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New Approach to Solving Fuzzy Multiobjective Linear Fractional Optimization Problems

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Abstract. In this paper, an iterative approach based on the use of fuzzy parametric functions is proposed to find the best preferred optimal solution to a fuzzy multiobjective linear fractional optimization problem. From this approach, the decision-maker imposes tolerance values or termination conditions for each parametric objective function. Indeed, the fuzzy parametric values are computed iteratively, and each fuzzy fractional objective is transformed into a fuzzy non-fractional parametric function using these values of parameters. The core value of fuzzy numbers is used to transform the fuzzy multiobjective non-fractional problem into a deterministic multiobjective non-fractional problem, and the ϵ-constraint approach is employed to obtain a linear single objective optimization problem. Finally, by setting the value of parameter ϵ, the Dangtzig simplex method is used to obtain an optimal solution. Therefore, the number of solutions is equal to the number of used values, and the optimal solution is chosen according to the preference of the decision-maker. We have provided a didactic example to highlight the step of our approach and its numerical performances.

1. Introduction

Fuzzy multiobjective linear fractional optimization problems occurs when there are several ratios of fuzzy quantities to be optimized simultaneously. It has a variety of important uses of this model in the solving of real-world decision problems. For instance for the practical applications, it is useful in corporate and financial planning (profit/investment ratios, debt/equity), production planning (output/employee ratios, investment/sales), hospital planning and healthcare planning (nurses/patients,costs/patients) and university planning (student/teacher, placement/admission, research output/teacher) or the quantities, which are used to measure the performance of any system, and so on. In general, the objective functions are conflicting, and it is not possible to obtain a unique solution that optimizes simultaneously these objective functions. Many solutions can be

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found by using the concept of Pareto optimality [7,23]. For this set of solutions, we cannot choose a better one without adding some criteria of preference. Unfortunately, there is no successful and universal method available in the literature that can help decision-makers solve these kinds of problems. Usually, each method is used for a particular type of model.

Indeed, Fuzzy multiobjective linear fractional optimization problems have attracted considerable research interest since recent few years. Several methods have been proposed in this context for the determination of the optimal solutions in the single objective case or the Pareto optimal solution in the multiple objective case. Here are some works on this topic: solving multiobjective fuzzy fractional programming problem was proposed by P. Durga et al. [1]; fuzzy solution of fully fuzzy multiobjective linear fractional programming problems was proposed by T. Loganathan et al. [2]; fuzzy mathematical programming for multiobjective linear fractional programming problems was proposed by M. Chakraborty et al. [3]; solving linear fractional programming problem under fuzzy environment: numerical approach was proposed by C. Veeramani et al. [4]; Taylor series approach to fuzzy multiobjective linear fractional programming was proposed by M. Duran [5]; An approach for solving fuzzy multiobjective linear fractional programming problems was proposed by F. A. Pramy [10]; A note on fuzzy multiobjective linear fractional programming problem was proposed by M. Deb et al. [14]; Interactive decision making for multiobjective linear fractional programming problem with fuzzy parameters was proposed by M. Sakawa et al. [21]; and so on. But most of these existing methods provide a unique solution without taking into account the preference of the decision-maker. And yet, the solutions must consider the waiting of the decision-maker.

In this work, we have proposed an iterative approach to solve a fuzzy multiobjective linear fractional optimization problem using the concept of fuzzy parametric functions and the ϵ-constraint approach simultaneously. It converts the fuzzy multiobjective linear fractional optimization problem into an equivalent fuzzy multiobjective linear optimization problem with certain fuzzy parameters. The deffuzziffication of the problem is done by the core value function. It consists of transforming the problem into a deterministic multiobjective linear problem. And at this stage, we have used the ϵ-constraint approach to aggregate multiple objective functions into one. The use of Dantzig simplex allows us to obtain the solutions. The choice of the best compromise is subject to the termination conditions, which are imposed on all parametric objectives functions by the decision-makers. To facilitate the explanation in practice, we will use triangular fuzzy numbers, and a didactic example will be done in order to show clearly the main step of the method. According to the obtained solutions, we have demonstrated the effectiveness of our method.

For a better presentation of this work, we will present some preliminaries in Section 2. Section 3 will be dedicated to the presentation of the results and discussion. Section 4 will be devoted to presenting the conclusion.

2. Preliminaries

2.1. **Fuzzy number.**
This part presents the notion of fuzzy numbers and some arithmetic operations.

Definition 2.1. *[4, 8, 10–12]. Let $\mathcal{X}$ be a set. A fuzzy subset $\tilde{a}$ of $\mathcal{X}$ is characterized by a membership function $\mu_{\tilde{a}} : \mathcal{X} \to [0,1]$ and represented by a set of ordered pairs defined as follows:*

$$\tilde{a} = \{(x, \mu_{\tilde{a}}(x))/x \in \mathcal{X}\}. \tag{2.1}$$

The value $\mu_{\tilde{a}}(x) \in [0,1]$ represents the degree of membership of x to the fuzzy set and is interpreted as the extent to which x belongs to $\tilde{a}$.

Definition 2.2. *[4, 8, 10–12]. Let $\tilde{a}$ be a fuzzy set on $\mathcal{X}$ and $\alpha \in [0,1]$. The $\alpha -$ level set of $\tilde{a}$ is the classical set noted $\tilde{a}_\alpha$ and is defined by:*

$$\tilde{a}_\alpha = \{x \in \mathcal{X}, \mu_{\tilde{a}}(x) \geq \alpha\}. \tag{2.2}$$

In the following, we will identify $\mathcal{X}$ to $\mathbb{R}$.

