mathcal{H}}(\tilde{e}) = t, \tau_{\mathcal{H}}(\tilde{e}) = 0$ and $\delta_{\mathcal{H}}(\tilde{e}) = 0$, for any $\tilde{e} \in E$ and $t \in \xi$.

For $h_z, l_y \in \mathcal{B}$, $h_z \mathcal{R}_y = \phi$ if $h_z \subseteq l_y$ and $h_z \mathcal{R}_y = h_z \cap (l_y)^c$ otherwise.

**Definition 1.1.** [17] Let $h_z, l_y$ be svnf sets over $\mathcal{B}$. The union of svnf sets $h_z, l_y$ is a svnf set $g_x$, where $x = z \cup y$ and for any $\tilde{e} \in x$ and $\sigma_{g} : E \to \xi$ ($\sigma_{g}$ called truth-membership) $\tau_{g} : E \to \xi$ ($\tau_{g}$ called indeterminacy), $\delta_{g} : E \to \xi$ ($\delta_{g}$ called falsity-membership) of $g_x$ are as next:

$$
\sigma_{g(\tilde{e})}(x) = \begin{cases} 
\sigma_{h(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{h(\tilde{e})}(x) \cup \sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z \cup y. 
\end{cases}
$$

$$
\tau_{g(\tilde{e})}(x) = \begin{cases} 
\sigma_{h(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{h(\tilde{e})}(x) \cap \sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z \cap y. 
\end{cases}
$$

$$
\delta_{g(\tilde{e})}(x) = \begin{cases} 
\sigma_{h(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{h(\tilde{e})}(x) \cap \sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z \cap y. 
\end{cases}
$$

**Definition 1.2.** [17] The intersection of svnf sets $h_z, l_y$ is a svnf set $g_x$, where $x = z \cap y$ and for any $\tilde{e} \in x$, $g_{\tilde{e}} = h_{\tilde{e}} \mathcal{R}_y l_{\tilde{e}}$. We write as next:

$$
\sigma_{g(\tilde{e})}(x) = \begin{cases} 
\sigma_{h(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{h(\tilde{e})}(x) \cap \sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z \cap y. 
\end{cases}
$$

$$
\tau_{g(\tilde{e})}(x) = \begin{cases} 
\sigma_{h(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{h(\tilde{e})}(x) \cup \sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z \cup y. 
\end{cases}
$$

$$
\delta_{g(\tilde{e})}(x) = \begin{cases} 
\sigma_{h(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z - y, \\
\sigma_{h(\tilde{e})}(x) \cup \sigma_{l(\tilde{e})}(x), & \text{if } \tilde{e} \in z \cup y. 
\end{cases}
$$

**Definition 1.3.** [17] Let $h_z, l_y \in \mathcal{B}$, $E$. Then,

1. $h_z$ is a svnf subset of $l_y$ (simply, $h_z \subseteq l_y$) iff for every $\tilde{e} \in E$,

$$
\sigma_{h}(\tilde{e}) \leq \sigma_{l}(\tilde{e}), \quad \tau_{h}(\tilde{e}) \geq \tau_{l}(\tilde{e}), \quad \delta_{h}(\tilde{e}) \geq \delta_{l}(\tilde{e}).
$$
(2) The complement of \( h_z \) (simply, \( h_z^c \)) \([\text{where } h^c : E \rightarrow \xi(B)]\) is given by:

\[
h^c = \{(\bar{e}, (\delta h(\bar{e}), \tau h(\bar{e}), \sigma h(\bar{e}))) \mid \bar{e} \in \mathcal{E}\}.
\]

**Theorem 1.1.** \([17]\) Let \( h_z, l_j, g_x \in (\overline{\mathcal{B}}, \overline{\mathcal{E}}) \) and \((h_z)_j = (h_j)_z\), \((l_j)_j = (l_j)_z \in (\overline{\mathcal{B}}, \overline{\mathcal{E}}) \) \( j \in \Gamma \), where \( \Gamma \) is called the index set. Then

1. \( h_z \cap l_y = l_y \cap h_z \) and \( h_z \cup l_y = l_y \cup h_z \).
2. \( h_z \cup (l_y \cap g_x) = (h_z \cup l_y) \cup g_x \) and \( h_z \cap (l_y \cap g_x) = (h_z \cap l_y) \cap g_x \).
3. \( h_z \cup (\bigcap_{j \in \Gamma} [l_j]) = \bigcap_{j \in \Gamma} (h_z \cup l_j) \).
4. \( h_z \cap (\bigcup_{j \in \Gamma} [l_j]) = \bigcup_{j \in \Gamma} (h_z \cap l_j) \).
5. \([h^c_z]^c = h^c_z\).
6. If \( h^c_z \subseteq l_y \), then \( h^c_z \subseteq l^c_y \).
7. \( h_z \cap h_z = h_z \) and \( h_z \cup h_z = h_z \).
8. \( \phi \leq h_z \subseteq \overline{\mathcal{E}} \).
9. \((\bigcup_{j \in \Gamma} [h_z])^c = \bigcap_{j \in \Gamma} [h_z]^c\).

**Definition 1.4.** \([17]\) A single-valued neutrosophic soft topological space is ordered as \((\mathcal{B}, \overline{\tau}, \overline{\tau}, \overline{\delta})\) where \( \overline{\tau}, \overline{\tau}, \overline{\delta} : E \rightarrow \xi(\overline{\mathcal{B}}, \overline{\mathcal{E}}) \) is a mapping that satisfies the following axioms, for every \( h_z, l_z \in (\overline{\mathcal{B}}, \overline{\mathcal{E}}) \) and \( \bar{e} \in E \):

\[
\begin{align*}
\text{(T1)} & \quad \overline{\tau}_z(\phi) = \overline{\tau}_z(\overline{\mathcal{E}}) = 1 \quad \text{and} \quad \overline{\delta}_z(\phi) = \overline{\delta}_z(\overline{\mathcal{E}}) = 0, \\
\text{(T2)} & \quad \overline{\tau}_z(h_z \cap l_z) \geq \overline{\tau}_z(h_z) \cap \overline{\tau}_z(l_z), \quad \overline{\tau}_z(h_z \cup l_z) \leq \overline{\tau}_z(h_z) \cup \overline{\tau}_z(l_z), \\
& \quad \overline{\tau}_z(h_z \cap l_z) \leq \overline{\tau}_z(h_z) \cup \overline{\tau}_z(l_z), \quad \overline{\delta}_z(h_z \cap l_z) \leq \overline{\delta}_z(h_z) \cup \overline{\delta}_z(l_z), \\
\text{(T3)} & \quad \overline{\tau}_z(\bigcup_{j \in \Gamma} [h_z]) \geq \bigcup_{j \in \Gamma} \overline{\tau}_z([h_z]) \quad \text{and} \quad \overline{\delta}_z(\bigcup_{j \in \Gamma} [h_z]) \leq \bigcup_{j \in \Gamma} \overline{\delta}_z([h_z]) \quad \text{where } \bigcup_{j \in \Gamma} [h_z] \subseteq \bigcup_{j \in \Gamma} \overline{\mathcal{E}}(\bigcup_{j \in \Gamma} [h_z]).
\end{align*}
\]

The svnft is termed to be stratified if it satisfies the following conditions:

\[
\begin{align*}
\text{(T4.1)} & \quad \overline{\tau}_z(\overline{\mathcal{E}}) = 1, \quad \overline{\tau}_z(\overline{\mathcal{E}}) = 0 \quad \text{and} \quad \overline{\delta}_z(\overline{\mathcal{E}}) = 0.
\end{align*}
\]

The Quadruple \((\mathcal{B}, \overline{\tau}, \overline{\tau}, \overline{\delta})\) is known as a single-valued neutrosophic soft topological space \((\text{svnft-space})\), representing the degree of openness \((\overline{\tau}_z(h_z))\), the degree of indeterminacy \((\overline{\delta}_z(h_z))\), and the degree of non-openness \((\overline{\tau}_z(h_z))\); of a svnfs \( h_z \) with respect to the parameter \( \bar{e} \in E \) respectively.

Occasionally, \((\overline{\tau}, \overline{\tau}, \overline{\delta})\) is written as \(\overline{\tau}_{\sigma\tau\delta}\) here into avoid ambiguity.

2. Single-Valued Neutrosophic Soft Grill

**Definition 2.1.** A mapping \( K^\sigma, K^\tau, K^\delta : E \rightarrow \xi(\overline{\mathcal{B}}, \overline{\mathcal{E}}) \) is called single-valued neutrosophic soft grill on \( B \) \([\text{abbreviated, svnfs-grill}]\) if it satisfies the following conditions \( \forall h_z, l_z \in (\overline{\mathcal{B}}, \overline{\mathcal{E}}) \) and \( \bar{e} \in E \):

\[
\begin{align*}
\text{Occasionally, } (\overline{\tau}, \overline{\tau}, \overline{\delta}) \text{ is written as } \overline{\tau}_{\sigma\tau\delta} \text{ here into avoid ambiguity.}
\end{align*}
\]

2. Single-Valued Neutrosophic Soft Grill
\((K_1)\) \(K^\sigma_\delta(\phi) = 0, K^\tau_\delta(\phi) = 1, K^{\delta_\delta}_\delta(\phi) = 1 \) and \(K^\delta_\delta(\tilde{E}) = 0, K^\tau_\delta(\tilde{E}) = 0,\)

\((K_2)\) If \(h \subseteq l\), then \(K_\delta^\sigma(h_2) \leq K_\delta^\sigma(l_2), K_\tau^\sigma(h_2) \geq K_\delta^\tau(l_2)\) and \(K^{\delta_\delta}_\delta(h_2) \geq K^{\delta_\delta}_\delta(l_2).\)

\((K_3)\) \(K^\sigma_\delta(h_2 \cup l_2) \leq K^\sigma_\delta(h_2) \vee K^\sigma_\delta(l_2), K^\tau_\delta(h_2 \cup l_2) \geq K^\tau_\delta(h_2) \wedge K^\tau_\delta(l_2)\) and \(K^{\delta_\delta}_\delta(h_2 \cup l_2) \geq K^{\delta_\delta}_\delta(h_2) \wedge K^{\delta_\delta}_\delta(l_2).\)

Let \(K^{\sigma_\tau_\delta}_\delta\) and \(K^{\sigma_\tau_\delta}_\tau\) be svnf-grills on \(B\), we say \(K^{\sigma_\tau_\delta}_\delta\) is finer than \(K^{\sigma_\tau_\delta}_\tau\) \((K^{\sigma_\tau_\delta}_\tau\) is coarser than \(K^{\sigma_\tau_\delta}_\delta\)) denoted by \(K^{\sigma_\tau_\delta}_\delta \subseteq K^{\sigma_\tau_\delta}_\tau\) if

\[
K^\sigma_\delta(h_2) \leq K^\sigma_\delta(h_2), \quad K^\tau_\delta(h_2) \geq K^\tau_\delta(h_2), \quad K^{\delta_\delta}_\delta(h_2) \geq K^{\delta_\delta}_\delta(h_2), \quad \forall h_2 \in (\tilde{B}, \tilde{E}), \ e \in E.
\]

The triple \((B, \tilde{\tau}^{\sigma_\tau_\delta}_\delta, K^{\sigma_\tau_\delta}_\delta)\) is termed the single-valued neutrosophic soft grill topological space (abbreviated, svnf-space).

