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On Interior Bases of Ordered Semigroups

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ABSTRACT. In this paper, the notions of interior bases of ordered semigroups are introduced, and some examples are also presented. We describe a characterization when a non-empty subset of an ordered semigroup is an interior base of an ordered semigroup. Finally, a characterization when an interior base of an ordered semigroup is a subsemigroup of an ordered semigroup will be given.

1. Introduction

A semigroup is one of algebraic structures which was widely studied. There are many generalizations, for example, LA-semigroup, Γ-semigroup, ordered semigroups, etc. The study of ordered semigroups began about 1950 by several authors, for example, Alimov [1], and Chehata [2]. The notion of one-sided bases of a semigroup was introduced by Tamura [3]. In 1972, Fabrici studied the structure of semigroups containing one-sided bases and he introduced the concept of two-sided bases of semigroups in 1975 [4,5]. Later, Changphas and Summaprab introduced the concept of two-sided bases of an ordered semigroup [6]. In 2017, Kummoon and Changphas introduced the concept of bi-bases of a semigroup and bi-bases of Γ-semigroups [7,8].

In this paper, the concepts of interior bases of ordered semigroups will be introduced. Moreover, we describe a characterization when a non-empty subset of an ordered semigroup is

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an interior base of an ordered semigroup and a characterization when an interior base of an ordered semigroup is a subsemigroup of an ordered semigroup.

An ordered semigroup (some authors called po-semigroup) $(S,\cdot,\leq)$ is a poset $(S,\leq)$ at the same time a semigroup $(S,\cdot)$ such that, for any $x,y,z\in S$,

$$x\leq y \text{ implies } xz\leq yz \text{ and } zx\leq zy.$$

Throughout this paper, unless stated otherwise, we write S instead of $(S,\cdot,\leq)$ and S stands for an ordered semigroup.

A non-empty subset A of an ordered semigroup S is called a subsemigroup of S if $AA\subseteq A$.

Let S be an ordered semigroup. For A and B are non-empty subsets of S, we denote

$$AB=\{ab \mid a\in A,\, b\in B\} \text{ and } (A]=\{b\in S \mid b\leq a \text{ for some } a\in A\}.$$

For $a\in S$, we write Ba for $B\{a\}$, similarly aB for $\{a\}B$, and $(a]$ for $(\{a\}]$.

Definition 1.1. [9] A subsemigroup A of an ordered semigroup S is called an interior ideal of S if it satisfies the following condition:

(1) $SAS\subseteq A$;

(2) if $a\in A$ and $b\in S$ such that $b\leq a$, then $b\in A$.

Lemma 1.2. [10,11] Let S be an ordered semigroup. Then the following statements hold.

(1) $A\subseteq (A]$, $(S=(S])$ for any $A\subseteq S$.

(2) $((A]]=(A]$ for any $A\subseteq S$.

(3) If $A\subseteq B\subseteq S$, then $(A]\subseteq (B]$.

(4) $(A](B]\subseteq (AB]$ for any $A,B\subseteq S$.

(5) $((A](B]]=(AB]$ for any $A,B\subseteq S$.

(6) If A is an interior ideal of S, then $A=(A]$.

(7) $(A\cup B]=(A]\cup(B]$ for any $A,B\subseteq S$.

(8) $A(B\cup C)=AB\cup AC$ and $(B\cup C)A=BA\cup CA$ for any $A,B,C\subseteq S$.

Lemma 1.3. [11] Let S be an ordered semigroup and A_i be a subsemigroup of S for all $i\in I$. If $\bigcap_{i\in I}A_i\neq\varnothing$, then $\bigcap_{i\in I}A_i$ is a subsemigroup of S.

Lemma 1.4. Let S be an ordered semigroup and A_i be an interior ideal of S for all $i\in I$. If $\bigcap_{i\in I}A_i\neq\varnothing$, then $\bigcap_{i\in I}A_i$ is an interior ideal of S.