Definition 2.3. *[18–20]. Let $\tilde{a}$ a fuzzy subset of $\mathbb{R}$. Then, $\tilde{a}$ is called a fuzzy number if the following conditions are satisfied:*

(i) *$\tilde{a}$ is normal, i.e, $\mu_{\tilde{a}}(x) = 1$ for some $x \in \mathbb{R}$;*
(ii) *$\tilde{a}$ is convex, i.e., the membership function $\mu_{\tilde{a}}(x)$ is quasi-concave;*
(iii) *$\mu_{\tilde{a}}(x)$ is upper semicontinuous, i.e., $\tilde{a}_\alpha$ is a closed subset of $\mathbb{R}$ for some $\alpha \in \mathbb{R}$;*
(vi) *the 0-level set $\tilde{a}_0$, is a compact subset of $\mathbb{R}$.*

We note by $\mathcal{L}(\mathbb{R})$, the set of all fuzzy numbers. Indeed, if $\tilde{a} \in \mathcal{L}(\mathbb{R})$, the α-level set of $\tilde{a}$ is a compact and convex subset of $\mathbb{R}$. Then $\tilde{a}_\alpha$ is a closed interval, denoted $\tilde{a}_\alpha = [\tilde{a}^L(\alpha), \tilde{a}^U(\alpha)]$ for $\alpha \in [0,1]$.

We say that $\tilde{a}$ is a crisp number with value m if its membership function is given by

$$\mu_{\tilde{a}}(r) = \begin{cases} 1, & \text{if } r = m \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

We also use the notation $\tilde{1}_m$ to represent the crisp number with value m. So, we see that $\left(\tilde{1}_m\right)^L(\alpha) = \left(\tilde{1}_m\right)^U(\alpha) = m$ for all $\alpha \in [0,1]$. Let us remark that a real number m can be regarded as a crisp number $\tilde{1}_m$.

Definition 2.4. *[4, 8, 15–17]. A triangular fuzzy number $\tilde{a}$ can be defined by a triple (a^L, a, a^U) whose membership function $\mu_{\tilde{a}}(x)$ given below:*

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a^L}{a - a^L}, & \text{if } a^L \leq x < a; \\ \dfrac{a^U - x}{a^U - a}, & \text{if } a \leq x < a^U; \\ 0, & \text{else.} \end{cases}$$

Then, the α-level set $\tilde{a}$ is

$$\tilde{a}_\alpha = \Big[(1-\alpha)a^L + \alpha a, (1-\alpha)a^U + \alpha a\Big] \tag{2.4}$$

such as,

$$\tilde{a}^L(\alpha) = (1-\alpha)a^L + \alpha a \ \ and \ \ \tilde{a}^U(\alpha) = (1-\alpha)a^U + \alpha a$$

Definition 2.5. *[4,8]. A triangular fuzzy number $\tilde{a} = (a^L, a, a^U)$, is assumed to be non negative triangular fuzzy number if and only if $a^L \geq 0$.*

Definition 2.6. *[24]. Let $\tilde{a} = (a^L, a, a^U)$ and $\tilde{b} = (b^L, b, b^U)$ be any two triangular fuzzy numbers, then the arithmetic operations are as follows:*

(i) *addition : $\tilde{a} \oplus \tilde{b} = (a^L + b^L, a + b, a^U + b^U)$,*
(ii) *subtraction: $\tilde{a} \ominus \tilde{b} = (a^L - b^U, a - b, a^U - b^L)$,*
(iii) *multiplication: $\tilde{b}$ to be nonnegative*

$$\tilde{a} \otimes \tilde{b} = \begin{cases} (a^L b^L, ab, a^U b^U), & si\ a^L \geq 0, \\ (a^L b^U, ab, a^U b^U), & si\ a^L < 0, a^U \geq 0, \\ (a^L b^U, ab, a^U b^L), & si\ a^U < 0, \end{cases}$$

and

$$k\tilde{a} = \begin{cases} (ka^L, ka, ka^U)\ if\ k \geq 0, \\ \\ (ka^U; ka; ka^L)\ if\ k < 0, \end{cases}$$

(iv) *division: $\tilde{a} \oslash \tilde{b} = (\frac{a^L}{b^U}; \frac{a}{b}; \frac{a^U}{b^L})$.*

2.2. **Defuzzification function.** A deffuzziffication function is an operator that transforms fuzzy data into a deterministic data. Let $\tilde{a}$ be a fuzzy subset of $\mathbb{R}$. The core set of $\tilde{a}$ is defined as

$$CORE(\tilde{a}) = \{x \in \mathbb{R} / \mu_{\tilde{a}}(x) = 1\}. \tag{2.5}$$

Indeed, if $\tilde{a} \in \mathcal{L}(\mathbb{R})$, then, by the definition of normality $CORE(\tilde{a}) \neq \emptyset$. In addition, $CORE(\tilde{a}) = [\tilde{a}^L(1), \tilde{a}^U(1)]$, the 1-level set of $\tilde{a}$. We can further define the core value of $\tilde{a}$, denoted by $core(\tilde{a})$, as the mean value of the core set $CORE(\tilde{a})$ by

$$core(\tilde{a}) = \frac{\tilde{a}^L(1) + \tilde{a}^U(1)}{2}. \tag{2.6}$$

This core value will be considered as the defuzzification of a fuzzy number in the fuzzy optimization problem.