**Definition 2.2.** Let \((B, \tilde{\tau}^{\sigma_\tau_\delta}_\delta, K^{\sigma_\tau_\delta}_\delta)\) be svnf-space, \(e \in E, r \in \xi_0\) and \(h_2 \in (\tilde{B}, \tilde{E})\). We define \(\varphi: E \times (B, \tilde{E}) \times \xi_0 \rightarrow (\tilde{B}, \tilde{E})\), indicated by \(\varphi(e, h_2, r)\) or \(\varphi(\tilde{\tau}^{\sigma_\tau_\delta}_\delta, K^{\sigma_\tau_\delta}_\delta)(e, h_2, r)\) and called the svnf-operator related to \((K^{\sigma_\delta}, K^\tau_\delta, K^{\delta_\delta}_\delta)\) and \((\tilde{\tau}^{\sigma_\delta}, \tilde{\tau}^\tau_\delta, \tilde{\tau}^{\delta_\delta}_\delta)\) can be defined as follows:

\[
\varphi(e, h_2, r) = \bigcap \{l_2 \in (\tilde{B}, \tilde{E}) | K^\sigma_\delta(h_2, l_2) < r, K^\tau_\delta(h_2, l_2) > 1 - r, K^{\delta_\delta}_\delta(h_2, l_2) > 1 - r \}
\]

and \(\tilde{\tau}^{\sigma_\tau_\delta}_\delta([l_2]_c) \geq r, \tilde{\tau}^\tau_\delta([l_2]_c) \leq 1 - r, \tilde{\tau}^{\delta_\delta}_\delta([l_2]_c) \leq 1 - r\).

Sometimes in this paper, we will write \(\varphi_{K^{\sigma_\tau_\delta}_\delta}(e, h_2, r)\) or \(\varphi(e, h_2, r)\) for \(\varphi(\tilde{\tau}^{\sigma_\tau_\delta}_\delta, K^{\sigma_\tau_\delta}_\delta)(e, h_2, r)\), and also, sometimes, we will write \(\varphi_{K^\sigma_\delta}(e, h_2, r), \varphi_{K^\tau_\delta}(e, h_2, r), \varphi_{K^{\delta_\delta}_\delta}(e, h_2, r)\) for \(\sigma_{[\varphi_{K^\sigma_\delta}(e, h_2, r)], \tau_{[\varphi_{K^\tau_\delta}(e, h_2, r)], \delta_{[\varphi_{K^{\delta_\delta}_\delta}(e, h_2, r)]}\))\) respectively.

If we take \(K^{\sigma_\tau_\delta}_\delta = (K^{\sigma_\tau_\delta}_\delta)_E\), then \(\varphi(e, h_2, r) = C_{\tilde{\tau}^{\sigma_\tau_\delta}_\delta}(e, h_2, r)\) for any \(e \in E, h_2 \in (\tilde{B}, \tilde{E}), r \in \xi_0\).

**Theorem 2.1.** Let \((B, \tilde{\tau}^{\sigma_\tau_\delta}_\delta)\) be svnf-space and \(K^{\sigma_\tau_\delta}_\delta, K^{\sigma_\tau_\delta}_\tau\) be two svnf-grills on \(B\). Therefore, for every \(e \in E, h_2 \subseteq (\tilde{B}, \tilde{E}), r \in \xi_0:\)

1. If \(h_2 \subseteq l_2\), then \(\varphi_{K^\sigma_\delta}(e, h_2, r) \leq \varphi_{K^\sigma_\delta}(e, l_2, r), \varphi_{K^\tau_\delta}(e, h_2, r) \geq \varphi_{K^\tau_\delta}(e, l_2, r)\) and \(\varphi_{K^{\delta_\delta}_\delta}(e, h_2, r) \geq \varphi_{K^{\delta_\delta}_\delta}(e, l_2, r)\).
2. If \(K^{\sigma_\delta}_\delta(h_2) < r, K^\tau_\delta(h_2) \geq 1 - r, K^{\delta_\delta}_\delta(h_2) \geq 1 - r\), then \(\varphi(e, h_2, r) = \phi\). Furthermore, \(\varphi(e, \phi, r) = \phi\).
3. If \(K^{\sigma_\tau_\delta}_\delta \subseteq K^{\sigma_\tau_\delta}_\delta\), then \(\varphi_{K^\sigma_\delta}(e, h_2, r) \leq \varphi_{K^\tau_\delta}(e, h_2, r), \varphi_{K^\tau_\delta}(e, h_2, r) \geq \varphi_{K^\tau_\delta}(e, h_2, r)\) and \(\varphi_{K^{\delta_\delta}_\delta}(e, h_2, r) \geq \varphi_{K^{\delta_\delta}_\delta}(e, h_2, r)\).
4. \(\varphi(e, h_2 \cap l_2, r) \subseteq \varphi(e, h_2, r) \cap \varphi(e, l_2, r)\).
5. \(\varphi(e, h_2 \cup l_2, r) \supseteq \varphi(e, h_2, r) \cup \varphi(e, l_2, r)\).
6. \(\varphi(e, h_2, r) = C_{\tilde{\tau}^{\sigma_\tau_\delta}_\delta}(e, \varphi(e, h_2, r), r) = C_{\tilde{\tau}^{\sigma_\tau_\delta}_\delta}(e, h_2, r)\).
7. \(\varphi(e, \varphi(e, h_2, r), r) \subseteq \varphi(e, h_2, r)\).

Proof. (1) Let

\[
\varphi_{K^\sigma_\delta}(e, h_2, r) \leq \varphi_{K^\sigma_\delta}(e, l_2, r), \quad \varphi_{K^\tau_\delta}(e, h_2, r) \geq \varphi_{K^\tau_\delta}(e, l_2, r), \quad \varphi_{K^{\delta_\delta}_\delta}(e, h_2, r) \geq \varphi_{K^{\delta_\delta}_\delta}(e, l_2, r)
\]
Then, there is $g_x \in (\mathcal{B}, \mathcal{E})$ with $K_\mathcal{E}^\sigma(l_x, \tilde{g} g_x) < r$, $K_\mathcal{E}^\tau(l_x, \tilde{g} g_x) > 1 - r$, $K_\mathcal{E}^\delta(l_x, \tilde{g} g_x) > 1 - r$ and $\tilde{\gamma}_\mathcal{E}^\sigma([g_x]^c) \geq r$, $\tilde{\gamma}_\mathcal{E}^\tau([g_x]^c) \leq 1 - r$, $\tilde{\gamma}_\mathcal{E}^\delta([g_x]^c) \leq 1 - r$, such that
\[
\varphi_{K^\sigma}(\tilde{e}, h_z, r) \geq g_x \geq \varphi_{K^\tau}(\tilde{e}, l_y, r), \quad \varphi_{K^\tau}(\tilde{e}, h_z, r) \leq g_x \leq \varphi_{K^\delta}(\tilde{e}, l_y, r),
\]
\[
\varphi_{K^\delta}(\tilde{e}, h_z, r) \leq g_x \leq \varphi_{K^\delta}(\tilde{e}, l_y, r).
\]

On another side, since $\varphi_{K^\sigma}(\tilde{e}, l_y, r) \geq g_x$, $\varphi_{K^\tau}(\tilde{e}, l_y, r) \leq g_x$, $\varphi_{K^\delta}(\tilde{e}, l_y, r) \leq g_x$ and $h_z \subseteq l_y$ we obtain $h_z, \tilde{g} g_x \subseteq l_y, \tilde{g} g_x$. So,
\[
K_\mathcal{E}^\sigma(h_z, \tilde{g} g_x) \leq K_\mathcal{E}^\sigma(l_x, \tilde{g} g_x) < r, \quad K_\mathcal{E}^\tau(h_z, \tilde{g} g_x) \geq K_\mathcal{E}^\tau(l_x, \tilde{g} g_x) > 1 - r, \quad K_\mathcal{E}^\delta(h_z, \tilde{g} g_x) \geq K_\mathcal{E}^\delta(l_x, \tilde{g} g_x) > 1 - r.
\]

Hence, $\varphi_{K^\sigma}(\tilde{e}, h_z, r) \geq g_x$, $\varphi_{K^\tau}(\tilde{e}, h_z, r) \geq g_x$, and $\varphi_{K^\delta}(\tilde{e}, h_z, r) \geq g_x$. A contradiction. Thus,
\[
\varphi_{K^\delta}(\tilde{e}, h_z, r) \leq \varphi_{K^\delta}(\tilde{e}, l_y, r), \quad \varphi_{K^\tau}(\tilde{e}, h_z, r) \geq \varphi_{K^\tau}(\tilde{e}, l_y, r), \quad \varphi_{K^\delta}(\tilde{e}, h_z, r) \geq \varphi_{K^\delta}(\tilde{e}, l_y, r).
\]

(2) Since $h_z, \tilde{g} g_x \subseteq h_x$ we get
\[
K_\mathcal{E}^\sigma(h_z, \tilde{g} g_x) \leq K_\mathcal{E}^\sigma(h_x) < r, \quad K_\mathcal{E}^\tau(h_z, \tilde{g} g_x) \geq K_\mathcal{E}^\tau(h_x) > 1 - r, \quad K_\mathcal{E}^\delta(h_z, \tilde{g} g_x) \geq K_\mathcal{E}^\delta(h_x) > 1 - r,
\]

for each $l_y \in (\mathcal{B}, \mathcal{E})$. Thus based on the concept of $\varphi$ and if $K_\mathcal{E}^\sigma(h_x) < r$, $K_\mathcal{E}^\tau(h_x) \geq 1 - r$, $K_\mathcal{E}^\delta(h_x) \geq 1 - r$, then $\varphi(\tilde{e}, h_x, r) = \phi$.

