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Improving the Performance of a Series-Parallel System Based on Gamma Distribution

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Abstract. The performance of a series-parallel system is improved by using the reliability equivalence factors technique. The lifetimes of the components are assumed to be gamma distributed. The system reliability is improved by using three different methods: (i) Reduction method, (ii) Hot duplication method, (iii) Cold duplication method. The reliability function and mean time to failure for each method are derived. Finally, the numerical application is introduced.

1. Introduction

The concept of reliability equivalence factors (REF) is addressed to improve the system reliability, [14]. Sarhan [17,18] improved the system reliability by:

(1) Reducing the failure rates by a factor ξ, $0 < \xi < 1$, is called reduction method (RM).
(2) Duplicating the system's components by hot redundant identical standby components. This method is named hot duplication method (HDM).
(3) The system's components are connected with an identical component via a perfect switch, so it is called the cold duplication method (CDM).
(4) Connecting some system's components with standby redundant component via an imperfect switch. It is called the imperfect duplication method (IDM).

For more details, see [2,4–13,15–23].

The series-parallel system is one of the important systems in reliability theory and has many applications in engineering sciences. This system has many special cases such as: serial, parallel and radar system.

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Therefore, studying and improving the performance of this system includes improving all these systems at the same time. The series-parallel system has been studied for several lifetime distributions, such as (i) exponential, (ii) linear exponential and (iii) modified Weibull distribution, see [1,3,6,13,19,21,22,24].

The gamma distribution (GD) with parameters β, ν, has the following probability density function.

$$\psi(t) = \frac{\nu^{\beta}}{\Gamma(\beta)} t^{\beta-1} e^{-\nu t}, \ t \geq 0, \ \beta, \nu > 0. \tag{1.1}$$

The parameters β, ν are called a shape and scale parameter, respectively. The GD has been employed in engineering to study the system reliability.

The GD has some special distributions, for some values of β and ν, as follows.

(1) The exponential distribution with constant failure rate ν can be obtained if $\beta = 1$.
(2) The chi-square (χ^2) distribution can be derived if $\beta = n/2$ and $\nu = 1/2$ (n is an integer).
(3) The GD is called an Erlang distribution, when β is an integer.

Erlang distribution is used in queuing theory to model waiting times. χ^2 distribution is used in statistical inference.

Let $h(t)$ be the hazard (failure) rate of GD, it is given by, [24],

$$h(t) = \frac{1}{\int_0^{\infty} \left(1 + \frac{s}{t}\right)^{\beta-1} e^{-\nu s} ds}. \tag{1.2}$$

The function $h(t)$ has different shapes based on β, $h(t)$ is: (i) increasing for $\beta > 1$, (ii) constant for $\beta = 1$, (iii) decreasing when $\beta < 1$, see Figure 1.

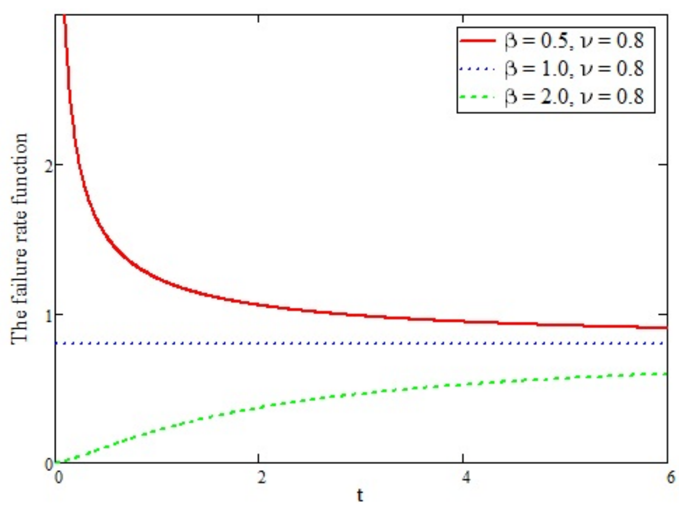


Figure 1. The $h(t)$ for some values of β.

The rest of the paper can be organized as follows.

A brief description of the original system is introduced in Section 2. The improved systems are discussed in Section 3. The α-fractiles are obtained in Section 4. In Section 5, the reliability equivalence factors are derived. A numerical application is discussed in Section 6. The conclusion of the paper is presented in Section 7.

2. The Original System

A series-parallel system has m subsystems connected in series. Each of them contains n_i elements connected in parallel, $i = 1, 2, \cdots, m$, see Figure 2.

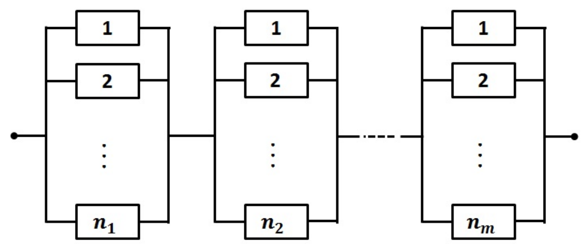


Figure 2. The original system diagram.