Proof. Assume that $\bigcap_{i\in I}A_i\neq\varnothing$. By Lemma 1.3, $\bigcap_{i\in I}A_i$ is a subsemigroup of S. Let $x\in S(\bigcap_{i\in I}A_i)S$. Then $x=s_1as_2$ for some $s_1,s_2\in S$ and $a\in\bigcap_{i\in I}A_i$. Since $a\in\bigcap_{i\in I}A_i$, we have $a\in A_i$ for all $i\in I$, where A_i is an interior ideal of S for all $i \in I$. So we have $x = s_1 a s_2 \in S(A_i)S \subseteq A_i$ for all $i \in I$. Thus $x \in \bigcap_{i \in I} A_i$. Next, let $y \in \bigcap_{i \in I} A_i$ and $z \in S$ be such that $z \le y$. Since $y \in \bigcap_{i \in I} A_i$, then $y \in A_i$ for all $i \in I$, where A_i is an interior ideal of S for all $i \in I$. Since $z \le y$ and $y \in A_i$ for all $i \in I$, we have $z \in A_i$ for all $i \in I$. So $z \in \bigcap_{i \in I} A_i$. Therefore, $\bigcap_{i \in I} A_i$ is an interior ideal of S.

Definition 1.5. Let S be an ordered semigroup and let A be a non-empty subset of S. Then the intersection of all interior ideals of S containing A is the smallest interior ideal of S generated by A, denoted by $(A)_I$.

Lemma 1.6. Let S be an ordered semigroup and let A be a non-empty subset of S. Then $(A)_I = (A \cup AA \cup SAS]$.

Proof. Let $B = (A \cup AA \cup SAS]$. Consider,

$$\begin{aligned} BB &= (A \cup AA \cup SAS](A \cup AA \cup SAS] \\ &\subseteq ((A \cup AA \cup SAS)(A \cup AA \cup SAS)] \\ &= (AA \cup AAA \cup ASAS \cup AAA \cup AAAA \cup AASAS \cup SASA \cup SASAA \cup SASSAS] \\ &\subseteq (AA \cup SAS] \subseteq B. \end{aligned}$$

Thus B is a subsemigroup of S. Next, consider

$$\begin{aligned} SBS &= S(A \cup AA \cup SAS]S \\ &= (S](A \cup AA \cup SAS)(S] \\ &\subseteq ((S)(A \cup AA \cup SAS)](S] \\ &= (SA \cup SAA \cup SSAS](S] \\ &\subseteq ((SA \cup SAA \cup SSAS)(S)] \\ &= (SAS \cup SAAS \cup SSASS] \\ &\subseteq (SAS] \\ &\subseteq B. \end{aligned}$$

Thus $SBS \subseteq B$. Clearly, if $x \in B = (A \cup AA \cup SAS]$ and $y \in S$ such that $y \le x$, then $y \in ((A \cup AA \cup SAS]] = (A \cup AA \cup SAS] = B$. Hence, B is an interior ideal of S containing A. Finally, let C be an interior ideal of S containing A. Clearly, $A \subseteq C$. Since C is a subsemigroup of S, we have $AA \subseteq CC \subseteq C$. Since C is an interior ideal of S, we have $SAS \subseteq SCS \subseteq C$. Thus $A \cup AA \cup SAS \subseteq C$, and so $B = (A \cup AA \cup SAS] \subseteq (C] = C$. Hence, B is the smallest interior ideal of S containing A. Therefore, $B = (A \cup AA \cup SAS]$.

2. Main Results

We begin this section with the following definition of interior bases of an ordered semigroup.

Definition 2.1. Let S be an ordered semigroup. A non-empty subset A of S is called an interior base of S if it satisfies the following two conditions:

(1) $S=(A\cup AA\cup SAS]$, i.e., $S=(A)_I$;

(2) if B is a subset of A such that $S=(B)_I$, then $B=A$.

Example 2.2. [12] Let $S=\{a,b,c,d,e\}$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\leq=\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.$$

The interior bases of S are $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, and $\{e\}$.