(3) Assume that,
\[
\varphi_{K^\sigma}(\tilde{e}, h_x, r) \leq \varphi_{K^\sigma}(\tilde{e}, h_z, r), \quad \varphi_{K^\tau}(\tilde{e}, h_x, r) \geq \varphi_{K^\tau}(\tilde{e}, h_z, r),
\]
\[
\varphi_{K^\delta}(\tilde{e}, h_x, r) \geq \varphi_{K^\delta}(\tilde{e}, h_z, r).
\]

Then, there is $g_x \in (\mathcal{B}, \mathcal{E})$ with $K_\mathcal{E}^\sigma(h_x, \tilde{g} g_x) < r$, $K_\mathcal{E}^\tau(h_x, \tilde{g} g_x) > 1 - r$, $K_\mathcal{E}^\delta(h_x, \tilde{g} g_x) > 1 - r$ and $\tilde{\gamma}_\mathcal{E}^\sigma([g_x]^c) \geq r$, $\tilde{\gamma}_\mathcal{E}^\tau([g_x]^c) \leq 1 - r$, $\tilde{\gamma}_\mathcal{E}^\delta([g_x]^c) \leq 1 - r$, such that
\[
\varphi_{K^\sigma}(\tilde{e}, h_x, r) > g_x \geq \varphi_{K^\sigma}(\tilde{e}, h_z, r), \quad \varphi_{K^\tau}(\tilde{e}, h_x, r) < g_x \leq \varphi_{K^\tau}(\tilde{e}, h_z, r),
\]
\[
\varphi_{K^\delta}(\tilde{e}, h_x, r) < g_x \leq \varphi_{K^\delta}(\tilde{e}, h_z, r).
\]

Since $\varphi_{K^\sigma}(\tilde{e}, h_z, r) \leq g_x$, $\varphi_{K^\tau}(\tilde{e}, h_z, r) \geq g_x$, $\varphi_{K^\delta}(\tilde{e}, h_z, r) \geq g_x$ and $K_\mathcal{E}^\sigma \supseteq K_\mathcal{E}^\tau \supseteq K_\mathcal{E}^\delta$, we get
\[
K_\mathcal{E}^\sigma(h_z, \tilde{g} g_x) \leq K_\mathcal{E}^\sigma(h_x, \tilde{g} g_x) < r, \quad K_\mathcal{E}^\tau(h_z, \tilde{g} g_x) \geq K_\mathcal{E}^\tau(h_x, \tilde{g} g_x) > 1 - r,
\]
\[
K_\mathcal{E}^\delta(h_z, \tilde{g} g_x) \geq K_\mathcal{E}^\delta(h_x, \tilde{g} g_x) > 1 - r.
\]

Hence, $\varphi_{K^\sigma}(\tilde{e}, h_z, r) \leq g_x$, $\varphi_{K^\tau}(\tilde{e}, h_z, r) \geq g_x$, $\varphi_{K^\delta}(\tilde{e}, h_z, r) \geq g_x$. A contradiction. Thus,
\[
\varphi_{K^\sigma}(\tilde{e}, h_z, r) \leq \varphi_{K^\sigma}(\tilde{e}, h_x, r), \quad \varphi_{K^\tau}(\tilde{e}, h_z, r) \geq \varphi_{K^\tau}(\tilde{e}, h_x, r) \text{ and } \varphi_{K^\delta}(\tilde{e}, h_z, r) \geq \varphi_{K^\delta}(\tilde{e}, h_x, r).
\]

(4) Since, $h_z \cap l_y \subseteq h_z$ and $h_z \cap l_y \subseteq l_y$. So, from (1), we get $\varphi(\tilde{e}, h_z \cap l_y, r) \subseteq \varphi(\tilde{e}, h_z, r)$ and $\varphi(\tilde{e}, h_z \cap l_y, r) \subseteq \varphi(\tilde{e}, l_y, r)$. Therefore,
\[
\varphi(\tilde{e}, h_z \cap l_y, r) \subseteq \varphi(\tilde{e}, h_z, r) \cap \varphi(\tilde{e}, l_y, r).
\]

(5) In a similar vein, we can demonstrate through a parallel line of reasoning that.
(6) From the concept of $\varphi(\tilde{e}, \mathcal{E}, r), C_{\mathcal{T} \sigma \delta}(\tilde{e}, \varphi(\tilde{e}, \mathcal{E}, r), r) = \varphi(\tilde{e}, \mathcal{E}, r)$. Now we will just verify $\varphi(\tilde{e}, \mathcal{E}, r) \subseteq C_{\mathcal{T} \sigma \delta}(\tilde{e}, \mathcal{E}, r)$. For each $s\nu\pi\sigma\text{-}\text{grill} \mathcal{K}_{\mathcal{E}}^{\sigma \tau \delta}$ we have $\mathcal{K}_{\mathcal{E}}^{\sigma \tau \delta} \subseteq \mathcal{K}_{\mathcal{E}}^{0 \sigma \tau \delta}$, so by (3), we have

$$\varphi_{\mathcal{K}_{\mathcal{E}}^{\sigma \tau \delta}}(\tilde{e}, \mathcal{E}, r) \subseteq \varphi_{\mathcal{K}_{\mathcal{E}}^{0 \sigma \tau \delta}}(\tilde{e}, \mathcal{E}, r) = C_{\mathcal{T} \sigma \delta}(\tilde{e}, \mathcal{E}, r).$$

Therefore,

$$\varphi(\tilde{e}, \mathcal{E}, r) \subseteq C_{\mathcal{T} \sigma \delta}(\tilde{e}, \mathcal{E}, r).$$

(7) Likewise, we can establish through a similar line of reasoning that.

\[\square\]

Example 2.1. Assume that, $B = \{x_1, x_2\}$ be a universal set, $E = \{\tilde{e}_1, \tilde{e}_2\}$ be a set of parameters. Define $\nu\pi\sigma\text{-}\text{topology (}^{\mathcal{T} \sigma \tau \delta}_{\mathcal{E}}\text{)}$ and $\nu\pi\sigma\text{-}\text{grill (}\mathcal{K}_{\mathcal{E}}^{\sigma \tau \delta}\text{)}$ as follow, for every $\tilde{e} \in E$

\[\begin{align*}
^{\mathcal{T} \sigma \tau}_{\mathcal{E}}(\tilde{e}) &= \begin{cases} 1, & \text{if } \mathcal{E} = \phi \text{ or } \tilde{E}, \\ \frac{1}{2}, & \text{if } \mathcal{E} = \{(\tilde{e}_1, (0,3,0,3,0.3)), (\tilde{e}_2, (0,6,0,6,0.6))\}, \\ 0, & \text{if otherwise}, \end{cases} \\
^{\mathcal{T} \tau}_{\mathcal{E}}(\tilde{e}) &= \begin{cases} 0, & \text{if } \mathcal{E} = \phi \text{ or } \tilde{E}, \\ \frac{1}{2}, & \text{if } \mathcal{E} = \{(\tilde{e}_1, (0,3,0,3,0.3)), (\tilde{e}_2, (0,6,0,6,0.6))\}, \\ 1, & \text{if otherwise}, \end{cases} \\
^{\mathcal{T} \delta}_{\mathcal{E}}(\tilde{e}) &= \begin{cases} 0, & \text{if } \mathcal{E} = \phi \text{ or } \tilde{E}, \\ \frac{1}{2}, & \text{if } \mathcal{E} = \{(\tilde{e}_1, (0,3,0,3,0.3)), (\tilde{e}_2, (0,6,0,6,0.6))\}, \\ 1, & \text{if otherwise}, \end{cases} \end{align*}\]

\[\begin{align*}
\mathcal{K}_{\mathcal{E}}^{\sigma \tau \delta}(\tilde{e}) &= \begin{cases} 1, & \text{if } \{(\tilde{e}_1, (1,0,0)), (\tilde{e}_2, (0,1,1))\} \subseteq \mathcal{E} \subseteq \tilde{E}, \\ 0.7, & \text{if } \{(\tilde{e}_1, (0.5,0,0)), (\tilde{e}_2, (0.5,0,0))\} \subseteq \mathcal{E} \subseteq \tilde{E}, \\ 0, & \text{if otherwise}, \end{cases} \\
\mathcal{K}_{\mathcal{E}}^{\tau \delta}(\tilde{e}) &= \begin{cases} 0, & \text{if } \{(\tilde{e}_1, (1,0,0)), (\tilde{e}_2, (0,1,1))\} \subseteq \mathcal{E} \subseteq \tilde{E}, \\ 0.3, & \text{if } \{(\tilde{e}_1, (0.5,0,0)), (\tilde{e}_2, (0.5,0,0))\} \subseteq \mathcal{E} \subseteq \tilde{E}, \\ 1, & \text{if otherwise}, \end{cases} \\
\mathcal{K}_{\mathcal{E}}^{\delta \tau \delta}(\tilde{e}) &= \begin{cases} 0, & \text{if } \{(\tilde{e}_1, (1,0,0)), (\tilde{e}_2, (0,1,1))\} \subseteq \mathcal{E} \subseteq \tilde{E}, \\ 0.2, & \text{if } \{(\tilde{e}_1, (0.5,0,0)), (\tilde{e}_2, (0.5,0,0))\} \subseteq \mathcal{E} \subseteq \tilde{E}, \\ 1, & \text{if otherwise}. \end{cases} \end{align*}\]