Form Figure 2, if $m = 1$, then we will get the parallel system, while if $n_i = 1$, then the series system will be obtained, but if $m = 2$ and $n_i = i, i = 1, 2$, we will have the radar system.
Consider the lifetime of the system components is independent gamma distribution. The reliability function (RF) for a component $j(j = 1, \cdots, n_i)$ is given by

$$\mathcal{R}_{ij}(t) = P[T_{ij} > t] = \int_{\nu t}^{\infty} \frac{1}{\Gamma(\beta)} u^{\beta-1} e^{-u} du = 1 - \Psi(\nu t, \beta),\ t \geq 0, \tag{2.1}$$

where

$$\Psi(\nu t, \beta) = \int_0^{\nu t} \frac{1}{\Gamma(\beta)} u^{\beta-1} e^{-u} du,$$

see [23].
The RF of the subsystem i, is given as,

$$\mathcal{R}_i(t) = 1 - \Psi(\nu t, \beta)^{n_i}. \tag{2.2}$$

Therefore, the RF of the original system is derived as

$$\mathcal{R}_s(t) = \prod_{i=1}^{m} \left[1 - \Psi(\nu t, \beta)^{n_i}\right]. \tag{2.3}$$

The mean time to failure (MTTF) is

$$\mathcal{M} = \int_0^{\infty} \mathcal{R}_s(t) dt = \int_0^{\infty} \prod_{i=1}^{m} \left[1 - \Psi(\nu t, \beta)^{n_i}\right] dt. \tag{2.4}$$

Some numerical techniques can be used to calculate MTTF numerically, from Equation (2.4), for given values of β, ν, n_i and m.

3. The Improved Systems

Three different methods will be applied to improve the performance of the original system.

3.1. **The RM.** The failure rate, $h(t)$, of the components of set $\mathcal{A}$ are reduced to $r(t)h(t)$, $0 < r(t) < 1$, where $|\mathcal{A}| = \ell$, $0 \le \ell \le M$, and $M = \sum_{i=1}^{m} n_i$. We shall reduce $h(t)$ by reducing the scale parameter only by the factor ξ.

Suppose $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_m$, where $\mathcal{A}_i$ denotes the set of components from subsystem i whose failure rates are reduced and $|\mathcal{A}_i| = \ell_i$ and $\ell = \sum_{i=1}^{m} \ell_i$. We denote such a set by $\mathcal{A}_{|\mathcal{A}|}^{(|\mathcal{A}_1|,|\mathcal{A}_2|,\cdots,|\mathcal{A}_m|)}$.
After reducing the failure rate of component j, it has the following RF, $\mathcal{R}_{ij,\xi}(t)$,

$$\mathcal{R}_{ij,\xi}(t) = \int_{\xi\nu t}^{\infty} \frac{u^{\beta-1}}{\Gamma(\beta)} e^{-u} du = 1 - \Psi(\xi\nu t, \beta). \tag{3.1}$$

The RF, $\mathcal{R}_{\mathcal{A}_i,\xi}(t)$, of the subsystem i after reducing the failure rates of $\mathcal{A}_i$ is given by

$$\begin{aligned} \mathcal{R}_{\mathcal{A}_i,\xi}(t) &= 1 - \left[\int_0^{\xi\nu t} \frac{u^{\beta-1}}{\Gamma(\beta)} e^{-u} du\right]^{\ell_i} \left[\int_0^{\nu t} \frac{u^{\beta-1}}{\Gamma(\beta)} e^{-u} du\right]^{n_i-\ell_i} \\ &= 1 - \Psi(\xi\nu t, \beta)^{\ell_i} \Psi(\nu t, \beta)^{n_i-\ell_i}. \end{aligned} \tag{3.2}$$

The RF of the improved system by RM, $R_{\mathcal{A},\xi}(t)$, is

$$\mathcal{R}_{\mathcal{A},\xi}(t) = \prod_{i=1}^{m} \mathcal{R}_{\mathcal{A}_i,\xi}(t) = \prod_{i=1}^{m} \left[1 - \Psi(\xi\nu t, \beta)^{\ell_i} \Psi(\nu t, \beta)^{n_i-\ell_i}\right]. \tag{3.3}$$

The MTTF of the reduction system, is calculated by

$$\mathcal{M}_{\mathcal{A},\xi} = \int_0^{\infty} \prod_{i=1}^{m} \left[1 - \Psi(\xi\nu t, \beta)^{\ell_i} \Psi(\nu t, \beta)^{n_i-\ell_i}\right] dt. \tag{3.4}$$

3.2. **The HDM.** Suppose the components are in a set $\mathcal{B}$, $|\mathcal{B}| = k$ and $0 \le k \le M$ will be improved according to HDM. Each component in $\mathcal{B}$ is duplicated by a hot standby component. The k can be distributed such that k_i components from subsystem i, where $k = \sum_{i=1}^{m} k_i$ and $0 \le k_i \le n_i$.

Let $\mathcal{B}_{|\mathcal{B}|}^{(|\mathcal{B}_1|,|\mathcal{B}_2|,\cdots,|\mathcal{B}_m|)}$ denote the improved set, $\mathcal{B} = \cup_{i=1}^{m} \mathcal{B}_i$. Where $|\mathcal{B}_i| = k_i$ components from subsystem i.