Example 2.3. [13] Let $S=\{a,b,c,d,f\}$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	f
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
f	d	d	c	d	c

$$\leq=\{(a,a),(a,b),(b,b),(c,c),(d,b),(d,c),(d,d),(f,c),(f,f)\}.$$

The interior bases of S are $\{a,c\}$, $\{a,f\}$, $\{b,c\}$, and $\{b,f\}$.

Lemma 2.4. Let A be an interior base of an ordered semigroup S, and let $a,b\in A$. If $a\in(bb\cup SbS]$, then $a=b$.

Proof. Assume that $a\in(bb\cup SbS]$, and suppose that $a\neq b$. Setting $B=A\setminus\{a\}$. Then $B\subset A$. Since $a\neq b$, we have $b\in B$. We will show that $(A)_I\subseteq(B)_I$. Let $x\in(A)_I$. Since $x\in(A)_I=(A\cup AA\cup SAS]$, we have $x\leq y$ for some $y\in A\cup AA\cup SAS$. We can consider the three following cases.

Case 1: $y\in A$. There are two subcases to consider.

Subcase 1.1: $y\neq a$.

So $y\in B\subseteq(B\cup BB\cup SBS]$. Since $x\leq y$ and $y\in(B\cup BB\cup SBS]$, we obtain

$$x\in((B\cup BB\cup SBS]]=(B\cup BB\cup SBS]=(B)_I.$$

Subcase 1.2: $y=a$.

By assumption, we have

$$y = a \in (bb \cup SbS] \subseteq (BB \cup SBS] \subseteq (B)_I.$$

Since $x \le y$ and $y \in (B)_I$, so we obtain $x \in ((B)_I] = (B)_I$.

Case 2: $y \in AA$. Then $y = a_1a_2$ for some $a_1, a_2 \in A$. There are four subcases to consider.

Subcase 2.1: $a_1 \ne a$ and $a_2 \ne a$.

We have $a_1, a_2 \in B$. So $y = a_1a_2 \in BB \subseteq (B)_I$. Since $x \le y$ and $y \in (B)_I$, we obtain $x \in ((B)_I] = (B)_I$.

Subcase 2.2: $a_1 = a$ and $a_2 \ne a$.

Then by assumption and $a_2 \in B$, we have

$$\begin{aligned} y = a_1a_2 \in (bb \cup SbS]B &\subseteq (BB \cup SBS](B] \\ &\subseteq ((BB \cup SBS)(B)] \\ &= (BBB \cup SBSB] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

Since $x \le y$ and $y \in (B)_I$, so $x \in ((B)_I] = (B)_I$.

Subcase 2.3: $a_1 \ne a$ and $a_2 = a$.

Then by assumption and $a_1 \in B$, we have

$$\begin{aligned} y = a_1a_2 \in B(bb \cup SbS] &\subseteq (B](BB \cup SBS] \\ &\subseteq ((B)(BB \cup SBS)] \\ &= (BBB \cup BSBS] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

Since $x \le y$ and $y \in (B)_I$, so $x \in ((B)_I] = (B)_I$.

Subcase 2.4: $a_1 = a$ and $a_2 = a$.

By assumption, we have

$$\begin{aligned} y = a_1a_2 \in (bb \cup SbS](bb \cup SbS] &\subseteq ((bb \cup SbS)(bb \cup SbS)] \\ &= (bbbb \cup bbSbS \cup SbSbb \cup SbSSbS] \\ &\subseteq (BBBB \cup BBSBS \cup SBSBB \cup SBSSBS] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

Since $x \le y$ and $y \in (B)_I$, so $x \in ((B)_I] = (B)_I$.

Case 3: $y \in SAS$. Then $y = s_1a_3s_2$ for some $s_1, s_2 \in S$ and $a_3 \in A$. There are two subcases to consider.

Subcase 3.1: $a_3 \ne a$.

We have $a_3 \in B$. So $y = s_1a_3s_2 \in SBS \subseteq (B)_I$. Since $x \le y$ and $y \in (B)_I$, we have $x \in ((B)_I] = (B)_I$.