Then \(\{(\tilde{e}_1, (0.7,0,7,0.7)), (\tilde{e}_2, (0.4,0.4,0.4))\} = \varphi(\tilde{e}, E^{0.6}, \frac{1}{2}) \neq \varphi(\tilde{e}, \varphi(\tilde{e}, E^{0.6} \frac{1}{2}), \frac{1}{2}) = \phi\)

Theorem 2.2. Let \((B, \mathcal{T}^{\sigma \tau \delta}_{\mathcal{E}}, \mathcal{T}^{\tau \delta}_{\mathcal{E}})\) be $\nu\pi\sigma\text{-}\text{grilt-space, } \{(\tilde{e}_i) \in (B, E) : i \in \Gamma\}, \tilde{e} \in E, r \in \xi_0$. Then:

1. \(\cup(\varphi(\tilde{e}, (\tilde{e}_i), r)) : i \in \Gamma \subseteq \varphi(\tilde{e}, \cup(\tilde{e}_i), r) : i \in \Gamma\).
2. \((\varphi(\tilde{e}, \cap(\tilde{e}_i), r) : i \in \Gamma \subseteq \cap(\varphi(\tilde{e}, (\tilde{e}_i), r)) : i \in \Gamma\).
Proof. (1) Since \( ((h_z)_i \sqsubseteq \sqcup(h_z)_i, \ \forall \ i \in \Gamma) \), so by theorem 2.1 (1), we have, \( \varphi(\tilde{e},(h_z)_i, r) \sqsubseteq \varphi(\tilde{e}, \sqcup(h_z)_i, r) \). Hence, \( \sqcup(\varphi(\tilde{e},(h_z)_i, r)) \sqsubseteq \varphi(\tilde{e}, \sqcup(h_z)_i, r) \), \( \forall \ i \in \Gamma \)

(2) Since \( (\cap(h_z)_i \sqsubseteq (h_z)_i, \ \forall \ i \in \Gamma) \), so by theorem 2.1 (1), we have, \( \cap(\varphi(\tilde{e},(h_z)_i, r)) \sqsubseteq \varphi(\tilde{e}, \cap(h_z)_i, r) \). Thus, \( \varphi(\tilde{e}, \cap(h_z)_i, r) \sqsubseteq \cap(\varphi(\tilde{e},(h_z)_i, r)) \), \( \forall \ i \in \Gamma \). \( \square \)

**Definition 2.3.** Let \( (B, \tilde{T}_{E, K}^{\sigma \tau \delta}, K_{E}^{\sigma \tau \delta}) \) be svnft-space. Then for all \( h_z \in (\widetilde{(B,E)}), \ \tilde{e} \in E \) and \( r \in \xi_0 \) we define a mapping \( C^* : E \times (\widetilde{(B,E)}) \times \xi_0 \rightarrow \xi(\widetilde{(B,E)}) \) as next:

\[
C^*(\tilde{e}, h_z, r) = h_z \cup \varphi(\tilde{e}, h_z, r).
\]

Clear that

\[
(\tilde{T}_{K}^{\sigma \tau \delta})_{\tilde{e}}(h_z) = \bigvee \{ r \mid C^*(\tilde{e}, h_z^c, r) = h_z^c \}.
\]

\[
(\tilde{T}_{K}^{\sigma \tau \delta})_{\tilde{e}}(h_z) = \bigwedge \{ 1 - r \mid C^*(\tilde{e}, h_z^c, 1 - r) = h_z^c \}.
\]

is a supra single-valued neutrosophic Soft topology generated by \( C^* \) and \( \tilde{T}_{E}^{\sigma \tau \delta} \subseteq (\tilde{T}_{K}^{\sigma \tau \delta})_E \). If \( K_{E}^{\sigma \tau \delta} = K_{E}^{0\sigma \tau \delta} \), therefor for any \( h_z \in (\widetilde{(B,E)}), \ \tilde{e} \in E \) and \( r \in \xi_0 \), we have,

\[
C^*(\tilde{e}, h_z, r) = h_z \cup \varphi(\tilde{e}, h_z, r) = h_z \cup C_{\tilde{T}_{E}^{\sigma \tau \delta}}(\tilde{e}, h_z, r) = C_{\tilde{T}_{E}^{\sigma \tau \delta}}(\tilde{e}, h_z, r).
\]

Thus in this case, \( \tilde{T}_{E}^{\sigma \tau \delta} \subseteq (\tilde{T}_{K}^{\sigma \tau \delta})_E \).

**Theorem 2.3.** For every \( \tilde{e} \in E, r \in \xi_0 \) and \( h_z, l_y \in (\widetilde{(B,E)}) \), the operator \( C^* \) fulfills the next conditions:

(1) \( C^*(\tilde{e}, \phi, r) = \phi \).

(2) \( h_z \subseteq C^*(\tilde{e}, h_z, r) = C_{\tilde{T}_{E}^{\sigma \tau \delta}}(\tilde{e}, h_z, r) \).

(3) If \( h_z \subseteq l_y \), then \( C^*(\tilde{e}, h_z, r) \subseteq C^*(\tilde{e}, l_y, r) \).

(4) \( C^*(\tilde{e}, h_z \cap l_y, r) \subseteq C^*(\tilde{e}, h_z, r) \cap C^*(\tilde{e}, l_y, r) \).

(5) \( C^*(\tilde{e}, h_z \cup l_y, r) \subseteq C^*(\tilde{e}, h_z, r) \cup C^*(\tilde{e}, l_y, r) \).

(6) \( C^*(\tilde{e}, h_z, r) \subseteq C^*(\tilde{e}, C^*(\tilde{e}, h_z, r), r) \).

Proof. (1) \( C^*(\tilde{e}, \phi, r) = \phi \cup \varphi(\tilde{e}, \phi, r) = \phi \cup \phi = \phi \).

(2) From the concept of \( C^* \), we get that \( h_z \subseteq h_z \cup \varphi(\tilde{e}, h_z, r) = C^*(\tilde{e}, h_z, r) \). Since \( h_z \subseteq C_{\tilde{T}_{E}^{\sigma \tau \delta}}(\tilde{e}, h_z, r) \) and by Theorem 2.1 (6), we obtain \( \varphi(\tilde{e}, h_z, r) \subseteq C_{\tilde{T}_{E}^{\sigma \tau \delta}}(\tilde{e}, h_z, r) \) implies that

\[
h_z \cup \varphi(\tilde{e}, h_z, r) = C^*(\tilde{e}, h_z, r) \subseteq C_{\tilde{T}_{E}^{\sigma \tau \delta}}(\tilde{e}, h_z, r).
\]

Therefore, \( h_z \subseteq C^*(\tilde{e}, h_z, r) = C_{\tilde{T}_{E}^{\sigma \tau \delta}}(\tilde{e}, h_z, r) \).

(3) Because \( h_z \subseteq l_y \) and by Theorem 2.1 (1), we obtain \( \varphi(\tilde{e}, h_z, r) \subseteq \varphi(\tilde{e}, l_y, r) \). Therefore, \( h_z \cup \varphi(\tilde{e}, h_z, r) \subseteq l_y \cup \varphi(\tilde{e}, l_y, r) \). Thus, \( C^*(\tilde{e}, h_z, r) \subseteq C^*(\tilde{e}, l_y, r) \).

(4) From (3), we get that \( C^*(\tilde{e}, h_z \cap l_y, r) \subseteq C^*(\tilde{e}, h_z, r) \) and \( C^*(\tilde{e}, h_z \cap l_y, r) \subseteq C^*(\tilde{e}, l_y, r) \) implies

\[
C^*(\tilde{e}, h_z \cap l_y, r) \subseteq C^*(\tilde{e}, h_z, r) \cap C^*(\tilde{e}, l_y, r).
\]

(5) Similarly, we can affirm through a corresponding argument that.
Theorem 2.4. Let \((\mathcal{B}, \bar{\mathcal{T}}^\sigma, \mathcal{T}^\delta, \bar{\mathcal{T}}^\delta)\) be svnfgt-space, \(\mathcal{H}_z \in (\mathcal{B}, \mathcal{E})\), \(\bar{\mathcal{E}} \in \mathcal{E}\), \(r \in \xi_0\). Then:

1. If \(h_z \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r)\), then
   \[
   C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) = C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) = C_{\mathcal{T}^\sigma}(\bar{e}, \varphi(h_z, r), r) = \varphi(h_z, r).
   \]

2. If \(\bar{\mathcal{T}}^\sigma([h_z]^c) \geq r\), \(\bar{\mathcal{T}}^\sigma([h_z]^c) \leq 1 - r\), then \(\varphi(h_z, r) \subseteq h_z\).

Proof. (1) Because \(h_z \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r)\) and \(\varphi(h_z, r) \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r)\), so we obtain,
   \[
   h_z \cup \varphi(h_z, r) = C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r).
   \]

In view of Theorem 2.1 (6), we get,
   \[
   \varphi(h_z, r) = C_{\mathcal{T}^\sigma}(\bar{e}, \varphi(h_z, r), r) \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r).
   \]

Because, \(h_z \subseteq \varphi(h_z, r)\) we have \(C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, \varphi(h_z, r), r)\) and since \(\varphi(h_z, r) \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r)\). Hence,
   \[
   C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) = C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) = C_{\mathcal{T}^\sigma}(\bar{e}, \varphi(h_z, r), r) = \varphi(h_z, r).
   \]

(2) Form Theorem 2.3 (2), we have
   \[
   \varphi(h_z, r) = C_{\mathcal{T}^\sigma}(\bar{e}, \varphi(h_z, r), r) \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) \subseteq C_{\mathcal{T}^\sigma}(\bar{e}, h_z, r) = h_z.
   \]
Remark 3.1. Any two r-svnf-separated sets are r-svnfg-separated sets. That is from
\[ C^*(\tilde{e}, g_x, r) \subseteq C^*_r(\tilde{e}, g_x, r), \quad \forall \ g_x \in (\mathcal{B}, \mathcal{E}), \ \tilde{e} \in \mathcal{E}, \ r \in \xi_0. \]

However, the converse is not true in general, as shown in the following example.