The RF of the improved subsystem, $\mathcal{R}_{\mathcal{B}_i}^{H}(t)$ is

$$\mathcal{R}_{\mathcal{B}_i}^{H}(t) = 1 - \left[\int_0^{\nu t} \frac{u^{\beta-1}}{\Gamma(\beta)} e^{-u} du\right]^{n_i+k_i} = 1 - \Psi(\nu t, \beta)^{n_i+k_i}, \tag{3.5}$$

and the RF of the improved system according to HDM, $\mathcal{R}_{\mathcal{B}}^{H}(t)$ is

$$\mathcal{R}_{\mathcal{B}}^{H}(t) = \prod_{i=1}^{m} R_{\mathcal{B}_i}^{H}(t) = \prod_{i=1}^{m} \left[1 - \Psi(\nu t, \beta)^{n_i+k_i}\right]. \tag{3.6}$$

The MTTF of the improved system by HDM, from Equation (3.6), $\mathcal{M}_{\mathcal{B}}^{H}$, is

$$\mathcal{M}_{\mathcal{B}}^{H} = \int_0^{\infty} \prod_{i=1}^{m} \left[1 - \Psi(\nu t, \beta)^{n_i+k_i}\right] dt. \tag{3.7}$$

3.3. **The CDM.** In this method, the set $\mathcal{B}$ of system components are duplicated each one with an identical component by a perfect switch, $|\mathcal{B}| = r$, $0 \le r \le M$. The set $\mathcal{B}$ can be expressed as $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_m$, $\mathcal{B}_i$ consists r_i components from the subsystem i, such that $r = \sum_{i=1}^{m} r_i$. The set $\mathcal{B}$ can be denoted by $\mathcal{B}_{|\mathcal{B}|}^{(|\mathcal{B}_1|,|\mathcal{B}_2|,\cdots,|\mathcal{B}_m|)}$.
The RF of the improved system by CDM, $\mathcal{R}_{\mathcal{B}}^{C}(t)$, is

$$\mathcal{R}_{\mathcal{B}}^{C}(t) = \prod_{i=1}^{m} \mathcal{R}_{\mathcal{B}_i}^{C}(t), \tag{3.8}$$

where $\mathcal{R}_{\mathcal{B}_i}^{C}(t)$ is the RF of subsystem i after improving the set $\mathcal{B}_i$ of components. $\mathcal{R}_{\mathcal{B}_i}^{C}(t)$ is given as

$$\begin{aligned} \mathcal{R}_{\mathcal{B}_i}^{C}(t) &= 1 - \left[\int_0^{\nu t} \frac{u^{2\beta-1}}{\Gamma(2\beta)} e^{-u} du\right]^{r_i} \left[\int_0^{\nu t} \frac{u^{\beta-1}}{\Gamma(\beta)} e^{-u} du\right]^{n_i - r_i} \\ &= 1 - \Psi(\nu t, 2\beta)^{r_i} \Psi(\nu t, \beta)^{n_i - r_i}. \end{aligned} \tag{3.9}$$

Using (3.8) and (3.9), then

$$\mathcal{R}_{\mathcal{B}}^{C}(t) = \prod_{i=1}^{m} \left[1 - \Psi(\nu t, 2\beta)^{r_i} \Psi(\nu t, \beta)^{n_i - r_i}\right]. \tag{3.10}$$

The MTTF, $\mathcal{M}_{\mathcal{B}}^{C}$, is calculated by

$$\mathcal{M}_{\mathcal{B}}^{C} = \int_0^{\infty} \prod_{i=1}^{m} \left[1 - \Psi(\nu t, 2\beta)^{r_i} \Psi(\nu t, \beta)^{n_i - r_i}\right] dt. \tag{3.11}$$

4. The α-Fractiles

Let $\mathcal{L}(\beta, \alpha)$, $\mathcal{L}_{\mathcal{B}}^{D}(\beta, \alpha)$, be the α-fractiles of the original and duplicated systems. Which can be calculated by solving the following equations, respectively.

$$\mathcal{R}\left(\frac{\mathcal{L}(\beta, \alpha)}{\nu}\right) = \alpha, \qquad \mathcal{R}_{\mathcal{B}}^{D}\left(\frac{\mathcal{L}(\beta, \alpha)}{\nu}\right) = \alpha, \ D = H, C. \tag{4.1}$$

Substituting Equation (2.3) into (4.1), the $\mathcal{L} = \mathcal{L}(\beta, \alpha)$ satisfies te following equation.

$$\sum_{i=1}^{m} \ln\left[1 - \Psi(\mathcal{L}, \beta)^{n_i}\right] - \ln(\alpha) = 0. \tag{4.2}$$

From Equations (3.6) and (4.1), $\mathcal{L} = \mathcal{L}_{\mathcal{B}}^{H}(\beta, \alpha)$ satisfies the equation.

$$\sum_{i=1}^{m} \ln\left[1 - \Psi(\mathcal{L}, \beta)^{n_i + k_i}\right] - \ln(\alpha) = 0. \tag{4.3}$$

For $D = C$, and from Equations (3.10) and (4.1), $\mathcal{L} = \mathcal{L}_{\mathcal{B}}^{C}(\beta, \alpha)$ is a solution of

$$\sum_{i=1}^{m} \ln\left[1 - \Psi(\mathcal{L}, 2\beta)^{r_i} \Psi(\mathcal{L}, \beta)^{n_i - r_i}\right] - \ln(\alpha) = 0. \tag{4.4}$$

Equations (4.2)-(4.4) must be solved numerically, to obtain $\mathcal{L}$.

