Bipolar Fuzzy Filters of Gamma-Near Rings

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Abstract. The main objective of this paper is to present the notation of bipolar fuzzy filters of Γ-near rings and ordered Γ-near rings. As a consequence, we deal with bipolar fuzzy prime ideals of Γ-near rings and ordered Γ-near rings. Also, we examine the one-to-one correspondence of bipolar fuzzy filters and crisp filters of Γ-near rings. Later, we define and study the homomorphism of ordered Γ-near rings.

1. Introduction

The near-ring theory was introduced by Pilz [6]. The concept of Γ-rings, a generalization of a ring, was introduced by Nobusawa [5]. Γ-near rings (GNRs) were defined by Satyanarayana [16], and the ideal theory in GNRs was studied by Satyanarayana [16] and Booth [1]. Further, several authors studied various algebraic structures on GNRs, like ideals, weak ideals, bi-ideals, quasi-ideals, and normal ideals on GNRs. The idea of bipolar-valued fuzzy sets (BFSs) was given by Zhang [20], which is the extension of the theory of Zadeh’s fuzzy sets (FSs) [19] to BFSs. Later, taking into consideration, many authors applied fuzzification on crisp sets, like Satyanarayana studied and invented the idea of fuzzy ideals, prime ideals of GNRs. Some results and properties on fuzzy ideals of GNRs are discussed by Jun [2]. In order to study uncertainty, the application of bipolar fuzzification, which is a generalization of FSs, has been developed by Jun and Lee [3]. Several researchers like Ragamayi [7–13, 17, 18] and Rao [14, 15] did their research on the development of the BFS theory on different algebraic structures like semigroups, groups, semirings, rings, etc.

As a continuity of all these, we introduced bipolar fuzzy ideals, bi-ideals, and weak bi-ideals on GNRs in 2023. Now, we are studying bipolar fuzzy filters and prime ideals of GNRs.
2. Preliminaries

This section reviews important definitions for research in this paper.

**Definition 2.1.** [6] A near ring is a nonempty set $R$ equipped with two binary operations $+$ and $\cdot$ such that
(i) $(R, +)$ is a group,
(ii) $(R, \cdot)$ is a semigroup,
(iii) $(a + b)c = ac + bc$, $\forall a, b, c \in R$ obeying only right distributive law over addition.

**Definition 2.2.** [16] A $\Gamma$-near ring (GNR) is a triple $(M_R, +, \Gamma)$ where
(i) $(M_R, +)$ is a group,
(ii) $\Gamma$ is a nonempty set of binary operators on $M_R$ such that for each $\alpha \in \Gamma, (M_R, +, \alpha)$ is a near ring,
(iii) $\psi_\alpha(\omega \beta \kappa) = (\psi_\alpha \omega)\beta \kappa$, $\forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma$.

**Definition 2.3.** [4] A GNR $M_R$ is said to be zero-symmetric if $\psi_\alpha 0 = 0, \forall \psi \in M_R, \alpha \in \Gamma$.

**Definition 2.4.** [2] An FS $\xi$ in a GNR $M_R$ is a fuzzy sub $\Gamma$-near ring of $M_R$ if
(i) $\xi(\psi - \omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi(\psi_\alpha \omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega, \alpha \in M_R$.

**Definition 2.5.** [3,20] Let $M_R$ be a GNR and $B_R$ be a BFS of $M_R$. We say that $B_R = (\xi^+_B, \xi^-_B)$ is a bipolar fuzzy sub $\Gamma$-near ring (BFSGNR) of $M_R$ if
(i) $\xi^+_B(\psi - \omega) = \min\{\xi^+_B(\psi), \xi^+_B(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi^-_B(\psi - \omega) = \max\{\xi^-_B(\psi), \xi^-_B(\omega)\}, \forall \psi, \omega \in M_R$,
(iii) $\xi^+_B(\psi_\alpha \omega) = \min\{\xi^+_B(\psi), \xi^+_B(\omega)\}, \forall \psi, \omega, \alpha \in M_R$,
(iv) $\xi^-_B(\psi_\alpha \omega) = \max\{\xi^-_B(\psi), \xi^-_B(\omega)\}, \forall \psi, \omega, \alpha \in M_R$.