Subcase 3.2: $a_3 = a$.

By assumption, we have

$$\begin{aligned} y = s_1 a_3 s_2 \in S(bb \cup SbS]S &\subseteq (S](BB \cup SBS](S] \\ &\subseteq ((S)(BB \cup SBS)](S] \\ &= (SBB \cup SSBS](S] \\ &\subseteq ((SBB \cup SSBS)(S)] \\ &= (SBBS \cup SSBSS] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

Since $x \leq y$ and $y \in (B)_I$, we have $x \in ((B)_I] = (B)_I$.

From both cases, we obtain $(A)_I \subseteq (B)_I$. Since A is an interior base of S, we have

$$S = (A)_I \subseteq (B)_I \subseteq S.$$

Thus $S = (B)_I$. This is a contradiction. Therefore, $a = b$.

Lemma 2.5. Let A be an interior base of an ordered semigroup S, and let $a,b,c \in A$. If $a \in (cb \cup ScbS]$, then $a = b$ or $a = c$.

Proof. Assume that $a \in (cb \cup ScbS]$. Suppose that $a \neq b$ and $a \neq c$. We set $B = A \setminus \{a\}$. Clearly, $B \subset A$. Since $a \neq b$ and $a \neq c$, so we have $b,c \in B$. We will show that $(B)_I = S$. Obviously, $(B)_I \subseteq S$. Next, to show that $S \subseteq (B)_I$. Let $x \in S$. Since A is an interior base of S, we have $S = (A)_I$. So $x \in (A)_I = (A \cup AA \cup SAS]$. Since $x \in (A \cup AA \cup SAS]$, we have $x \leq y$ for some $y \in A \cup AA \cup SAS$. We can consider the three following cases.

Case 1: $y \in A$. There are two subcases to consider.

Subcase 1.1: $y \neq a$. So $y \in B \subseteq (B)_I$.

Subcase 1.2: $y = a$. By assumption, we have

$$y = a \in (cb \cup ScbS] \subseteq (BB \cup SBBS] \subseteq (BB \cup SBS] \subseteq (B)_I.$$

Case 2: $y \in AA$. Then $y = a_1 a_2$ for some $a_1, a_2 \in A$. There are four subcases to consider.

Subcase 2.1: $a_1 \neq a$ and $a_2 \neq a$. We have $a_1, a_2 \in B$. So $y = a_1 a_2 \in BB \subseteq (B)_I$.

Subcase 2.2: $a_1 = a$ and $a_2 \neq a$. By assumption and $a_2 \in B$, we have

$$\begin{aligned} y = a_1 a_2 \in (cb \cup ScbS]B &\subseteq (BB \cup SBBS)(B] \\ &\subseteq ((BB \cup SBBS)(B)] \\ &= (BBB \cup SBBSB] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

Subcase 2.3: $a_1 \neq a$ and $a_2 = a$. By assumption and $a_1 \in B$, we have

$$y = a_1 a_2 \in B(cb \cup ScbS] \subseteq (B](BB \cup SBBS] \subseteq ((B)(BB \cup SBBS)] = (BBB \cup BSBBS] \subseteq (SBS] \subseteq (B)_I.$$

Subcase 2.4: $a_1 = a$ and $a_2 = a$. By assumption, we have

$$\begin{aligned} y = a_1a_2 \in (cb \cup ScbS](cb \cup ScbS] &\subseteq ((cb \cup ScbS)(cb \cup ScbS)] \\ &= (cbcb \cup cbScbS \cup ScbScb \cup ScbSScbS] \\ &\subseteq (BBBB \cup BBSBBS \cup SBBSBB \cup SBBSSBBS] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

Case 3: $y \in SAS$. Then $y = s_1a_3s_2$ for some $s_1, s_2 \in S$ and $a_3 \in A$. There are two subcases to consider.

Subcase 3.1: $a_3 \neq a$. We have $a_3 \in B$. So $y = s_1a_3s_2 \in SBS \subseteq (B)_I$.