5. The Reliability Equivalence Factors

Since the failure rate of GD(β,ν) is non-constant, the REFs of GD will be a function of time t.

Definition [17]: The REF is defined as the factor that must be used to reduce the failure rates of the set, $\mathcal{A}$, of system components in order to obtain the reliability of the system, which is improved by improving the set, $\mathcal{B}$, of system components by the duplication method.
The failure rate, $h(t)$, of GD will be reduced by $r(t)$, only by reducing the scale parameter from ν to $\xi\nu$ only.

$$r(t)h(t) = \frac{1}{\int_0^{\infty}\left(1+\frac{s}{t}\right)^{\beta-1} e^{-\xi\nu s}ds}. \tag{5.1}$$

We will discuss how ξ can be calculated, and $r(t)$ can be obtained by taking ξ in Equation (5.1).

The factor $\xi = \xi^{D}_{\mathcal{A},\mathcal{B}}(\alpha)$ is the solution of

$$\mathcal{R}_{\mathcal{A},\xi}(t) = \alpha, \qquad \mathcal{R}^{D}_{\mathcal{B}}(t) = \alpha, \quad \alpha \in (0,1),\ D = H, C. \tag{5.2}$$

Substituting from Equations (3.3) and (3.6) into (5.2), the factor $\xi = \xi^{H}_{\mathcal{A},\mathcal{B}}(\alpha)$ satisfies the following system.

$$\left.\begin{aligned} \sum_{i=1}^{m} \ln\left[1-\Psi(\xi\nu t,\beta)^{\ell_i}\Psi(\nu t,\beta)^{n_i-\ell_i}\right] - \ln(\alpha) = 0,\\ \sum_{i=1}^{m} \ln\left[1-\psi(\nu t,\beta)^{n_i+k_i}\right] - \ln(\alpha) = 0 \end{aligned}\right\}. \tag{5.3}$$

The Hot REF:

$$r^{H}_{\mathcal{A},\mathcal{B}}(\alpha,t) = \frac{\int_0^{\infty}\left(1+\frac{s}{t}\right)^{\beta-1} e^{-\nu s}ds}{\int_0^{\infty}\left(1+\frac{s}{t}\right)^{\beta-1} e^{-\xi\nu s}ds}, \tag{5.4}$$

where $\xi = \xi^{H}_{\mathcal{A},\mathcal{B}}(\alpha)$.
Using Equation (5.2) with Equations (3.3) and (3.10), $\xi = \xi^{C}_{\mathcal{A},\mathcal{B}}(\alpha)$ satisfies the following equations.

$$\left.\begin{aligned} \sum_{i=1}^{m} \ln\left[1-\Psi(\xi\nu t,\beta)^{\ell_i}\Psi(\nu t,\beta)^{n_i-\ell_i}\right] - \ln(\alpha) = 0,\\ \sum_{i=1}^{m} \ln\left[1-\Psi(\nu t,2\beta)^{r_i}\Psi(\nu t,\beta)^{n_i-r_i}\right] - \ln(\alpha) = 0 \end{aligned}\right\}. \tag{5.5}$$

The Cold REF:

$$r^{C}_{\mathcal{A},\mathcal{B}}(\alpha,t) = \frac{\int_0^{\infty}\left(1+\frac{s}{t}\right)^{\beta-1} e^{-\nu s}ds}{\int_0^{\infty}\left(1+\frac{s}{t}\right)^{\beta-1} e^{-\xi\nu s}ds}, \tag{5.6}$$

where $\xi = \xi^{C}_{\mathcal{A},\mathcal{B}}(\alpha)$.
The systems (5.3) and (5.5) have no closed form solutions, so $\xi = \xi^{D}_{\mathcal{A},\mathcal{B}}(\alpha)$ can be obtained by using some numerical techniques.

6. Numerical Application

Under the following assumptions the REFs of a series-parallel system are calculated:

(1) There are two subsystems, $m = 2$.
(2) The lifetimes are assumed independent and identical with GD(β, ν).
(3) The number of components in the subsystems are $n_1 = 1$ and $n_2 = 2$ (Radar system), see Figure 3.
(4) The parameters $\nu = 0.7$ and $\beta = 1, 3, 5$ $(\beta \geq 1)$.

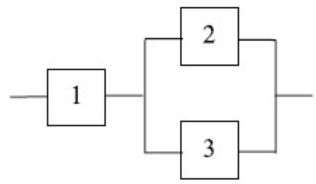


Figure 3. The radar system.

Table 1 views the values of $\mathcal{M}$ and $\mathcal{M}^D_{\mathcal{B}}$ for different $\mathcal{B}$, $D = H$ and C.

Table 1. The $\mathcal{M}$ and $\mathcal{M}^D_{\mathcal{B}}$, for different values of $|\mathcal{B}|$.