Remark 2.1. *If $\tilde{1}_m$ is a crisp number with value m, then $core(\tilde{1}_m) = m$.*

Proposition 2.1. *[13]. Let $\tilde{a}$, $\tilde{b}$ be fuzzy numbers and $\tilde{1}_m$, $\tilde{1}_{m_1}$, $\tilde{1}_{m_2}$ be crisp some numbers with values m, m_1 and m_2 respectively. Then, the following properties hold true:*

(i) *$core(\tilde{a} \oplus \tilde{b}) = core(\tilde{a}) + core(\tilde{b})$,*
(ii) *$core(\tilde{a} \otimes \tilde{1}_m) = m.core(\tilde{a})$,*
(iii) *$core\Big((\tilde{a} \otimes \tilde{1}_{m_1}) \oplus (\tilde{b} \otimes \tilde{1}_{m_2})\Big) = m_1.core(\tilde{a}) + m_2.core(\tilde{b})$.*

2.3. **Fuzzy multiobjective linear fractional optimization problem.** A fuzzy multiobjective linear fractional optimization problem is defined as follows:

$$\min_{x\in\Omega} \tilde{f}(x) = \left(\tilde{f}_1(x), \tilde{f}_2(x), ..., \tilde{f}_p(x)\right) \tag{2.7}$$

where

- $\Omega = \left\{x \in \mathbb{R}^n : \tilde{g}_k(x) \preceq \tilde{0}\right\}, k = 1, ..., m$
- $\tilde{f}_i(x) = \dfrac{\sum_{j=1}^{n} \tilde{c}_{ij}x_j \oplus \tilde{\alpha}_i}{\sum_{j=1}^{n} \tilde{d}_{ij}x_j \oplus \tilde{\beta}_i} = \dfrac{\tilde{P}_i(x)}{\tilde{Q}_i(x)}, i = \overline{1, p};$

  with $\tilde{c}_{ij}, \tilde{d}_{ij}, \tilde{\alpha}_i, \tilde{\beta}_i \in \mathcal{L}(\mathbb{R})$ and $\tilde{Q}_i(x) \succ \tilde{0}$.

In the following, we will assume that Ω is the non-empty compact feasible region in which $\tilde{g}_k$ are linear functions with fuzzy coefficients.

Definition 2.7. *[2, 22]. A feasible solution $\overline{x}$ is said to be Pareto optimal solution of the (2.7) if there does not exist another feasible solution $x \in \Omega$ such as $\tilde{f}_i(x) \preceq \tilde{f}_i(\overline{x})$ for all i and $\tilde{f}_j(x) \prec \tilde{f}_j(\overline{x})$ at least one j.*

Definition 2.8. *[2, 22]. $\overline{x} \in \Omega$ is a weakly Pareto optimal solution of (2.7) if there does not exist another feasible solution $x \in \Omega$ such that $\tilde{f}_i(x) \prec \tilde{f}_i(\overline{x})$ for all i.*

Definition 2.9. *[2, 22]. $\tilde{\lambda}$ is called an ideal objective vector when its coordinates are obtained by evaluating the values of objective functions at their respective individual minimum points. In other words*

$$\tilde{\lambda}_i = \min_{x\in\Omega} \tilde{f}_i, i = 1, 2, \cdots, p.$$

3. Results and Discussion

3.1. **New approach.** Our method is an iterative approach to solving some decision problems that give a fuzzy multiobjective fractional optimization program. The main steps can be summarized as follows.

Step 1 : Linearization.

It consists of the transformation the fuzzy multiobjective fractional linear optimization problem into a fuzzy multiobjective linear optimization problem. We have proposed a fuzzy version of the Dinkelback theorem in order to do this operation.

Theorem 3.1. . *Let us consider the problem (2.7). Let us assume that x^* be a Pareto optimal solution of the problem (2.7). Then, $\tilde{\lambda}_i^* = \dfrac{\tilde{P}_i(x^*)}{\tilde{Q}_i(x^*)} = \min\left\{\dfrac{\tilde{P}_i(x)}{\tilde{Q}_i(x)}, x \in \tilde{\Omega}\right\}$ if and only if $\min\left\{\tilde{P}_i(x) - \tilde{\lambda}_i^*\tilde{Q}_i(x), x \in \Omega\right\} \approx \tilde{0}$.*

Proof. Let x^* be a Pareto optimal solution of Problem (2.7). We have

$$\forall x \in \tilde{\Omega},\ \tilde{\lambda}_i^* = \frac{\tilde{P}_i(x^*)}{\tilde{Q}_i(x^*)} \preceq \frac{\tilde{P}_i(x)}{\tilde{Q}_i(x)} \ for\ all\ i\ and$$

$$\tilde{\lambda}_j^* = \frac{\tilde{P}_j(x^*)}{\tilde{Q}_j(x^*)} \prec \frac{\tilde{P}_j(x)}{\tilde{Q}_j(x)} \text{ at least one } j.$$

Hence

$$\tilde{P}_i(x) - \tilde{\lambda}_i^* \tilde{Q}_i(x) \geq \tilde{0} \; for\ all\ i\ and \tag{3.1}$$

$$\tilde{P}_i(x) - \tilde{\lambda}_i^* \tilde{Q}_i(x) \succ \tilde{0} \; at\ east\ one\ j. \tag{3.2}$$

Form (3.2), we have $\min\{\tilde{P}_i(x) - \tilde{\lambda}_i^* \tilde{Q}_i(x)\} \approx \tilde{0}$.
Conversely, let $x^\star$ be a solution of problem

$$\min_{x \in \tilde{\Omega}} \left(\tilde{P}_i(x) - \tilde{\lambda}_i^* \tilde{Q}_i(x) \right), \quad i = 1, ..., p \tag{3.3}$$

where
$\tilde{\lambda}_i^* = \dfrac{\tilde{P}_i(x^*)}{\tilde{Q}_i(x^*)}$ such that $\tilde{P}_i(x^*) - \tilde{\lambda}_i^* \tilde{Q}_i(x^*) \approx \tilde{0}$.
The definition of (3.3) implies that $for\ all\ x \in \tilde{\Omega}$

$$\tilde{P}_i(x^*) - \tilde{\lambda}_i^* \tilde{Q}_i(x^*) \leq \tilde{P}_i(x) - \tilde{\lambda}_i^* \tilde{Q}_i(x). \tag{3.4}$$