Subcase 3.2: $a_3 = a$. By assumption, we have

$$\begin{aligned} y = s_1a_3s_2 \in S(cb \cup ScbS]S &\subseteq (S](BB \cup SBBS](S] \\ &\subseteq ((S)(BB \cup SBBS)](S] \\ &= (SBB \cup SSBBS](S] \\ &\subseteq ((SBB \cup SSBBS)(S)] \\ &= (SBBS \cup SSBBSS] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

From both cases, we obtain $y \in (B)_I$. Since $x \leq y$ and $y \in (B)_I$, we have $x \in ((B)_I] = (B)_I$. Thus $S \subseteq (B)_I$ and hence $S = (B)_I$. This is a contradiction. Therefore, $a = b$ or $a = c$.

Beside the partial order $\leq$ on an ordered semigroup S, we define quasi-order $\leq_I$ on S as follows:

Definition 2.6. Let S be an ordered semigroup. We define a quasi-order on S by for any $a, b \in S$,

$$a \leq_I b \Leftrightarrow (a)_I \subseteq (b)_I.$$

The following example shows that $\leq_I$ defined above is not, in general, a partial order.

Example 2.7. From Example 2.2, we have that $(a)_I \subseteq (b)_I$ (i.e., $a \leq_I b$) and $(b)_I \subseteq (a)_I$ (i.e., $b \leq_I a$), but $a \neq b$. Thus $\leq_I$ is not a partial order on S.

Lemma 2.8. Let S be an ordered semigroup. For any $x, y \in S$, if $x \leq y$, then $x \leq_I y$.

Proof. For any $x, y \in S$, let $x \leq y$. We will show that $(x)_I \subseteq (y)_I$. Since $x \leq y$ and $y \in (y)_I$, we have $x \in ((y)_I] = (y)_I$. Since $\{x\} \subseteq (y)_I = (y \cup yy \cup SyS]$, then

$$\begin{aligned} &x \cup xx \cup SxS \\ &\subseteq (y \cup yy \cup SyS] \cup (y \cup yy \cup SyS](y \cup yy \cup SyS] \cup S(y \cup yy \cup SyS]S \\ &\subseteq (y \cup yy \cup SyS] \cup ((y \cup yy \cup SyS)(y \cup yy \cup SyS)] \cup (S](y \cup yy \cup SyS](S] \\ &\subseteq (y \cup yy \cup SyS] \cup (yy \cup yyy \cup ySyS \cup yyy \cup yyyy \cup yySyS \cup SySy \cup SySyy \cup SySSyS] \\ &\quad \cup ((S)(y \cup yy \cup SyS)](S] \\ &\subseteq (y \cup yy \cup SyS] \cup (yy \cup SyS] \cup (Sy \cup Syy \cup SSyS](S] \\ &\subseteq (y \cup yy \cup SyS] \cup (yy \cup SyS] \cup ((Sy \cup Syy \cup SSyS)(S)] \\ &= (y \cup yy \cup SyS] \cup (yy \cup SyS] \cup (SyS \cup SyyS \cup SSySS] \\ &\subseteq (y \cup yy \cup SyS] \cup (yy \cup SyS] \cup (SyS] \end{aligned}$$

$=(y\cup yy\cup SyS]=(y)_I.$

So $(x)_I=(x\cup xx\cup SxS]\subseteq((y)_I]=(y)_I$. Thus $(x)_I\subseteq(y)_I$. Therefore, $x\leq_I y$.

Nevertheless, the converse of Lemma 2.8, is not valid in general. By Example 2.2, we have $b\leq_I a$, but $b\leq a$ is false.

Lemma 2.9. Let A be an interior base of an ordered semigroup S. If $a,b\in A$ such that $a\neq b$, then neither $a\leq_I b$ nor $b\leq_I a$.