Example 3.1. Assume that, \( \mathcal{B} = \{x_1, x_2\} \) be a universal set, \( \mathcal{E} = \{\tilde{e}_1, \tilde{e}_2\} \) be a set of parameters. Define \textit{svnf-topology} \( \hat{\tau}^\delta \) and \textit{svnfg-grill} \( \hat{K}^\delta \) as follow, for every \( \tilde{e} \in \mathcal{E} \)
\[ \hat{\tau}^\delta_e(\tilde{e}) = \begin{cases} 1, & \text{if } h_e = \phi \text{ or } \tilde{E}, \\ \frac{1}{2}, & \text{if } h_e = \{(\tilde{e}_1, (1, 0.4, 0.4)), (\tilde{e}_2, (0.5, 1, 1))\}, \\ 0, & \text{if otherwise}, \end{cases} \]
\[ \hat{K}^\delta_e(\tilde{e}) = \begin{cases} 0, & \text{if } h_e = \phi \text{ or } \tilde{E}, \\ \frac{1}{2}, & \text{if } h_e = \{(\tilde{e}_1, (1, 0.4, 0.4)), (\tilde{e}_2, (0.5, 1, 1))\}, \\ 1, & \text{if otherwise}, \end{cases} \]
\[ \hat{\tau}^\gamma_e(\tilde{e}) = \begin{cases} 0, & \text{if } h_e = \phi \text{ or } \tilde{E}, \\ \frac{1}{2}, & \text{if } h_e = \{(\tilde{e}_1, (1, 0.4, 0.4)), (\tilde{e}_2, (0.5, 1, 1))\}, \\ 1, & \text{if otherwise}, \end{cases} \]
\[ \hat{K}^\gamma_e(\tilde{e}) = \begin{cases} 1, & \text{if } \{(\tilde{e}_1, (1, 0, 0)), (\tilde{e}_2, (0, 1, 1))\} \subseteq h_e \subseteq \tilde{E}, \\ 0.5, & \text{if } \{(\tilde{e}_1, (0, 0, 0.3)), (\tilde{e}_2, (0, 1, 1))\} \subseteq h_e \subseteq \tilde{E}, \\ 0, & \text{if otherwise}, \end{cases} \]
\[ \hat{K}^\delta_e(\tilde{e}) = \begin{cases} 0, & \text{if } \{(\tilde{e}_1, (1, 0, 0)), (\tilde{e}_2, (0, 1, 1))\} \subseteq h_e \subseteq \tilde{E}, \\ 0.5, & \text{if } \{(\tilde{e}_1, (0, 0.3, 0.3)), (\tilde{e}_2, (0, 1, 1))\} \subseteq h_e \subseteq \tilde{E}, \\ 1, & \text{if otherwise}, \end{cases} \]
\[ \hat{K}^\delta_e(\tilde{e}) = \begin{cases} 0, & \text{if } \{(\tilde{e}_1, (1, 0, 0)), (\tilde{e}_2, (0, 1, 1))\} \subseteq h_e \subseteq \tilde{E}, \\ 0.25, & \text{if } \{(\tilde{e}_1, (0, 0.3, 0.3)), (\tilde{e}_2, (0, 1, 1))\} \subseteq h_e \subseteq \tilde{E}, \\ 1, & \text{if otherwise}. \end{cases} \]

Let \( l_\tilde{e} = \{(\tilde{e}_1, (0.8, 0, 0)), (\tilde{e}_2, (0, 0.5, 0.5))\} \) and \( g_\tilde{e} = \{(\tilde{e}_1, (0, 0, 0.2)), (\tilde{e}_2, (0.5, 0.5, 0))\} \). Since \( \hat{K}^\delta_e(l_\tilde{e}) < \frac{1}{2}, \hat{K}^\gamma_e(l_\tilde{e}) \geq 1 - \frac{1}{2}, \hat{K}^\delta_e(g_\tilde{e}) < \frac{1}{2}, \hat{K}^\gamma_e(g_\tilde{e}) \geq 1 - \frac{1}{2}, \) we have \( \varphi(\tilde{e}, l_\tilde{e}, \frac{1}{2}) = \varphi(\tilde{e}, g_\tilde{e}, \frac{1}{2}) = \phi \). So, \( \text{cl}^*(\tilde{e}, l_\tilde{e}, \frac{1}{2}) = l_\tilde{e} \) and \( \text{cl}^*(\tilde{e}, g_\tilde{e}, \frac{1}{2}) = g_\tilde{e} \). Thus,
\[ \text{cl}^*(\tilde{e}, l_\tilde{e}, \frac{1}{2}) \cap g_\tilde{e} = l_\tilde{e} \cap g_\tilde{e} = l_\tilde{e} \cap \text{cl}^*(\tilde{e}, g_\tilde{e}, \frac{1}{2}) = \phi. \]

Hence, \( l_\tilde{e} \) and \( g_\tilde{e} \) are r-svnfg-separated sets. However, \( l_\tilde{e} \) and \( g_\tilde{e} \) are not r-svnf-separated sets where \( C^*_r(\tilde{e}, l_\tilde{e}, \frac{1}{2}) = \tilde{E} \) and thus \( C^*_r(\tilde{e}, l_\tilde{e}, \frac{1}{2}) \cap g_\tilde{e} \neq \phi. \)
Definition 3.2. Let \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{\varpi}^{\sigma \tau \delta})\) be r-svnfgt-space, and let \(h_z, l_y \in \tilde{(B, E)}\) be nonempty svnf sets, such that

1. \(h_z, l_y\) are r-svnfg-separated with \(h_z \cup l_y = \tilde{E}\). Therefore, \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{K}^{\sigma \tau \delta})\) is termed r-single-valued neutrosophic grill disconnected (abbreviated r-svnfg-disconnected space).

2. \(h_z, l_y\) are r-svnfg-separated with \(h_z \cup l_y = g_x\). Therefore, \(g_x\) is termed r-svnfg-disconnected on \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{K}^{\sigma \tau \delta})\).

Theorem 3.1. Let \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{K}^{\sigma \tau \delta})\) be r-svnfgt-space. Therefore, the following statements are equivalent.

1. \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{K}^{\sigma \tau \delta})\) is r-svnfg-connected.

2. If \(h_z \cup l_y = \tilde{E}\) and \(h_z \cap l_y = \phi\) with \(\tilde{T}^g(h_z) \geq r, \tilde{T}^g(l_y) \leq 1 - r, \tilde{K}^g(h_z) \leq 1 - r\) and \(\tilde{T}^g(l_y) \geq r, \tilde{K}^g(h_z) \leq 1 - r\), then \(h_z = \phi = l_y = \phi\).

3. \(h_z \cup l_y = \tilde{E}\) and \(h_z \cap l_y = \phi\) with \(\tilde{T}^g([h_z]_c) \geq r, \tilde{T}^g([l_y]_c) \leq 1 - r, \tilde{K}^g([h_z]_c) \leq 1 - r\) and \(\tilde{T}^g([l_y]_c) \geq r, \tilde{K}^g([l_y]_c) \leq 1 - r\), then \(h_z = \phi = l_y = \phi\).

Proof. (1)\(\implies\)(2) Suppose there exist \(h_z, l_y \in \tilde{(B, E)}\) with \(\tilde{T}^g(h_z) \geq r, \tilde{T}^g(l_y) \leq 1 - r, \tilde{K}^g(h_z) \leq 1 - r\), \(\tilde{T}^g(l_y) \geq r, \tilde{K}^g([h_z]_c) \geq r, \tilde{K}^g([l_y]_c) \leq 1 - r\), such that \(h_z \cup l_y = \tilde{E}\) and \(h_z \cup l_y = \phi\), which implies \(h_z = [l_y]_c\) and \(l_y = [h_z]_c\). Then, by Theorem 2.3 (2) and Theorem 2.4 (2) we have:

\[ C^*(\bar{e}, \bar{[l_y]_c}, r) \cap [h_z]_c \subseteq C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, [l_y]_c, r) \cap [h_z]_c = [l_y]_c \cap [h_z]_c = h_z \cap l_y = \phi, \]

and

\[ C^*(\bar{e}, \bar{[h_z]_c}, r) \cap [l_y]_c \subseteq C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, [h_z]_c, r) \cap [l_y]_c = [h_z]_c \cap [l_y]_c = l_y \cap h_z = \phi. \]

Therefore, \([l_y]_c\) and \([h_z]_c\) are r-svnfg-separated sets with \([l_y]_c \cup [h_z]_c = h_z \cup l_y = \tilde{E}\). But \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{K}^{\sigma \tau \delta})\) is r-svnfg-connected implies \([l_y]_c = \phi = [h_z]_c = \phi\) and hence, \(l_y = \phi\) or \(h_z = \phi\).

(2)\(\implies\)(1) Clear.

(3)\(\implies\)(1) Let \(h_z, l_y \in \tilde{(B, E)}\), \(h_z \neq \phi, l_y \neq \phi\) such that \(h_z \cup l_y = \tilde{E}\). Assume that \(g_x = C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, h_z, r)\) and \(w_d = C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, l_y, r), \bar{e} \in E, r \in \xi_0\), then \(g_x \cup w_d = \tilde{E}\) with \(\tilde{T}^g([g_x]_c) \geq r, \tilde{T}^g([w_d]_c) \leq 1 - r, \tilde{K}^g([g_x]_c) \leq 1 - r, \tilde{K}^g([w_d]_c) \leq 1 - r\), \(\tilde{T}^g([g_x]_c) \geq r, \tilde{K}^g([w_d]_c) \leq 1 - r, \tilde{T}^g([w_d]_c) \leq 1 - r, \tilde{K}^g([g_x]_c) \leq 1 - r, \tilde{K}^g([w_d]_c) \leq 1 - r\), \(\bar{e} \in E, r \in \xi_0\). Now, suppose that (3) is not satisfied. That is, \(g_x \neq \phi, w_d \neq \phi, g_x \cup w_d = \phi\). Thus, by Theorem 2.3 (2), we obtain:

\[ C^*(\bar{e}, h_z, r) \cap l_y \subseteq C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, h_z, r) \cap C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, l_y, r) = g_x \cap w_d = \phi. \]

and

\[ h_z \cap C^*(\bar{e}, l_y, r) \subseteq C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, h_z, r) \cap C_{\tilde{T}^{\sigma \tau \delta}}(\bar{e}, l_y, r) = g_x \cap w_d = \phi. \]

Therefore, \(l_y\) and \(h_z\) are r-svnfg-separated sets, \(l_y = \phi\) or \(h_z = \phi\), which implies \(h_z \cup l_y = \tilde{E}\). Hence, \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{K}^{\sigma \tau \delta})\) is not r-svnfg-connected.