		$\mathcal{B}_1^{(1,0)}$	$\mathcal{B}_1^{(0,1)}$	$\mathcal{B}_2^{(1,1)}$	$\mathcal{B}_2^{(0,2)}$	$\mathcal{B}_3^{(1,2)}$
β	$\mathcal{M}$	$\mathcal{M}^H_{\mathcal{B}}$				
1	0.95238	1.30952	1.07143	1.50000	1.14286	1.61905
3	3.48839	4.26218	3.72022	4.62251	3.84873	4.83255
5	6.13475	7.19054	6.44083	7.66235	6.60657	7.93168
		$\mathcal{M}^C_{\mathcal{B}}$				
1	0.95238	1.50794	1.15079	1.95767	1.24339	2.19577
3	3.48839	5.052	4.03079	6.8354	4.1652	7.47507
5	6.13475	8.47595	6.94963	11.9289	7.07169	12.8969

Figures 4–6 compare the RF of the original and duplicated systems, when $|\mathcal{B}| = 1, 2$ and 3, respectively.

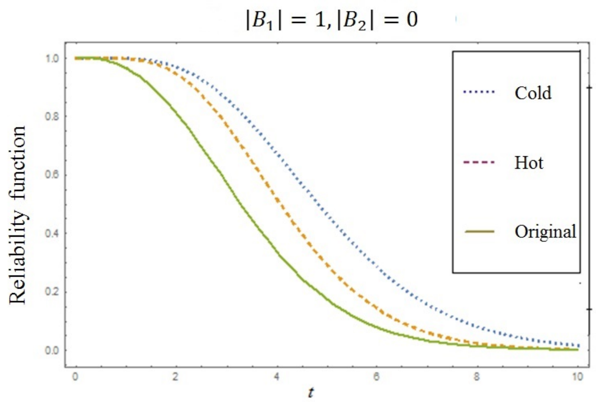
Figure 7 displays the RF of the original and duplicated systems for $|\mathcal{B}| = 1, 2, 3$ and different methods.

For the level $\alpha = 0.1, 0.2, \cdots, 0.9$ and $\beta = 3$, Mathematica Program System is used to calculate the α-fractiles and the REFs.

Table 2 introduces the values of α-fractiles, $\mathcal{L}(\beta, \alpha)$ and $\mathcal{L}^D_{\mathcal{B}}(\beta, \alpha)$, for $D = H$ and C.

From the results shown in Tables 1, 2 and Figures 5–7, we can conclude that:

(1) $\mathcal{R}(t) < \mathcal{R}^H_{\mathcal{B}}(t) < \mathcal{R}^C_{\mathcal{B}}(t)$, $\forall\, |\mathcal{B}| = 1, 2$ and 3.
(2) $\mathcal{M} < \mathcal{M}^H_{\mathcal{B}} < \mathcal{M}^C_{\mathcal{B}}$, for all $|\mathcal{B}| = 1, 2$ and 3.



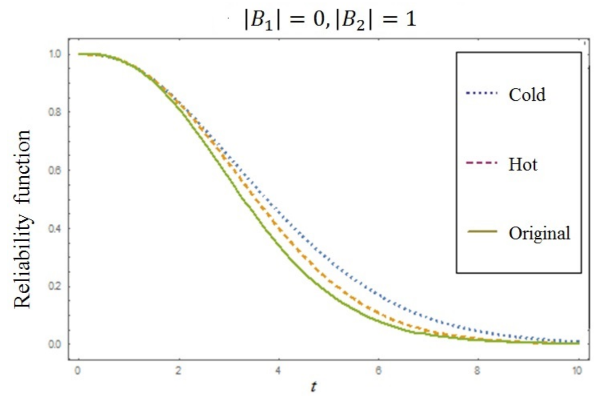
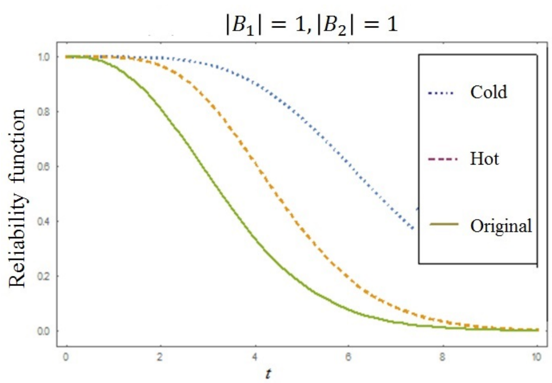


FIGURE 4. The RF, of the original and duplicated systems, when $|\mathcal{B}| = 1$.



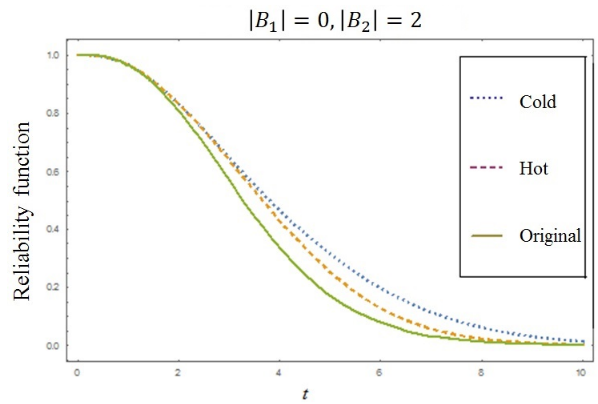


FIGURE 5. The $\mathcal{R}(t), \mathcal{R}^D_{\mathcal{B}}(t)$, for $|\mathcal{B}| = 2$.