Hence,

$$\tilde{P}_i(x) - \tilde{\lambda}_i^* \tilde{Q}_i(x) \geq \tilde{0} \; for\ all\ x \in \tilde{\Omega}. \tag{3.5}$$

This implies that

$$\tilde{\lambda}_i^* \leq \frac{\tilde{P}_i(x)}{\tilde{Q}_i(x)} \tag{3.6}$$

for all $x \in \tilde{\Omega}$, $i = 1, ..., p$. From (3.6), $\tilde{\lambda}_i^*$, $i = 1, ..., p$ is a minimum of problem (2.7), consequently $\tilde{\lambda}_i^* = \dfrac{\tilde{P}_i(x^*)}{\tilde{Q}_i(x^*)} = \min \left\{ \dfrac{\tilde{P}_i(x)}{\tilde{Q}_i(x)} \right\}$ for $x \in \tilde{\Omega}$. □

In the rest of the work, we will build the sequence $\left(\tilde{\lambda}_i^{(k)}, i = 1, \cdots, p \right)_{k \in \mathbb{N}^*}$, by posing for $k > 1$ with $\tilde{\lambda}_i^k = \dfrac{\tilde{P}_i(\overline{X}^{(k-1)})}{\tilde{Q}_i(\overline{X}^{(k-1)})}$ where $\overline{X}^{(k-1)} = \arg\min \left(\tilde{P}_i(x) - \tilde{\lambda}_i^{(k-1)} \tilde{Q}_i(x) \right)$.
For $k = 1$, we calculate $\tilde{\lambda}_i^{(1)}$ as follows:

$$\tilde{\lambda}_i^{(1)} = \frac{\tilde{P}_i(\overline{X}^{(0)})}{\tilde{Q}_i(\overline{X}^{(0)})},$$

where $\overline{X}^{(0)} = \sum_{i=1}^{p} \omega_i \overline{X}_i$, $\sum_{i=1}^{p} \omega_i = 1$, $\omega_i > 0$ and $\overline{X}_i$ are the individual minimal solutions of the objectives of problem (2.7).
Now, let us consider the k^{th} iteration, then the problem can be formulate as follows:

$$\min_{x \in \tilde{\Omega}} \left(\tilde{P}_i(x) - \tilde{\lambda}_1^{(k)} \tilde{Q}_i(x) \right), \; i = \overline{1, p} \tag{3.7}$$

Theorem 3.2. *$\overline{x}$ is Pareto optimal solution of problem (2.7) if and only if $\overline{x}$ is Pareto optimal solution of (3.7).*

Proof. Consider the following notations for convenience of the proof, $\tilde{I}_i(x) = \dfrac{\tilde{P}_i(x)}{\tilde{Q}_i(x)}$ and $\tilde{J}_i(x) = \left\{\tilde{P}_i(x) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(x)\right\}$ with $\overline{\tilde{\lambda}}_i^{(k)} = \dfrac{\tilde{P}_i(\overline{x})}{\tilde{Q}_i(\overline{x})}$, $i = 1, ..., p$ at the k^{th} step.
Let $\overline{x}$ is a Pareto optimal solution of the problem (2.7). Suppose on contrary, $\overline{x}$ is not Pareto optimal for problem (3.7), i.e, by definition, $\exists x\ \tilde{\Omega}$ such that, $\tilde{J}_i(x) \leq \tilde{J}_i(\overline{x})$, $\forall i = 1, ..., p$ and $\tilde{J}_i(x) < \tilde{J}_i(\overline{x})$ for at least one $j \in \{1, ..., p\}$. Otherwise $\tilde{P}_i(x) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(x) \leq \tilde{P}_i(\overline{x}) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(\overline{x})$ $\forall i$ and $\tilde{P}_i(x) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(x) < \tilde{P}_i(\overline{x}) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(\overline{x})$ for at least one j. Since $\overline{\tilde{\lambda}}_i^{(k)} = \dfrac{\tilde{P}_i(\overline{x})}{\tilde{P}_i(\overline{x})}$, we have
$\tilde{P}_i(\overline{x}) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(\overline{x}) \approx \tilde{0}$, i.e
$\tilde{P}_i(x) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(x) \leq \tilde{0}$ and $\tilde{P}_i(x) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(x) < \tilde{0}$ for at least one j. Which implies $\dfrac{\tilde{P}_i(x)}{\tilde{Q}_i(x)} \leq \dfrac{\tilde{P}_i(\overline{x})}{\tilde{Q}_i(\overline{x})}$ $\forall i$ and $\dfrac{\tilde{P}_i(x)}{\tilde{Q}_i(x)} < \dfrac{\tilde{P}_i(\overline{x})}{\tilde{Q}_i(\overline{x})}$ for at least one j. It contradicts the Pareto optimality of $\overline{x}$ for the problem (2.7). Therefore, $\overline{x}$ is Pareto optimal solution for problem(3.7). Conversely, let $\overline{x}$ is Pareto optimal for problem (3.7). Suppose on contrary, $\overline{x}$ is not Pareto optimal for problem(2.7), i.e., $\exists x \in \tilde{\Omega}$ such that $\tilde{I}_i(x) \leq \tilde{I}_i(\overline{x})$ $\forall i$ and $\tilde{I}_j(x) < \tilde{I}_j(\overline{x})$ for at least one j. After simplification, we get $\tilde{P}_i(x) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(x) \leq \tilde{0}$ and
$\tilde{P}_i(x) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(x) < \tilde{0}$ for at least one j, i.e., $\tilde{J}_i(x) \leq \tilde{0}$, $\forall i$ and
$\tilde{J}_j(x) < \tilde{0}$. But, $\tilde{J}_i(\overline{x}) = \tilde{P}_i(\overline{x}) - \overline{\tilde{\lambda}}_i^{(k)}\tilde{Q}_i(\overline{x}) \approx \tilde{0}$ $\forall i$. So, $\tilde{J}_i(x) \leq \tilde{J}_i(\overline{x})$ $\forall i$ and $\tilde{J}_j(x) < \tilde{J}_j(\overline{x})$ for at least one j. It contradicts the Pareto optimality of $\overline{x}$ for problem(3.7). Thus, $\overline{x}$ is Pareto optimal solution of problem(2.7). □