Theorem 3.2. Let \((B, \tilde{T}^{\sigma \tau \delta}, \tilde{K}^{\sigma \tau \delta})\) be r-svnfgt-space and \(h_z, l_y, g_x \in \tilde{(B, E)}\). If \(l_y\) and \(g_x\) are r-svnfg-separated sets, then \(h_z \cap l_y, h_z \cap g_x\) are r-svnfg-separated sets.
Proof. Let \( l_y \) and \( g_x \) be r-svnfg-separated sets, that is,

\[
C^*(\bar{\epsilon}, l_y, r) \cap g_x = \phi = \text{cl}^*(\bar{\epsilon}, g_x, r) \cap l_y, \forall \, \bar{\epsilon} \in E, \ r \in \xi_0.
\]

Then, from Theorem 2.3 (4) we get that

\[
C^*(\bar{\epsilon}, \mathcal{H} \cap l_y, r) \cap [h_z \cap g_x] = [C^*(\bar{\epsilon}, h_z, r) \cap C^*(\bar{\epsilon}, l_y, r)] \cap [h_z \cap g_x] \\
= h_z \cap \phi = \phi
\]

and

\[
C^*(\bar{\epsilon}, \mathcal{H} \cap g_x, r) \cap [h_z \cap l_y] = [C^*(\bar{\epsilon}, h_z, r) \cap C^*(\bar{\epsilon}, g_x, r)] \cap [h_z \cap l_y] \\
= h_z \cap \phi = \phi
\]

Therefore, \( h_z \cap l_y, h_z \cap g_x \) are r-svnfg-separated sets. \( \square \)

**Theorem 3.3.** Let \((B, \hat{\tau}^\delta, E, K^\tau)\) be r-svnfgt-space and \( h_z \in (\hat{B}, \hat{E}) \). Therefore, the following statements are equivalent.

(1) \( h_z \) is r-svnfg-connected.

(2) If \( l_y \) and \( g_x \) are r-svnfg-separated with \( h_z \subseteq l_y \cup g_x \), then \( h_z \cap l_y = \phi \) or \( h_z \cap g_x = \phi \).

(3) If \( l_y \) and \( g_x \) are r-svnfg-separated with \( h_z \subseteq l_y \cup g_x \), then \( h_z \subseteq l_y \) or \( h_z \subseteq g_x \).

**Proof.**

(1)\( \Rightarrow \) (2) \( l_y \) and \( g_x \) are r-svnfg-separated such that \( h_z \subseteq l_y \cup g_x \). Form Theorem 3.2, \( h_z \cap l_y \) and \( h_z \cap g_x \) are r-svnfg-separated. So, \( h_z = h_z \cap (l_y \cup g_x) = (h_z \cap l_y) \cup (h_z \cap g_x) \). But \( h_z \) is r-svnfg-connected. Therefore, \( h_z \cap l_y = \phi \) or \( h_z \cap g_x = \phi \).

(2)\( \Rightarrow \) (3) If \( h_z \cap l_y = \phi \), then \( h_z \subseteq h_z \cap (l_y \cup g_x) = (h_z \cap l_y) \cup (h_z \cap g_x) = h_z \cap g_x \), and hence, \( h_z \subseteq g_x \). Similarly, if \( h_z \cap g_x = \phi \), then \( h_z \subseteq l_y \).

(3)\( \Rightarrow \) (1) Let \( l_y \) and \( g_x \) be r-svnfg-separated such that \( h_z \subseteq l_y \cup g_x \), by (3), we have \( h_z \subseteq l_y \) or \( h_z \subseteq g_x \).

If \( h_z \subseteq l_y \) and \( l_y, g_x \) are r-svnfg-separated sets, then \( g_x = g_x \cap h_z \subseteq g_x \cap l_y \subseteq g_x \cap C^*(\bar{\epsilon}, l_y, r) = \phi \). Thus, \( g_x = \phi \).

If \( h_z \subseteq g_x \), similarly, we have \( l_y = \phi \). Therefore, \( h_z \) is r-svnfg-connected. \( \square \)

**Theorem 3.4.** Let \((B, \hat{\tau}^\delta, E, K^\tau)\) be svnfgt-space, \( h_z, l_y, \bar{\epsilon} \in (\hat{B}, \hat{E}), \bar{\epsilon} \in E \) and \( r \in \xi_0 \). If \( h_z \neq \phi \) is r-svnfg-connected and \( l_y \subseteq h_z \subseteq C^*(\bar{\epsilon}, h_z, r) \), then \( l_y \) is r-svnfg-separated.

**Proof.** Assume that, \( l_y \) is not r-svnfg-separated. So, there exist non-empty r-svnfg-separated \( g_x, w_x \in (\hat{B}, \hat{E}) \) such that \( l_y = g_x \cup w_x \). that is,

\[
C^*(\bar{\epsilon}, g_x, r) \cap w_x = \phi = C^*(\bar{\epsilon}, w_x, r) \cap g_x, \forall \, \bar{\epsilon} \in E, \ r \in \xi_0.
\]
Because, $h_z \subseteq l_y = g_x \cup w_D$ and $h_z$ is r-svnfg-connected, and by Theorem 3.3 (3), we obtain either $h_z \subseteq g_x$ or $h_z \subseteq w_D$. Form $l_y \subseteq C^*(\tilde{\sigma}, h_z, r)$, we have

if $h_z \subseteq g_x$, then

$$w_D = (g_x \cap w_D) \cap w_D = l_y \cap w_D \subseteq C^*(\tilde{\sigma}, h_z, r) \cap w_D \subseteq C^*(\tilde{\sigma}, g_x, r) \cap w_D = \phi$$

which contradicts to $w_D \neq \phi$.

If $h_z \subseteq w_D$, then

$$g_x = (w_D \cap g_x) \cap g_x = l_y \cap g_x \subseteq C^*(\tilde{\sigma}, h_z, r) \cap g_x \subseteq C^*(\tilde{\sigma}, w_D, r) \cap g_x = \phi$$

which contradicts to $g_x \neq \phi$. Hence, $l_y$ is r-svnfg-separated. \qed

**Theorem 3.5.** Let $(B, \tilde{T}_E, K_E)$ be svnfg-space, $h_z, l_y \in \widehat{(B, E)}$, $\tilde{\sigma} \in E$ and $r \in \xi_0$. If $h_z$, $l_y$ are r-svnfg-connected which are not r-svnfg-separated, therefore, $h_z \cup l_y$ is r-svnfg-connected.

**Proof.** Let $w_D$ and $g_x$ be r-svnfg-connected with $h_z \cup l_y = w_D \cup g_x$. Because $h_z$ is r-svnfg-connected and by theorem 3.3 (3), $h_A \subseteq g_x$ or $h_z \subseteq w_D$. Say $h_z \subseteq w_D$. Assume that $l_y \subseteq g_x$. Because

$$(h_z \cup l_y) \cap w_D = (h_A \cup w_D) \cap (l_y \cap w_D) = h_z \cup \phi = h_z$$

and

$$(h_z \cup l_y) \cap g_x = (h_z \cup g_x) \cup (l_y \cap g_x) = g_x \cup \phi = g_x.$$ Form Theorem 7, we obtain, $h_z$ and $l_y$ are r-svnfg-connected. Which is a contradiction. Therefore, $l_y \subseteq w_D$. Thus, $h_z \cup l_y \subseteq w_D$. In the same way, if $h_z \subseteq g_x$, we obtain that $h_z \cup l_y \subseteq g_x$. Therefore by Theorem 8, we have, $h_z \cup l_y$ is r-svnfg-connected. \qed

**Theorem 3.6.** Let $(B, \tilde{T}_E, K_E)$ be svnfg-space and let $\mathcal{L} = \{(h_{i}), i \in \Gamma\}$ be a collection of r-svnfg-connected sets in $B$, such that no two members of $\mathcal{L}$ are r-svnfg-separated. Then, $\bigcup_{i \in \Gamma}(h_{i})$ is r-svnfg-connected.

**Proof.** Put $h_z = \bigcup_{i \in \Gamma}(h_{i})$ and let $l_y, g_x \in \widehat{(B, E)}$ be r-svnfg-separated sets such that $h_z = l_y \cup g_x$. Because every two members $(h_{i}), (h_{j}) \in \mathcal{L}$ are not r-svnfg-separated, by Theorem 3.5, $(h_{i}) \cup (h_{j})$ is r-svnfg-connected. Form Theorem 3.3 (3), we have $(h_{i}) \cup (h_{j}) \subseteq l_y$ or $(h_{i}) \cup (h_{j}) \subseteq g_x$, say $(h_{i}) \cup (h_{j}) \subseteq l_y$. Thus $h_z$ is r-svnfg-connected. \qed

**Theorem 3.7.** Let $(B, \tilde{T}_E, K_E)$ be svnfg-space and $\{(h_{i}), i \in \Gamma\}$ be a collection of r-svnfg-connected sets and $\bigcap_{i \in \Gamma}(h_{i}) \neq \phi$. Then, $\bigcup_{i \in \Gamma}(h_{i})$ is r-svnfg-connected.

**Proof.** Clear. \qed

**Definition 3.3.** Let $(B, \tilde{T}_E, K_E)$ be svnfg-space. A non empty set $h_z \subseteq \widehat{(B, E)}$ is r-svnfg-component if $h_z$ is a maximal r-svnfg-connected set in $B$, that is if $h_z \subseteq l_y$ and $l_y$ is r-svnfg-connected set, then $h_z = l_y$. 
**Theorem 3.8.** Let \((B, \overline{\mathcal{T}_E^\sigma \delta}, K_E^\sigma \delta)\) be r-svnfgt-space and \(h_z, l_y \in (\overline{B}, E), \tilde{\epsilon} \in E, r \in \xi_0\). Therefore, 
(1) if \(h_z\) is r-svnfg-component, then \(C^*(\tilde{\epsilon}, h_z, r) = h_z\). 
(2) If \(l_y\) and \(h_z\) are r-svnfg-components in \(B\) with \(l_y \cap h_z = \phi\), then \(l_y\) and \(h_z\) are r-svnfg-separated sets.