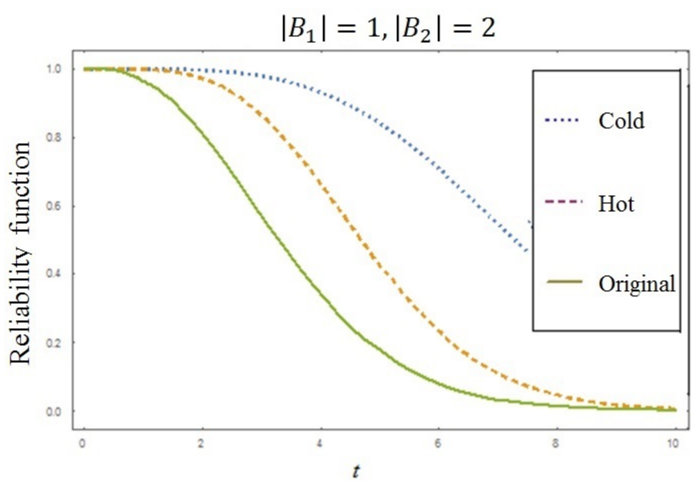


FIGURE 6. The $\mathcal{R}(t), \mathcal{R}^D_{\mathcal{B}}(t)$, for $|\mathcal{B}| = 3$.

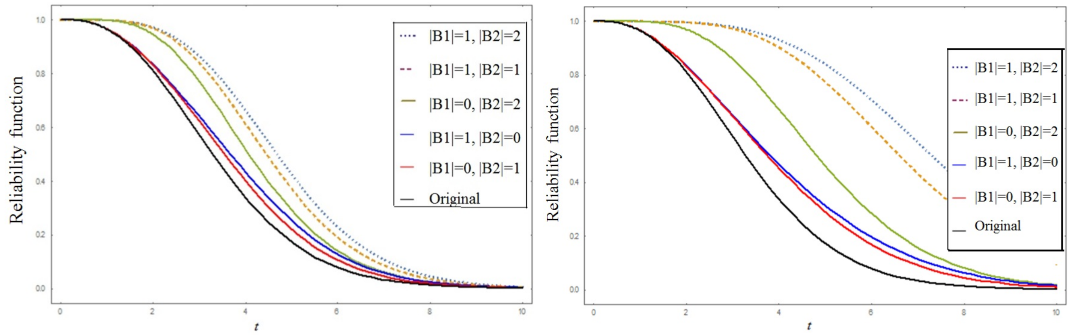


FIGURE 7. The $\mathcal{R}(t), \mathcal{R}^D_{\mathcal{B}}(t)$, for $|\mathcal{B}| = 1, 2, 3,\ D = H$ (left panel) and $D = C$ (right panel).

TABLE 2. The α-fractiles.

		$\mathcal{B}_1^{(1,0)}$		$\mathcal{B}_1^{(0,1)}$		$\mathcal{B}_1^{(1,1)}$		$\mathcal{B}_1^{(0,2)}$		$\mathcal{B}_1^{(1,2)}$	
α	$\mathcal{L}$	$\mathcal{L}^H$	$\mathcal{L}^C$	$\mathcal{L}^H$	$\mathcal{L}^C$	$\mathcal{L}^H$	$\mathcal{L}^C$	$\mathcal{L}^H$	$\mathcal{L}^C$	$\mathcal{L}^H$	$\mathcal{L}^C$
0.1	3.999	4.5043	5.3855	4.2576	4.7988	4.7815	6.9603	4.4313	5.0724	4.9698	7.5293
0.2	3.3579	3.8782	4.6286	3.6006	3.9945	4.1552	6.0919	3.7574	4.1818	4.3391	6.6477
0.3	2.9333	3.4644	4.1255	3.1578	3.4507	3.7391	5.5091	3.2955	3.5753	3.9169	6.0462
0.4	2.5944	3.1345	3.7231	2.7978	3.011	3.4053	5.0387	2.9141	3.0892	3.5757	5.5531
0.5	2.2965	2.8449	3.3695	2.4756	2.6229	3.1104	4.6214	2.5681	2.6681	3.2717	5.1083
0.6	2.0165	2.5729	3.0371	2.1669	2.2601	2.8312	4.2249	2.2338	2.2832	2.9809	4.6782
0.7	1.7363	2.3007	2.7048	1.8532	1.9035	2.5489	3.8232	1.8941	1.9133	2.6837	4.2337
0.8	1.4341	2.0059	2.3458	1.5117	1.5319	2.2395	3.3815	1.5301	1.5349	2.3534	3.7339
0.9	1.0631	1.6381	1.9009	1.0977	1.1016	1.8462	2.8181	1.1016	1.1021	1.9277	3.0807

(3) $\mathcal{L}(\beta,\alpha) < \mathcal{L}^H_{\mathcal{B}}(\beta,\alpha) < \mathcal{L}^C_{\mathcal{B}}(\beta,\alpha),\ \forall\ |\mathcal{B}|$.
(4) The MTTF and $\mathcal{L}^D_{\mathcal{B}}(\beta,\alpha)$ are increasing, when β increases.
(5) A better design is obtained by improving $\mathcal{B}_1^{(1,0)}$ than improve $\mathcal{B}_1^{(0,1)}$ according to the same method.
(6) Improving two components, $\mathcal{B}_2^{(1,1)}$ produces a better design than improving two components $\mathcal{B}_2^{(0,2)}$.
(7) The best design is obtained by improving all components, $\mathcal{B}_3^{(1,2)}$.
(8) Cold duplication method gives the best improvement than other methods.