Proof. Assume that $a,b\in A$ such that $a\neq b$. Suppose that $a\leq_I b$. Setting $B=A\setminus\{a\}$. We have $b\in B$ and $B\subset A$. First, we claim that, for any $x\in S$ there exists $y\in A$ such that $(x)_I\subseteq(y)_I$. Since $x\in S$ and $S=(A)_I$, we have $x\in(A)_I$. Since $x\in(A)_I$, we have $x\in(y)_I$ for some $y\in A$. Since $x\in(y)_I$, it follows that $(x)_I\subseteq(y)_I$. So $(x)_I\subseteq(y)_I$ for some $y\in A$. Next, we will show that $S=(B)_I$. Let $x_1\in S$. There exists $y_1\in A$ such that $(x_1)_I\subseteq(y_1)_I$. There are two cases to consider. If $y_1\neq a$, then $y_1\in B$. We have

$$x_1\in(x_1)_I\subseteq(y_1)_I\subseteq(B)_I.$$

If $y_1=a$, then $y_1\leq_I b$, i.e., $(y_1)_I\subseteq(b)_I$. We have

$$x_1\in(x_1)_I\subseteq(y_1)_I\subseteq(b)_I\subseteq(B)_I.$$

Thus $S\subseteq(B)_I$, and so $S=(B)_I$. This is a contradiction. Hence, $a\leq_I b$ is false. The case $b\leq_I a$ can be proved similarly.

Lemma 2.10. Let A be an interior base of an ordered semigroup S. Let $a,b,c\in A$ and $s\in S$.

(1) If $a\in(bc\cup bcbc\cup SbcS]$, then $a=b$ or $a=c$.

(2) If $a\in(sbcs\cup sbcssbcs\cup SsbcsS]$, then $a=b$ or $a=c$.

Proof. (1) Assume that $a\in(bc\cup bcbc\cup SbcS]$. Suppose that $a\neq b$ and $a\neq c$. We set $B=A\setminus\{a\}$. Clearly, $B\subset A$. Since $a\neq b$ and $a\neq c$, we have $b,c\in B$. We will show that $(A)_I\subseteq(B)_I$, it suffices to show that $A\subseteq(B)_I$. Let $x\in A$. If $x\neq a$, then $x\in B\subseteq(B)_I$. If $x=a$, then by assumption, we have

$$\begin{aligned}x=a\in(bc\cup bcbc\cup SbcS] &\subseteq (BB\cup BBBB\cup SBBS]\\ &\subseteq (BB\cup SBS]\\ &\subseteq (B)_I.\end{aligned}$$

So $A\subseteq(B)_I$. It follows that $(A)_I\subseteq(B)_I$. Since A is an interior base of S, so we have

$$S=(A)_I\subseteq(B)_I\subseteq S.$$

Thus $S=(B)_I$. This is a contradiction. Therefore, $a=b$ or $a=c$.

(2) Assume that $a\in(sbcs\cup sbcssbcs\cup SsbcsS]$. Suppose that $a\neq b$ and $a\neq c$. We set $B=A\setminus\{a\}$. Then $B\subset A$. Since $a\neq b$ and $a\neq c$, we have $b,c\in B$. We will show that $(A)_I\subseteq(B)_I$, it suffices to show that $A\subseteq(B)_I$. Let $x\in A$. If $x\neq a$, then $x\in B\subseteq(B)_I$. If $x=a$, then by assumption, we have

$$\begin{aligned} x = a \in (sbcs \cup sbcssbcs \cup SsbcsS] &\subseteq (SBBS \cup SBBSSBBS \cup SSBBSS] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

So $A \subseteq (B)_I$. This implies that $(A)_I \subseteq (B)_I$. Thus

$$S = (A)_I \subseteq (B)_I \subseteq S.$$

Hence, $S = (B)_I$. This is a contradiction. Therefore, $a = b$ or $a = c$.

Lemma 2.11. Let A be an interior base of an ordered semigroup S.

(1) For any $a,b,c \in A$, if $a \neq b$ and $a \neq c$, then $a \nleq_I bc$.

(2) For any $a,b,c \in A$ and $s \in S$, if $a \neq b$ and $a \neq c$, then $a \nleq_I sbcs$.