**Proof.** (1) Because \(h_z\) is r-svnfg-connected set and \(h_z \sqsubseteq C^*(\tilde{\epsilon}, h_z, r)\), from Theorem 3.4, we obtain \(C^*(\tilde{\epsilon}, h_z, r)\) is r-svnfg-connected. On the other hand \(h_z\) is r-svnfg-component, it implies \(h_z = C^*(\tilde{\epsilon}, h_z, r)\).

(2) Because \(l_y\) and \(h_z\) are r-svnfg-components in \(B\) such that \(l_y \cap h_z = \phi\). So, Form (1), we obtain \(l_y = C^*(\tilde{\epsilon}, l_y, r)\) and \(h_z = C^*(\tilde{\epsilon}, h_z, r)\). Hence

\[
C^*(\tilde{\epsilon}, h_z, r) \cap l_y = \phi = h_z \cap C^*(\tilde{\epsilon}, l_y, r).
\]

Therefore, \(l_y\) and \(h_z\) are r-svnfg-separated sets. \(\square\)

4. Single-Valued Neutrosophic Soft \(\gamma\)-Connected Spaces

Here, we present the single-valued neutrosophic soft \(\gamma\)-connected spaces r-svnf-connected of space \(B\) relative to a r-svnf operator \(\gamma\). Suppose [with respect to any r-svnft \(\overline{\mathcal{T}_E^\sigma \delta}\) defined on \(B\) and \(cl_{\sigma \delta}\) is the single-valued neutrosophic soft closure operator on \((B, \overline{\mathcal{T}_E^\sigma \delta})\)] that:

\[
h_z \sqsubseteq \gamma(\tilde{\epsilon}, h_z, r) \sqsubseteq C_{\overline{\mathcal{T}_E^\sigma \delta}}(\tilde{\epsilon}, h_z, r) \forall h_z \in (\overline{B}, E), \tilde{\epsilon} \in E, r \in \xi_0.
\]

Also, suppose that \(\gamma\) is a monotone operator, that is, \(l_y \sqsubseteq g_x\) implies \(\gamma(\tilde{\epsilon}, l_y, r) \sqsubseteq C_{\overline{\mathcal{T}_E^\sigma \delta}}(\tilde{\epsilon}, g_x, r)\), \(l_y, g_x \in (\overline{B}, E), \tilde{\epsilon} \in E, r \in \xi_0\).

**Definition 4.1.** Let \(B\) be a non-null set and \(E\) be a set of parameters. Therefore, 
(1) the svnf-sets \(h_z, l_y \in (\overline{B}, E)\) are called r-single-valued neutrosophic \(\gamma\)-separated (abbreviated r-svnf\(\gamma\)-separated) if \(h_z\) and \(l_y\) satisfy the following condition

\[
\gamma(\tilde{\epsilon}, h_z, r) \cap l_y = \phi = h_z \cap \gamma(\tilde{\epsilon}, l_y, r), \text{ for every } \tilde{\epsilon} \in E, r \in \xi_0.
\]

(2) \(B\) is termed r-single-valued neutrosophic \(\gamma\)-connected (abbreviated r-svnf\(\gamma\)-connected space) if one cannot find two svnf-sets \(h_z, l_y \in (\overline{B}, E)\) \(h_z \neq \phi, l_y \neq \phi\) and \(h_z \sqcup l_y = E\). That is, there do not exist r-svnf\(\gamma\)-separated sets \(h_z, l_y \in (\overline{B}, E)\), except \(h_z = \phi, l_y = \phi\).

**Definition 4.2.** Let \(h_z, l_y \in (\overline{B}, E)\), \(h_z \neq \phi, l_y \neq \phi\), such that:
(1) \(h_z, l_y\) are r-svnf\(\gamma\)-separated with \(h_z \sqcup l_y = E\). Therefore, \(B\) is termed r-single-valued neutrosophic \(\gamma\)-disconnected (abbreviated r-svnf\(\gamma\)-disconnected space).
(2) \(h_z, l_y\) are r-svnf\(\gamma\)-separated with \(h_z \sqcup l_y = g_x\). Therefore, \(g_x\) is termed r-svnf\(\gamma\)-disconnected space in \((B, E)\).

For a r-svnfgt-space \((B, \overline{\mathcal{T}_E^\sigma \delta}, \mathcal{T}_E^\sigma \delta)\).

If \(\gamma = \mathcal{C}_{\overline{\mathcal{T}_E^\sigma \delta}}\), then we obtain the r-svnf- connectedness.

If \(\gamma = \mathcal{C}_{\mathcal{T}_E^\sigma \delta}\), then we obtain the r-svnfg- connectedness.
Example 4.1. Assume that, \( \mathcal{B} = \{a, b\} \), \( \mathbf{E} = \{\tilde{e}_1, \tilde{e}_2\} \) and \((\mathfrak{h}_E)_{\mathcal{B}}, (\mathfrak{h}_E)_{\tilde{e}} \in (\overline{\mathcal{B}}, \mathbf{E})\) where \((\mathfrak{h}_E)_{\mathcal{B}} = \{((\tilde{e}_1, (1, 1, 0)), (\tilde{e}_2, (0, 0, 1)))\} \) and \((\mathfrak{h}_E)_{\tilde{e}} = \{((\tilde{e}_1, (0, 0, 1)), (\tilde{e}_2, (1, 1, 0)))\} \) for \( \tilde{e} \in \mathbf{E}, r \in \xi_0 \), we define the single valued soft operator \( \gamma \) as follows:

\[
\gamma(\tilde{e}, \mathfrak{h}_E, r) = \begin{cases} 
\phi, & \text{if } \mathfrak{h}_E = \phi \forall r \in \xi_0, \\
(\mathfrak{h}_E)_{\mathcal{B}}, & \text{if } \phi \neq \mathfrak{h}_E \subseteq (\mathfrak{h}_E)_{\mathcal{B}}, r \leq \frac{1}{2}, \\
(\mathfrak{h}_E)_{\tilde{e}}, & \text{if } \phi \neq \mathfrak{h}_E \subseteq (\mathfrak{h}_E)_{\tilde{e}}, r \leq \frac{2}{3}, \\
\mathbf{E}, & \text{if otherwise,}
\end{cases}
\]

Thus, \( \mathfrak{h}_E \) and \( \mathfrak{g}_E \) are r-svn\( \gamma \)-separated sets. At \( \mathfrak{h}_E = (\mathfrak{h}_E)_{\mathcal{B}}, \mathfrak{g}_E = (\mathfrak{h}_E)_{\tilde{e}} \) and \( r \leq \frac{1}{3} \), then we have

\[
\gamma(\tilde{e}, \mathfrak{h}_E, r) \cap \mathfrak{g}_E = \phi = \mathfrak{g}_E \cap \gamma(\tilde{e}, \mathfrak{g}_E, r).
\]

The following theorem is similarly proved, as in Theorem 3.1.

Theorem 4.1. Let \((\mathcal{B}, \overline{\mathcal{T}}_{\mathcal{E}})\) be r-svn\( \mathfrak{g} \)-space. Therefore, the following statements are equivalent.

1. \((\mathcal{B}, \overline{\mathcal{T}}_{\mathcal{E}})\) is r-svn\( \mathfrak{g} \)-connected.
2. If \( \mathfrak{h}_z \cup l_y = \mathbf{E} \) and \( \mathfrak{h}_z \cap l_y = \phi \) with \( \overline{\mathcal{T}}_{\mathcal{E}}(\mathfrak{h}_z) \geq r \), \( \overline{\mathcal{T}}_{\mathcal{E}}(\mathfrak{h}_z) \leq 1 - r \mathfrak{h}_z \subseteq \mathfrak{h}_z \leq 1 - r \), \( \overline{\mathcal{T}}_{\mathcal{E}}(l_y) \geq r \), \( \overline{\mathcal{T}}_{\mathcal{E}}(l_y) \leq 1 - r \overline{\mathcal{T}}_{\mathcal{E}}(l_y) \leq 1 - r, \tilde{e} \in \mathbf{E}, r \in \xi_0 \), then \( \mathfrak{h}_z = \phi \) or \( l_y = \phi \).
3. If \( \mathfrak{h}_z \cup l_y = \mathbf{E} \) and \( \mathfrak{h}_z \cap l_y = \phi \) with \( \overline{\mathcal{T}}_{\mathcal{E}}([\mathfrak{h}_z]^c) \geq r \), \( \overline{\mathcal{T}}_{\mathcal{E}}([\mathfrak{h}_z]^c) \leq 1 - r \mathfrak{h}_z \subseteq \mathfrak{h}_z \leq 1 - r \), \( \overline{\mathcal{T}}_{\mathcal{E}}([l_y]^c) \geq r \), \( \overline{\mathcal{T}}_{\mathcal{E}}([l_y]^c) \leq 1 - r \overline{\mathcal{T}}_{\mathcal{E}}([l_y]^c) \leq 1 - r, \tilde{e} \in \mathbf{E}, r \in \xi_0 \), then \( \mathfrak{h}_z = \phi \) or \( l_y = \phi \).

The following theorem is similarly proved, as in Theorem 3.2.

Theorem 4.2. Let \( \mathcal{B} \) be a non-empty set, \( \mathbf{E} \) be a set of parameters and \( \mathfrak{h}_z, l_y, g_x \in (\overline{\mathcal{B}}, \mathbf{E}) \). If \( l_y \) and \( g_x \) are r-svn\( \mathfrak{g} \)-separated sets, then \( \mathfrak{h}_z \cap l_y, \mathfrak{h}_z \cap g_x \) are r-svn\( \mathfrak{g} \)-separated sets.

The following theorem is similarly proved, as in Theorem 3.3.