Step 2 : Defuzzification.

It aims to transform the fuzzy multiobjective linear optimization problem into a deterministic multiobjective linear optimization problem. Let us consider the problem (3.7). Using the core value defined below on the fuzzy numbers of the problem (3.7), the objective functions are written as follows:
$\left(\sum_{j=1}^{n} core(\tilde{c}_{ij})x_j + core(\tilde{\alpha}_i)\right) - core(\tilde{\lambda}_i^{(k)}) \times \left(\sum_{j=1}^{n} core(\tilde{d}_{ij})x_j + core(\tilde{\beta}_i)\right) =$
$core(\tilde{P}_i(x)) - core(\tilde{\lambda}_i^{(k)})core(\tilde{Q}_i(x)) = P_i(x) - \lambda_i^{(k)}Q_i(x)$,
and for each constraint function, we have
$core(\tilde{g}_k(x)) = g_k(x),\ \forall k = 1, \cdots, m.$
For $k \in \{1, 2, \cdots, p\}$ So, the problem (3.7) is defuzzified as follows:

$$\min_{x\in\Omega, i} T_i(x, \lambda^{(k)}) = \left(P_i(x) - \lambda_i^{(k)}Q_i(x)\right) \tag{3.8}$$

The problem (3.8) is a deterministic linear multi-objective parametric optimization problem.

Theorem 3.3. *Any Pareto optimal solution of (3.7) is Pareto optimal of (3.8) and vice versa.*

Proof. Let $\overline{x}$ be a solution of problem (3.7) i.e., $\tilde{T}_i(\overline{x},\tilde{\lambda}_i^{(k)}) \preceq \tilde{T}_i(x,\tilde{\lambda}_i^{(k)}), \forall i$ and $\tilde{T}_j(\overline{x},\tilde{\lambda}_j^{(k)}) \prec \tilde{T}_j(x,\tilde{\lambda}_j^{(k)})$ for at least one $j = \{1,...,p\}$ with $x \in \tilde{\Omega}$. This means $\tilde{P}_i(\overline{x}) - \tilde{\lambda}_i^{(k)}\tilde{Q}_i(\overline{x}) \preceq \tilde{P}_i(x) - \tilde{\lambda}_i^{(k)}\tilde{Q}_i(x)$, $\forall i$ and $\tilde{P}_j(\overline{x}) - \tilde{\lambda}_i^{(k)}\tilde{Q}_j(\overline{x}) \prec \tilde{P}_j(x) - \tilde{\lambda}_j^{(k)}\tilde{Q}_j(x)$ for at least one j. By using the core of fuzzy number, we have:

$$\begin{cases} \tilde{P}_i(\overline{x}) - \tilde{\lambda}_i^{(k)}\tilde{Q}_i(\overline{x}) & \preceq & \tilde{P}_i(x) - \tilde{\lambda}_i^{(k)}\tilde{Q}_i(x) \\ \tilde{P}_j(\overline{x}) - \tilde{\lambda}_i^{(k)}\tilde{Q}_j(\overline{x}) & \prec & \tilde{P}_j(x) - \tilde{\lambda}_j^{(k)}\tilde{Q}_j(x) \end{cases} \Rightarrow$$

$$\begin{cases} P_i(\overline{x}) - \lambda_i^{(k)}Q_i(\overline{x}) \leq P_i(x) - \lambda_i^{(k)}Q_i(x), \\ P_j(\overline{x}) - \lambda_i^{(k)}Q_j(\overline{x}) < P_j(x) - \lambda_j^{(k)}Q_j(x). \end{cases}$$

So $T_i(\overline{x},\lambda_i^{(k)}) \leq T_i(x,\lambda_i^{(k)})$, $\forall i$ and $T_j(\overline{x},\lambda_j^{(k)}) < T_j(x,\lambda_j^{(k)})$ for at least one $j = \{1,...,p\}$, with $x \in \Omega$. Consequently $\overline{x}$ is a Pareto optimal of problem(3.8).

Conversely, let $\overline{x}$ be a solution of problem (3.8) i.e., $T_i(\overline{x},\lambda_i^{(k)}) \leq T_i(x,\lambda_i^{(k)})$, $\forall i$ and $T_j(\overline{x},\lambda_j^{(k)}) < T_j(x,\lambda_j^{(k)})$ for at least one $j = \{1,...,p\}$, with $x \in \Omega$. By definition of core, then, we have $\tilde{T}_i(\overline{x},\tilde{\lambda}_i^{(k)}) \preceq \tilde{T}_i(x,\tilde{\lambda}_i^{(k)})$, $\forall i$ and $\tilde{T}_j(\overline{x},\tilde{\lambda}_j^{(k)}) \prec \tilde{T}_j(x,\tilde{\lambda}_j^{(k)})$ for at least one $j = \{1,...,p\}$, with $x \in \tilde{\Omega}$. Therefore $\overline{x}$ is a Pareto solution of problem (3.7). □

Step 3: Aggregation.

t consists of converting the multiple objective functions into a single objective function using an aggregation function. Here, ϵ-constraint approach that is used.