Proof. (1) For any $a,b,c \in A$, let $a \neq b$ and $a \neq c$. Suppose that $a \leq_I bc$, i.e., $(a)_I \subseteq (bc)_I$. We have

$$a \in (a)_I \subseteq (bc)_I = (bc \cup bcbc \cup SbcS].$$

By Lemma 2.10(1), we obtain $a = b$ or $a = c$. This contradicts to assumption. Therefore, $a \nleq_I bc$.

(2) For any $a,b,c \in A$ and $s \in S$, let $a \neq b$ and $a \neq c$. Suppose that $a \leq_I sbcs$. We have

$$a \in (a)_I \subseteq (sbcs)_I = (sbcs \cup sbcssbcs \cup SsbcsS].$$

By Lemma 2.10(2), $a = b$ or $a = c$. This contradicts to assumption. Therefore, $a \nleq_I sbcs$.

Lemma 2.12. Let A be an interior base of an ordered semigroup S. For any $a,b \in A$ and $s_1, s_2 \in S$, if $a \neq b$, then $a \nleq_I s_1bs_2$.

Proof. For any $a,b \in A$ and $s_1, s_2 \in S$, let $a \neq b$. Suppose that $a \leq_I s_1bs_2$, i.e., $(a)_I \subseteq (s_1bs_2)_I$. We have

$$a \in (a)_I \subseteq (s_1bs_2)_I = (s_1bs_2 \cup s_1bs_2s_1bs_2 \cup Ss_1bs_2S].$$

We set $B = A \setminus \{a\}$. Then $b \in B$ and $B \subset A$. We will show that $(A)_I \subseteq (B)_I$, it suffices to show that $A \subseteq (B)_I$. Let $x \in A$. If $x \neq a$, then $x \in B \subseteq (B)_I$. If $x = a$, then by assumption, we have

$$\begin{aligned} x = a \in (s_1bs_2 \cup s_1bs_2s_1bs_2 \cup Ss_1bs_2S] &\subseteq (SBS \cup SBSSBS \cup SSBSS] \\ &\subseteq (SBS] \\ &\subseteq (B)_I. \end{aligned}$$

So $x \in (B)_I$. Thus $A \subseteq (B)_I$. It follows that $(A)_I \subseteq (B)_I$. Since A is an interior base of S, then

$$S = (A)_I \subseteq (B)_I \subseteq S.$$

Hence, $S = (B)_I$. This is a contradiction. Therefore, $a \nleq_I s_1bs_2$.

We now prove the main result of this paper.

Theorem 2.13. A non-empty subset A of an ordered semigroup S is an interior base of S if and only if A satisfies the following conditions:

(1) For any $x \in S$,

(1.1) there exists $a \in A$ such that $x \leq_I a$; or

(1.2) there exist $a_1, a_2 \in A$ such that $x \leq_I a_1a_2$; or

(1.3) there exist $a_3 \in A$ and $s_1, s_2 \in S$ such that $x \leq_I s_1a_3s_2$.

(2) For any $a,b,c \in A$, if $a \neq b$ and $a \neq c$, then $a \nleq_I bc$.

(3) For any $a,b \in A$ and $s_1,s_2 \in S$, if $a \neq b$, then $a \nleq_I s_1bs_2$.

Proof. Assume that A is an interior base of S. We have $S=(A)_I$. To show that (1) holds. Let $x \in S$. Then $x \in (A)_I = (A \cup AA \cup SAS]$. Since $x \in (A \cup AA \cup SAS]$, we have $x \leq y$ for some $y \in A \cup AA \cup SAS$. We consider three cases:

Case 1: $y \in A$. Then $y=a$ for some $a \in A$. This implies $(y)_I \subseteq (a)_I$, and so $y \leq_I a$. Since $x \leq y$, by Lemma 2.8, we have $x \leq_I y$. Thus $x \leq_I y \leq_I a$, and hence $x \leq_I a$.