Table 3 displays the values of the REFs for different $\mathcal{A}$ and $\mathcal{B}$.
From the results presented in Tables 2 and 3, at $\beta = 3$:

(1) The $\mathcal{L}(3, 0.1)$ is increased from $3.9990/\nu$ to $4.5043/\nu$ when the set $\mathcal{B}_1^{(1,0)}$ improved by HDM, see Table 2. We can get the same effect by reducing the failure rates of (i) $\mathcal{A}_1^{(1,0)}$ by $\xi^H = 0.7826$, (ii) $\mathcal{A}_1^{(0,1)}$ by $\xi^H = 0.6033$, (iii) $\mathcal{A}_2^{(1,1)}$ by $\xi^H = 0.8582$, (iv) $\mathcal{A}_2^{(0,2)}$ by $\xi^H = 0.7430$, (v) $\mathcal{A}_3^{(1,2)}$ by $\xi^H = 0.8878$, see Table 3.

Table 3. The REF, $\xi^{D}_{\mathcal{A},\mathcal{B}}(\alpha)$, $D = H$ and C

		$\mathcal{B}_1^{(1,0)}$		$\mathcal{B}_1^{(0,1)}$		$\mathcal{B}_1^{(1,1)}$		$\mathcal{B}_1^{(0,2)}$		$\mathcal{B}_1^{(1,2)}$	
α	$\mathcal{A}$	ξ^H	ξ^C	ξ^H	ξ^C	ξ^H	ξ^C	ξ^H	ξ^C	ξ^H	ξ^C
0.1	$\mathcal{A}_1^{(1,0)}$	0.7826	0.4607	0.8845	0.6993	0.6758	–	0.8120	0.6701	0.6068	–
	$\mathcal{A}_1^{(0,1)}$	0.6033	–	0.7693	0.4364	0.4457	–	0.6490	0.2858	0.3438	–
	$\mathcal{A}_2^{(1,1)}$	0.8582	0.6888	0.9221	0.7926	0.7962	0.5147	0.8762	0.7404	0.7591	0.4731
	$\mathcal{A}_2^{(0,2)}$	0.7430	0.0111	0.8653	0.5979	0.6065	0.0857	0.7788	0.4481	0.5080	0.2996
	$\mathcal{A}_3^{(1,2)}$	0.8878	0.7426	0.9393	–	0.8364	0.5745	0.9025	0.7884	0.8047	0.5311
0.2	$\mathcal{A}_1^{(1,0)}$	0.7464	0.4288	0.8763	0.6648	0.6256	–	0.8016	0.6144	0.5486	–
	$\mathcal{A}_1^{(0,1)}$	0.4899	–	0.7274	0.3980	0.2601	–	0.5879	0.2329	0.1319	–
	$\mathcal{A}_2^{(1,1)}$	0.8324	0.6689	0.9146	0.7802	0.7635	0.4867	0.8663	0.7374	0.7237	0.4424
	$\mathcal{A}_2^{(0,2)}$	0.6579	0.0086	0.8401	0.5754	0.4373	0.0489	0.7384	0.4075	0.5054	0.2065
	$\mathcal{A}_3^{(1,2)}$	0.8658	0.7255	0.9326	–	0.8081	0.5512	0.8937	0.7030	0.7739	0.5051
0.3	$\mathcal{A}_1^{(1,0)}$	0.7172	0.4047	0.8742	0.5240	0.5861	–	0.8019	0.6035	0.5032	–
	$\mathcal{A}_1^{(0,1)}$	0.3520	–	0.6923	0.3679	0.1912	–	0.5375	0.1945	–	–
	$\mathcal{A}_2^{(1,1)}$	0.8107	0.6532	0.9110	0.6147	0.7371	0.4655	0.8632	0.7294	0.6960	0.4199
	$\mathcal{A}_2^{(0,2)}$	0.5414	0.0031	0.8187	0.5565	0.1021	0.0049	0.7039	0.3752	0.2995	0.2058
	$\mathcal{A}_3^{(1,2)}$	0.8467	0.7110	0.9289	–	0.7845	0.5325	0.8901	0.0204	0.7489	0.4852
0.4	$\mathcal{A}_1^{(1,0)}$	0.6903	0.3835	0.8763	0.4564	0.5502	–	0.8098	0.5714	0.4624	–
	$\mathcal{A}_1^{(0,1)}$	–	–	0.6587	0.3406	–	–	0.4897	0.1626	–	–
	$\mathcal{A}_2^{(1,1)}$	0.7898	0.6387	0.9101	0.5304	0.7127	0.4467	0.8650	0.7043	0.6710	0.4005
	$\mathcal{A}_2^{(0,2)}$	0.5122	0.0016	0.7978	0.5382	0.0787	–	0.6702	0.3456	0.2047	0.1450
	$\mathcal{A}_3^{(1,2)}$	0.8277	0.6968	0.9273	–	0.7619	0.5149	0.8703	–	0.7255	0.4672
0.5	$\mathcal{A}_1^{(1,0)}$	0.6634	0.3631	0.8824	0.4202	0.5151	–	0.8251	0.5653	0.4228	–
	$\mathcal{A}_1^{(0,1)}$	–	–	0.6239	0.3137	–	–	0.4409	0.1344	–	–
	$\mathcal{A}_2^{(1,1)}$	0.7682	0.6239	–	0.5192	0.6882	0.4285	0.8717	0.6832	0.6468	0.3824
	$\mathcal{A}_2^{(0,2)}$	0.1037	–	0.7757	0.5190	0.0247	–	0.6344	0.3169	0.0275	0.0440
	$\mathcal{A}_3^{(1,2)}$	0.8072	0.6816	0.8766	–	0.7383	0.4969	0.7542	–	0.7019	0.4496
0.6	$\mathcal{A}_1^{(1,0)}$	0.6350	0.3423	0.8930	0.3067	0.4785	–	0.8480	0.5156	0.3819	–
	$\mathcal{A}_1^{(0,1)}$	–	–	0.5855	0.2857	–	–	0.3885	0.1087	–	–
	$\mathcal{A}_2^{(1,1)}$	0.7442	0.6077	–	0.4715	0.6621	0.4097	–	0.6081	0.6217	0.3644
	$\mathcal{A}_2^{(0,2)}$	0.0164	–	0.7508	0.4979	0.0136	–	0.5943	0.2876	0.0184	0.0409
	$\mathcal{A}_3^{(1,2)}$	0.7837	0.6639	0.8305	–	0.7122	0.4773	0.7027	–	0.6765	0.4310
0.7	$\mathcal{A}_1^{(1,0)}$	0.6028	0.3196	0.9091	0.1923	0.4381	–	0.8788	0.4648	0.3373	–
	$\mathcal{A}_1^{(0,1)}$	–	–	0.5401	0.2552	–	–	0.3296	0.0843	–	–
	$\mathcal{A}_2^{(1,1)}$	0.7156	0.5885	–	0.3976	0.6323	0.3889	–	0.5921	0.5941	0.3455
	$\mathcal{A}_2^{(0,2)}$	0.0067	–	0.7205	0.4731	0.0124	–	0.5464	0.2561	0.0038	0.0051
	$\mathcal{A}_3^{(1,2)}$	0.7547	0.6419	0.7369	–	0.6812	0.4541	0.6167	–	0.6470	0.4101
0.8	$\mathcal{A}_1^{(1,0)}$	0.5630	0.2926	0.9320	0.0150	0.3894	–	0.7166	0.4125	0.2845	–
	$\mathcal{A}_1^{(0,1)}$	–	–	0.4818	0.2193	–	–	0.2605	0.0603	–	–
	$\mathcal{A}_2^{(1,1)}$	0.6782	0.5629	–	0.2777	0.5950	0.3638	–	0.4257	0.5608	0.3239
	$\mathcal{A}_2^{(0,2)}$	0.0055	–	0.6800	0.4415	0.0052	–	0.4852	0.2196	–	0.0021
	$\mathcal{A}_3^{(1,2)}$	0.7149	0.6113	0.6486	–	0.6403	0.4241	0.5372	–	0.6093	0.3841
0.9	$\mathcal{A}_1^{(1,0)}$	0.5046	0.2549	0.9629	0.0088	0.3209	–	0.6588	0.3058	0.2131	–
	$\mathcal{A}_1^{(0,1)}$	–	–	0.3946	0.1714	–	–	0.1728	0.0353	–	–
	$\mathcal{A}_2^{(1,1)}$	0.6187	0.5210	–	0.0623	0.5390	0.3275	–	0.3619	0.5126	0.2949
	$\mathcal{A}_2^{(0,2)}$	0.0010	–	0.6152	0.3939	0.0036	–	0.3956	0.1715	–	0.0004
	$\mathcal{A}_3^{(1,2)}$	0.6490	0.5593	0.4685	–	0.5758	0.3772	0.5058	–	0.5515	0.3451