Using the ϵ-constraint approach, the problem (3.8) is transformed into the following single objective optimization problem, as follows:

$$\begin{cases} \min T_s^{\epsilon}(x,\lambda^{(k)}), \\ x \in \Omega, \\ T_i(x,\lambda_i^{(k)}) \leq \epsilon_i, \quad i = 1,2,...,p, \\ \epsilon_i \in [\epsilon_i^L,\epsilon_i^U], \qquad i = 1,2,...p, \end{cases} \tag{3.9}$$

where $\epsilon_i^L = \min\{T_i(\overline{X}_i)|i = 1,...,p\}$ and $\epsilon_i^U = \max\{T_i(\overline{X}_i)|i = 1,...,p\}$ with $\overline{X}_i(i = 1,...,p)$ are the individual optimal solution of the objectives $T_i(x,\lambda^{(k)})$ obtained by minimizing them individually over the set of constraints Ω.

Theorem 3.4. *For a value of ϵ fixed in $[\epsilon_i^L,\epsilon_i^U]$ if $\overline{x}$ is an optimal solution of (3.9), then $\overline{x}$ is the Pareto optimal solution of (3.8). Conversely, if $\overline{x}$ is a Pareto optimal solution of (3.8) then there exists a value of ϵ in $[\epsilon_i^L,\epsilon_i^U]$ such that $\overline{x}$ is an optimal solution of (3.9).*

Proof. By posing $\epsilon_i = T_i(\overline{x},\lambda^{(k)})$, $i \neq s$, we have $\overline{x}$ is then an optimal solution of (3.8) if it is also a solution of (3.9). □

Step 4: Resolution.

It consists in the determination of the optimal solutions of the problem (3.9) by using Dantzig simplex method. These solutions are generated for each value of ϵ that meets the following two situations:

⋄ $\epsilon_i \in [-\sigma_i, \sigma_i]$ for satisfying the termination conditions $|T_i(x, \lambda_i^{(k)})| \leq \sigma_i$ as $T_i(x, \lambda_i^{(k)}) \approx \epsilon_i$, $i = 1, ..., s-1, s+1, ..., p$.

⋄ $[-\sigma_i, \sigma_i]$ and $[\epsilon_i^L, \epsilon_i^U]$ are disjoint, we choose $\epsilon_i \in [\epsilon_i^L, \epsilon_i^U]$ only.

where the $\sigma_i (i = 1, ..., p)$ are positive values and are proposed by the decision-maker. Indeed, these parameters are also considered as tolerance values acceptable for the objectives $\tilde{f}_i$ and their values are chosen next to zero. The objective with the smallest termination condition will be considered the most important objective function for the ϵ-constraint approach.

Srep 5 :: Test of termination conditions.

It is the test that allows you to choose the Pareto optimal solution that meets the termination conditions that are defined by the following equation:

$$|T_i(x, \lambda_i^{(k)})| \leq \sigma_i, i = 1, ..., p \tag{3.10}$$

In this process, there are two possible cases: when the Pareto optimal solutions satisfy the termination conditions, and when the termination is not all satisfied. In the first case, the solution is deemed as the preferred optimal solution. In the second case, we select the optimal solution with the smallest value of $\{|T_i(x^{(k)}, \lambda_i^{(k)})| - \sigma_i\}$ to initialize $\tilde{\lambda}_i^k$ in order to go back to the first step.

We summaries our iterative approach as the following algorithm:

Algorithm 3.1

Input: Problem (2.7); $\omega_i > 0$; $\sigma_i > 0$

Initialisation: Compute

$\overline{X}_i = \min_{x \in \tilde{\Omega}} f_i, i = 1, ..., p$

$\overline{X}^{(0)} = \sum_{i=1}^{p} \omega_i \overline{X}_i$

$\tilde{\lambda}_i^{(1)} = \tilde{f}_1(\overline{X}_i^{0}), i = 1, ..., p$

Whill: no solution

Compute at the k^{th} iteration

$\tilde{\lambda}_i^{(k)} = \tilde{f}_1(\overline{X}_i^{k-1}), i = 1, ..., p$

set $\tilde{T}_i(x^{(k)}, \lambda) = \tilde{P}_i(x^{(k)}) - \tilde{\lambda}^{(k)} \tilde{Q}_i(x^{(k)}, i = 1, ..., p$

: Use core value to obtain

$\min_{x \in \Omega} T_i(x^{(k)}, \lambda^{(k)}), i = 1, ..., p,$

: Find $\sigma_s = \min\{\sigma_1, \sigma_2, ..., \sigma_p\}$

choose T_s as a priority function.

formulate the problem as follows:

$$\begin{cases} \min T_s^{\epsilon}(x^{(k)}, \lambda^{(k)}), \\ x \in \Omega, \\ T_i(x^{(k)}, \lambda_i^{(k)}) \leq \epsilon_i, \quad i = 1, 2, ..., p, \\ \epsilon_i \in [\epsilon_i^L, \epsilon_i^U], \qquad i = 1, 2, ...p, \end{cases}$$