Theorem 4.3. Let \( \mathfrak{h}_z \in (\overline{\mathcal{B}}, \mathbf{E}) \). Then, the following statements are equivalent.

1. \( \mathfrak{h}_z \) is r-svn\( \mathfrak{g} \)-connected.
2. If \( l_y \) and \( g_x \) are r-svn\( \mathfrak{g} \)-separated with \( \mathfrak{h}_z \subseteq l_y \cup g_x \), then \( \mathfrak{h}_z \cap l_y = \phi \) or \( \mathfrak{h}_z \cap g_x = \phi \)
3. If \( l_y \) and \( g_x \) are r-svn\( \mathfrak{g} \)-separated with \( \mathfrak{h}_z \subseteq l_y \cup g_x \), then \( \mathfrak{h}_z \subseteq l_y \) or \( \mathfrak{h}_z \subseteq g_x \).

The following theorem is similarly proved, as in Theorem 3.4.

Theorem 4.4. Let \( \mathfrak{h}_z, l_y \in (\overline{\mathcal{B}}, \mathbf{E}), r \in \xi_0 \). If \( \mathfrak{h}_z \neq \phi \) is r-svn\( \mathfrak{g} \)-connected and \( \mathfrak{h}_z \subseteq l_y \subseteq \gamma(\tilde{e}, \mathfrak{h}_z, r), \tilde{e} \in \mathbf{E}, \) then \( l_y \) is r-svn\( \mathfrak{g} \)-connected.

Theorem 4.5. Let \( \mathfrak{h}_z, l_y \in (\overline{\mathcal{B}}, \mathbf{E}), r \in \xi_0 \). If \( \mathfrak{h}_z \) and \( l_y \) are r-svn\( \mathfrak{g} \)-connected which are not r-svn\( \mathfrak{g} \)-separated, then \( \mathfrak{h}_z \cup l_y \) is r-svn\( \mathfrak{g} \)-connected.
Proof. Let \( g_x \) and \( w_\phi \) be r-svn\( \gamma \)-separated, such that, \( \mathcal{h}_z \sqcup I_y = g_c \sqcup w_\phi \). Since, \( \mathcal{h}_A \) is r-svn\( \gamma \)-connected, by Theorem 4.3 (3), \( \mathcal{h}_z \subseteq g_x \) or \( \mathcal{h}_z \subseteq w_\phi \). Let \( \mathcal{h}_z \subseteq w_\phi \). Suppose \( I_y \subseteq g_x \). Since \( (\mathcal{h}_z \sqcup I_y) \sqcap w_\phi = (\mathcal{h}_z \sqcap w_\phi) \sqcup (I_y \sqcap w_\phi) = \mathcal{h}_z \sqcup \phi = \mathcal{h}_z \), by Theorem 4.2, \( \mathcal{h}_A \) and \( I_b \) are r-svn\( \gamma \)-separated. Which is a contradiction. Hence we have \( (\mathcal{h}_z \sqcup I_y) \sqcap w_\phi = \mathcal{h}_z \sqcup \phi = \mathcal{h}_z \). Then by Theorem 4.3 (3), r-svn\( \gamma \)-separated, then \( \mathcal{h}_z \sqcup I_y \) is r-svn\( \gamma \)-connected. 

The following theorem is similarly proved, as in Theorem 3.6.

**Theorem 4.6.** Let \( \zeta = \{ (\mathcal{h}_z) ; \in (\mathcal{B}, \mathcal{E}) \}, \ i \in \Gamma \) be a collection of r-svn\( \gamma \)-connected sets in \( \mathcal{B} \) such that no two members of \( \zeta \) are r-svn\( \gamma \)-separated. Then, \( \bigcup_{\i \in \Gamma} (\mathcal{h}_z) \) is r-svn\( \gamma \)-connected.

The following corollary follows from Theorem 4.6.

**Corollary 4.1.** Let \( \{ (\mathcal{h}_z) ; \in (\mathcal{B}, \mathcal{E}) \}, \ i \in \Gamma \) be a family of r-svn\( \gamma \)-connected sets and \( \cap_{\i \in \Gamma} (\mathcal{h}_z) \neq \phi \). Then, \( \bigcup_{\i \in \Gamma} (\mathcal{h}_z) \) is r-svn\( \gamma \)-connected.

**Theorem 4.7.** Let \( \theta_\psi : (\mathcal{B}, \mathcal{E}) \rightarrow (\mathcal{L}, \mathcal{F}) \) be a mapping such that,

\[
\gamma(\tilde{e}, \theta_\psi^{-1}(l_r), r) \subseteq \theta_\psi^{-1}(\theta(\psi(\tilde{e})), l_r), r) \forall \ l_r \in (\mathcal{L}, \mathcal{F}), r \in E, \rho \in E,
\]

where \( \gamma \) is a svn\( \gamma \)-operator on \( \mathcal{B} \) and \( \theta \) is a r-svn\( \gamma \)-operator on \( \mathcal{L} \). Then, the set \( \theta_\psi(h_z) \in (\mathcal{L}, \mathcal{F}) \) is r-svn\( \theta \)-connected if the set \( h_z \in (\mathcal{B}, \mathcal{E}) \) is r-svn\( \gamma \)-connected.

**Proof.** Let \( l_r \neq \phi \) and \( g_x \neq \phi \) be a r-svn\( \theta \)-separated sets in \( (\mathcal{L}, \mathcal{F}) \) with \( \theta_\psi(h_z) = l_r \sqcup g_x \). That is \( \theta(\psi(\tilde{e}), g_x, r) \sqcup l_r \subseteq \theta(\psi(\tilde{e}), l_r, r) \sqcup g_x = \phi \), for all \( r \in E, \tilde{e} \in E \), then we have \( h_z \subseteq \theta_\psi^{-1}(\theta_\psi(h_z)) = \theta_\psi^{-1}(l_r \sqcup g_x) = \theta_\psi^{-1}(l_r) \sqcup \theta_\psi^{-1}(g_x), \)

\[
\gamma(\tilde{e}, \theta_\psi^{-1}(l_r), r) \cap \theta_\psi^{-1}(g_x) \subseteq \theta_\psi^{-1}(\theta(\psi(\tilde{e}), l_r, r)) \cap \theta_\psi^{-1}(g_x) = \theta_\psi^{-1}(\theta(\psi(\tilde{e}), l_r, r) \cap g_x) = \theta_\psi^{-1}(\phi) = \phi.
\]

Also

\[
\gamma(\tilde{e}, \theta_\psi^{-1}(g_x), r) \cap \theta_\psi^{-1}(l_r) \subseteq \theta_\psi^{-1}(\theta(\psi(\tilde{e}), g_x, r)) \cap \theta_\psi^{-1}(l_r) = \theta_\psi^{-1}(\theta(\psi(\tilde{e}), g_x, r) \cap l_r) = \theta_\psi^{-1}(\phi) = \phi.
\]

Hence \( \theta_\psi^{-1}(l_r) \) and \( \theta_\psi^{-1}(g_x) \) r-svn\( \gamma \)-separated sets in \( \mathcal{B} \). So that, \( h_z \subseteq \theta_\psi^{-1}(l_r) \sqcup \theta_\psi^{-1}(g_x) \). But \( h_z \) is r-svn\( \gamma \)-connected, by Theorem 3.3 (3), \( h_z \subseteq \theta_\psi^{-1}(l_r) \) or \( h_z \subseteq \theta_\psi^{-1}(g_x) \), which means, \( \theta_\psi(h_z) \subseteq l_r \) or \( \theta_\psi(h_z) \subseteq g_x \). Hence, by using Theorem 3.3 (3), we have \( \theta_\psi(h_z) \) is r-svn\( \theta \)-connected. 

\( \square \)
Corollary 4.2. Let \((B, \overset{\sigma \delta}{\top}_E)\) and \((\mathcal{L}, \overset{\sigma \delta}{\top}_F)\) be two svnft-spaces. If \(\vartheta_{\psi} : (\overset{\sigma \delta}{\top}_E) \rightarrow (\overset{\sigma \delta}{\top}_F)\) is a svnft-continuous mapping and \(l_r \in (\overset{\sigma \delta}{\top}_E)\) is r-svnft-connected in \(B\), then \(\vartheta_{\psi}(l_r)\) is r-svnft-connected in \(\mathcal{L}\).

Note, if \(\gamma = \overset{\sigma \delta}{\top}_E\) and \(\theta = \overset{\sigma \delta}{\top}_F\). Then, the result follows from Theorem 4.7.

Corollary 4.3. Let \((B, \overset{\sigma \delta}{\top}_E, \overset{\sigma \delta}{K}_E)\) and \((\mathcal{L}, \overset{\sigma \delta}{\top}_F, \overset{\sigma \delta}{K}_F)\) be two svnfgt-spaces and \(\vartheta_{\psi} : (\overset{\sigma \delta}{\top}_E) \rightarrow (\overset{\sigma \delta}{\top}_F, \overset{\sigma \delta}{K}_F)\) be a mapping satisfying the condition,

\[
(1) \overset{\sigma \delta}{\top}_E(\overset{\sigma \delta}{K}_E(l_r, l), r) \subseteq \overset{\sigma \delta}{\top}_F(\overset{\sigma \delta}{K}_F(\psi(\overset{\sigma \delta}{K}_E(l), l, r)) \lor \overset{\sigma \delta}{\top}_F, r \in \mathcal{E}, \overset{\sigma \delta}{E}.
\]

Then, the set \(\vartheta_{\psi}(l_r) \in (\overset{\sigma \delta}{\top}_F)\) is r-svnft-connected if the set \(l_r \in (\overset{\sigma \delta}{\top}_E)\) is r-svnft-connected.

Proof. Note, if \(\gamma = \overset{\sigma \delta}{\top}_E\) and \(\theta = \overset{\sigma \delta}{\top}_F\). Then, the result follows from Theorem 4.7.

Acknowledgments: The authors would like to thank Deanship of Scientific Research at Majmaah University for supporting this work under Project Number No: R-2023-847.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


[23] A.A. Salama, F. Smarandache, Neutrosophic Crisp Set Theory, Educational Publisher, Columbus, 2015.