(2) When the set $\mathcal{B}_1^{(1,0)}$ is improved by CDM, the $\mathcal{L}(3, 0.1)$ increases from $3.9990/\nu$ to $5.3855/\nu$, see Table 2. We have the same effect by reducing the failure rates of (i) $\mathcal{A}_1^{(1,0)}$ by $\xi^C = 0.4607$, (ii) $\mathcal{A}_1^{(0,1)}$ by $\xi^C = 0.6888$, (iii) $\mathcal{A}_2^{(0,2)}$ by $\xi^C = 0.0111$, (iv) $\mathcal{A}_3^{(1,2)}$ by $\xi^C = 0.7426$, see Table 3.

(3) The results in Tables 2 and 3 can be interpreted by the same way.

(4) The symbol "–" means that there is no equivalence between both the reduction and duplication methods in this numerical study.

7. Conclusion

The reliability performance of a serial-parallel system based on a gamma distribution has been improved. This system is one of the important systems in reliability because it can be reduced to the series, parallel and radar systems. The system components have gamma lifetime distribution. Lifetimes are assumed independent and identical. The gamma distribution is an important distribution that is used in engineering to study system reliability. The gamma distribution has some special distributions based on the values of its parameters. The original system was improved using three different methods. Some reliability measures are derived for each method, such as RF and MTTF. The REFs and α-fractiles were established. Numerical application was discussed to interpret how the theoretical results can be applied. The cold duplication method gives the best improvement than other methods.

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