: For some ϵ value solve with Dantzig simplex

method for each obtained solution x^k

verify if $|T_s(x^{(k)}, \lambda_i^{(k)})| \leq \sigma_s$ save it

else

compute $\sum_{i=1}^{p} \left\{ |T_i(x^{(k)}, \lambda_i^{(k)})| - \sigma_i \right\}$

solve $x_m^{(k)} = arg \min \sum_{i=1}^{p} \left\{ |T_i(x^{(k)}, \lambda_i^{(k)})| - \sigma_i \right\}$

put $\overline{X}^{(k)} = x_m^{(k)}$

End:

3.2. **Didactic experiment.** We have dealt with the didactic example taken from P. Durga et al.'s works [1] Let us consider the following fuzzy multiobjective linear fractional optimization problem where:

$$\begin{cases} \max\ \tilde{z}_1(x) = \dfrac{(5.6,5.7,6.2)x_1 \oplus (4.7,4.9,5.5)x_2}{(1.7,1.9,2.5)x_1 \oplus (6.6,6.8,7.2)} and \\ \max\ \tilde{z}_2(x) = \dfrac{(1.6,1.7,2.2)x_1 \oplus (2.7,2.8,3.1)x_2}{(0.7,0.9,1.5)x_1 \oplus (0.8,0.9,1.4)x_2 \oplus (6.7,6.9,7.5)} \\ st \\ (0.8,0.9,1.4)x_1 \oplus (1.8,1.9,1.4)x_2 \leq (1.8,1.9,1.4) \\ (2.7,2.9,3.5)x_1 \oplus (1.7,1.8,2.1)x_2 \leq (5.7,5.8,6.1) \\ x_1, x_2 \geq 0 \end{cases}$$

with
$\tilde{6} = (5.6,5.7,6.2)$, $\tilde{5} = (4.7,4.9,5.5)$, $\tilde{2} = (1.7,1.9,2.5)$, $\tilde{7} = (6.6,6.8,7.2)$,
$\tilde{2} = (1.6,1.7,2.2)$, $\tilde{3} = (2.7,2.8,3.1)$, $\tilde{1} = (0.7,0.9,1.5)$, $\tilde{1} = (0.7,0.8,1.1)$,
$\tilde{7} = (6.7,6.9,7.5)$, $\tilde{1} = (0.8,0.9,1.4)$, $\tilde{2} = (1.8,1.9,1.4)$, $\tilde{3} = (2.7,2.8,3.1)$,
$\tilde{3} = (2.7,2.9,3.5)$, $\tilde{2} = (1.7,1.8,2.1)$, $\tilde{6} = (5.7,5.8,6.1)$. We first formulate the problem as the following minimization problem:

$$\begin{cases} \min\ \tilde{f_1}(x) = \dfrac{\ominus\tilde{6}x_1 \ominus \tilde{5}x_2}{\tilde{2}x_1 \oplus \tilde{7}}, \\ \min\ \tilde{f_2}(x) = \dfrac{\ominus\tilde{2}x_1 \ominus \tilde{3}x_2}{\tilde{1}x_1 \oplus \tilde{1}x_2 \oplus \tilde{7}} \\ st \\ \tilde{1}x_1 \oplus \tilde{2}x_2 \leq \tilde{3} \\ \tilde{3}x_1 \oplus \tilde{2}x_2 \leq \tilde{6} \\ x_1, x_2 \geq 0 \end{cases} \tag{3.11}$$

Let's assign equal weights $\omega_1 = \omega_2 = \frac{1}{2}$ and the termination constants for the two objectives are defined as $(\sigma_1, \sigma_2) = (0.03, 0.05)$.

Step 1: By defuzzifying and applying, the Charnes and Cooper variable transformation technique, we obtain $\overline{X}_1 = (1.53, 0.74)$ and $\overline{X}_2 = (1.53, 0.74)$ the individual optimal solutions of the objectives $\tilde{f_1}$ and $\tilde{f_1}$ respectively. The starting point of the approach iterative is obtained as
$\overline{X}^{(0)} = \omega_1\overline{X}_1 + \omega_2\overline{X}_2 = (1.53, 0.74)$.
For $k = 1$, $x^{(1)} = (1.53, 0.74)$ and the initial vectors of parameters are
$\tilde{\lambda}_1^{(1)} = (-1.47, -1.27, -1.09)$
$\tilde{\lambda}_2^{(1)} = (-0.68, -0.52, -0.41)$.
The fractional objectives can be parametrically linearized as:
$\tilde{T}_1(x^{(1)}, \tilde{\lambda}_1^{(1)}) = -(\tilde{6}x_1 + \tilde{5}x_2) - \tilde{\lambda}_1^{(1)}(\tilde{2}x_1 + \tilde{7})$
$\tilde{T}_2(x^{(1)}, \tilde{\lambda}_2^{(1)}) = -(\tilde{2}x_1 + \tilde{3}x_2) - \tilde{\lambda}_2^{(1)}(\tilde{1}x_1 + \tilde{1}x_2 + \tilde{7})$

Step 2: Defuzzifying this objectives, we obtain
$T_1(x^{(1)}, \lambda_1^{(1)}) = -3.287x_1 - 4.9x_2 + 8.636$
$T_2(x^{(1)}, \lambda_2^{(1)}) = -1.232x_1 - 2.384x_2 + 3.588.$

So, we can write the deterministic multi objective non fractional problem at 1 th iteration as:

$$\begin{cases} \min T_1(x^{(1)}, \lambda_1^{(1)}) = -3.287x_1 - 4.9x_2 + 8.636 \\ \min T_2(x^{(1)}, \lambda_2^{(1)}) = -1.232x_1 - 2.384x_2 + 3.588 \\ st \\ 0.9x_1 + 1.9x_2 \leq 2.8 \\ 2.9x_1 + 1.8x_2 \leq 5.8 \\ x_1, x_2 \geq 0. \end{cases}$$

Let Ω be the admissible domain of this problem.
By solving $\min_{x \in \Omega} T_2(x^{(1)}, \lambda_2^{(1)})$, we have $[\epsilon_2^L, \epsilon_2^U] = [-0.08319794, 3.588]$. Also we have $[-\sigma_2, \sigma_2] \subseteq [\epsilon_2^L, \epsilon_2^U]$. Then, for $\epsilon_2 \in [-0.05, 0.05]$ we have
$|T_2(x^{(1)}, \lambda_2^{(1)})| \leq \sigma_2$.