Case 2: $y \in AA$. Then $y=a_1a_2$ for some $a_1,a_2 \in A$. This implies $(y)_I \subseteq (a_1a_2)_I$, and so $y \leq_I a_1a_2$. Since $x \leq y$, by Lemma 2.8, we have $x \leq_I y$. Thus $x \leq_I y \leq_I a_1a_2$, and hence $x \leq_I a_1a_2$.

Case 3: $y \in SAS$. Then $y=s_1a_3s_2$ for some $a_3 \in A$, $s_1,s_2 \in S$. We obtain $(y)_I \subseteq (s_1a_3s_2)_I$. So $y \leq_I s_1a_3s_2$. Since $x \leq y$, by Lemma 2.8, we have $x \leq_I y$. Thus $x \leq_I y \leq_I s_1a_3s_2$, and hence $x \leq_I s_1a_3s_2$.

The validity of (2) and (3) follow, respectively, from Lemma 2.11(1), and Lemma 2.12.

Conversely, assume that the conditions (1), (2) and (3) hold. We will show that A is an interior base of S. First, We will show that $S=(A)_I$. Clearly, $(A)_I \subseteq S$. By (1.1), it follows that $S \subseteq A$. We have

$$S \subseteq A \cup AA \cup SAS \subseteq (A \cup AA \cup SAS] = (A)_I.$$

Thus $S \subseteq (A)_I$, and so $S=(A)_I$. Next, it remains to show that A is a minimal subset of S with the property $S=(A)_I$. Suppose that $S=(B)_I$ for some $B \subset A$. Since $B \subset A$, there exists $x \in A$ such that $x \notin B$. Since $x \in A \subseteq S = (B)_I = (B] \cup (BB \cup SBS]$, we have $x \in (B]$ or $x \in (B \cup BB \cup SBS]$. If $x \in (B]$, then $x \leq y$ for some $y \in B$. Since $x \leq y$, by Lemma 2.8, we have $x \leq_I y$ where $x,y \in A$. This contradicts to Lemma 2.9. Thus $x \notin (B]$, and so $x \in (BB \cup SBS]$. Since $x \in (BB \cup SBS]$, we have $x \leq z$ for some $z \in BB \cup SBS$. We consider two cases:

Case 1: $z \in BB$. Then $z=a_1a_2$ for some $a_1,a_2 \in B$. We have $a_1,a_2 \in A$. Since $x \notin B$, then $x \neq a_1$ and $x \neq a_2$. Since $z=a_1a_2$, we obtain $(z)_I \subseteq (a_1a_2)_I$, i.e., $z \leq_I a_1a_2$. Since $x \leq z$, by Lemma 2.8, we have $x \leq_I z$. So $x \leq_I z \leq_I a_1a_2$. Thus $x \leq_I a_1a_2$. This contradicts to (2).

Case 2: $z \in SBS$. Then $z=s_1a_3s_2$ for some $s_1,s_2 \in S$ and $a_3 \in B$. We have $a_3 \in A$. Since $x \notin B$, we have $x \neq a_3$. Since $z=s_1a_3s_2$, we obtain $(z)_I \subseteq (s_1a_3s_2)_I$, i.e., $z \leq_I s_1a_3s_2$. Since $x \leq z$, by Lemma 2.8, we have $x \leq_I z$. So $x \leq_I z \leq_I s_1a_3s_2$. Thus $x \leq_I s_1a_3s_2$. This contradicts to (3).

Therefore, A is an interior base of S.

The following theorem characterization when an interior base of an ordered semigroup S is a subsemigroup of S.

Theorem 2.14. Let A be an interior base of an ordered semigroup S. Then A is a subsemigroup of S if and only if for any $a,b \in A$, $ab=a$ or $ab=b$.

Proof. Assume that A is a subsemigroup of S. Suppose that $ab \neq a$ and $ab \neq b$. Let $c = ab$. Then $c \neq a$ and $c \neq b$. Since $c = ab \in (ab \cup SabS]$, by Lemma 2.5, we have $c = a$ or $c = b$. This is a contradiction. The converse statement is clear.

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