As $\sigma_1 < \sigma_2$, using the ϵ-constraint method, the problem is formulated as:

$$\begin{cases} \min T_1(x^{(1)}, \lambda_1^{(1)}) = -3.287x_1 - 4.9x_2 + 8.636 \\ st \\ 0.9x_1 + 1.9x_2 \leq 2.8 \\ 2.9x_1 + 1.8x_2 \leq 5.8 \\ -1.232x_1 - 2.384x_2 + 3.588 \leq \epsilon_2 \\ \epsilon_2 \in [\epsilon_2^L, \epsilon_2^U] \\ x_1, x_2 \geq 0. \end{cases}$$

Step 4: This final formulation is a linear programming problem. With the Dantzig's Simplex method, we find the best Pareto solution for each value of ϵ_2.

Step 5: According to the $(\sigma_1, \sigma_2) = (0.03, 0.05)$ we can notice that it is S_1, S_2 and S_3 that verify the termination conditions.

The representation of all obtained solutions in the same figure give this allows:

3.3. **Discussion.** Through the Table 3.2, we can see that the Pareto optimal solutions S_1, S_2 and S_3 satisfy the termination conditions $|T_1(x^{(1)}, \lambda_1^{(1)})| \leq \sigma_1$. As before, we had $|T_2(x^{(1)}, \lambda_2^{(1)})| \leq \sigma_2$, then we choose its solutions as the best preferred optimal solution. In one iteration, we have obtained the best preferred optimal solution as follows : $\chi_E = \{S_1, S_2, S_3\}$.

The Figure 1 shows a representation of each obtained solution that is on the analytic Pareto front.

Table 1. Pareto optimal solutions of step-1

	ϵ_2	x_1	x_2	$T_1(x^{(1)}, \lambda_1^{(1)})$
S_1	-0.05	1.55000000	0.72500000	-0.01135000
S_2	-0.04	1.55383305	0.71882453	0.00631056
S_3	-0.03	1.55766610	0.71264906	0.02397112
S_4	-0.02	1.56149915	0.70647359	0.77647359
S_5	-0.01	1.56533220	0.70029813	0.05929225
S_6	0.00	1.56916525	0.69412266	0.07695281
S_7	0.01	1.57299830	0.68794719	0.09461337
S_8	0.02	1.57683135	0.68177172	0.11227394
S_9	0.03	1.58066440	0.67559625	0.12993450
S_10	0.04	1.58449744	0.66942078	0.14755906
S_{11}	0.05	1.58833049	0.66324532	0.16525562

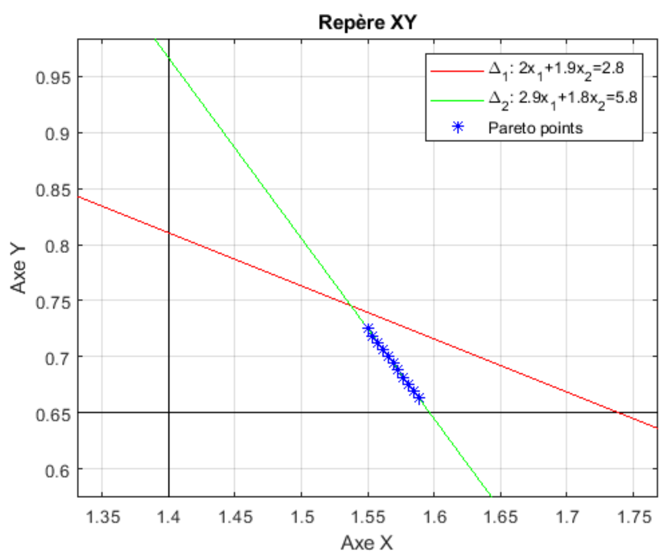

Figure 1. Pareto front

Note that the proposed approach starts with an initial solution and converges to the best preferred optimal solution or best preferred optimal solutions. At each iteration, the admissible domain changes with the change of ϵ_i. Then we have a set of Pareto optimal solutions that are generated during iterations. The possibility of satisfying the termination conditions in Step $(k+1)$ is greater than that of due to the proceeding at step k because of the selection procedure followed to determine the compromise solution. Indeed, in the event that we fail to attain the optimal solution in the various iterations, it is imperative to reshape the termination conditions of the objectives.

4. Conclusion

We have proposed a new way to solve fuzzy multiobjective linear fractional programming problems. In this approach, we transform fuzzy fractional objectives into fuzzy non-fractional form using a vector of fuzzy parameters. Then, the problem has been defuzzified by using the core value function. After that, the ϵ-constraint has been used to get a linear single-objective programming problem, and finally, the Dantzig simplex method has been used to find the solutions. According to the preference of the decision-maker, a test with termination conditions is conducted in order to choose the best compromise. The solution obtained for a didactic problem has enabled us to demonstrate that our approach is a suitable choice for resolving fuzzy multiobjective linear fractional programming problems.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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