If $B = (\xi^+_B, \xi^-_B)$ satisfies the conditions (i) and (ii), then it is called a bipolar fuzzy subgroup (BFSG) of $M_R$.

**Definition 2.6.** [16] Let $M_R$ be a GNR and $A_R$ be a nonempty subset of $M_R$. Then $A_R$ is said to be an ideal of $M_R$ if
(i) $\psi - \omega \in A_R, \forall \psi, \omega \in A_R$,
(ii) $\omega + \psi - \omega \in A_R, \forall \psi \in I_R, \omega \in M_R$,
(iii) $a(a + b) - ab \in A_R, \forall \psi \in A_R, a, b \in M_R, \alpha \in \Gamma$,
(iv) $\psi a \in A_R, \forall \psi \in A_R, a, b \in M_R, \alpha \in \Gamma$.

**Definition 2.7.** [16] An ideal $A_R$ of a GNR $M_R$ is said to be a prime ideal of $M_R$ if $\psi_\alpha \omega \in A_R \Rightarrow \psi \in A_R$ or $\omega \in A_R, \forall \psi, \omega \in M_R, \alpha \in \Gamma$.

**Definition 2.8.** [14] Let $M_R$ be a GNR and $A_R$ be a nonempty subset of $M_R$. Then $A_R$ is said to be a filter of $M_R$ if
(i) $\psi_\alpha \omega \in A_R, \forall \psi \in A_R, a, b \in M_R, \alpha \in \Gamma$,
(ii) $\psi \leq \omega \Rightarrow \omega \in A_R, \forall \psi \in A_R, \omega \in M_R$. 
Definition 2.9. [2] An FS $\xi$ in a GNR $M_R$ is called a fuzzy ideal of $M_R$ if
(i) $\xi(\psi - \omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi(\omega + \psi - \omega) \geq \xi(\psi), \forall \psi, \omega \in M_R$,
(iii) $\xi(a\alpha(\psi + b) - a\alpha b) \geq \xi(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$,
(iv) $\xi(\psi a\alpha) \geq \xi(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$.

Definition 2.10. [2] A BFS $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ of a GNR $M_R$ is called a bipolar fuzzy ideal (BFI) of $M_R$ if
(i) $\xi_{B_R}^+(\psi - \omega) \geq \min\{\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi_{B_R}^+(\omega + \psi - \omega) \geq \xi_{B_R}^+(\psi), \forall \psi, \omega \in M_R$,
(iii) $\xi_{B_R}^+(a\alpha(\psi + b) - a\alpha b) \geq \xi_{B_R}^+(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$,
(iv) $\xi_{B_R}^+(\psi a\alpha) \geq \xi_{B_R}^+(\psi), \forall \psi, a \in M_R, \alpha \in \Gamma$,
(v) $\xi_{B_R}^-(\psi - \omega) \leq \max\{\xi_{B_R}^-(\psi), \xi_{B_R}^-(\omega)\}, \forall \psi, \omega \in M_R$,
(vi) $\xi_{B_R}^-(\omega + \psi - \omega) \leq \xi_{B_R}^-(\psi), \forall \psi, \omega \in M_R$,
(vii) $\xi_{B_R}^-(a\alpha(\psi + b) - a\alpha b) \leq \xi_{B_R}^-(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$,
(viii) $\xi_{B_R}^-(\psi a\alpha) \leq \xi_{B_R}^-(\psi), \forall \psi, a \in M_R, \alpha \in \Gamma$.

Definition 2.11. [9] A BFI $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ of a GNR $M_R$ is said to be a bipolar fuzzy prime ideal (BFPI) of $M_R$ if
(i) $\xi_{B_R}^+(\psi a\alpha) = \max\{\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$,
(ii) $\xi_{B_R}^-(\psi a\alpha) = \min\{\xi_{B_R}^-(\psi), \xi_{B_R}^-(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$.

Definition 2.12. The $(t,s)$ cut of an BFS $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ is a crisp set defined by $B_R(t,s) = \{\psi | \xi_{B_R}^+(\psi) \geq t, \xi_{B_R}^-(\psi) \leq s\}$ for $t \in [0,1], s \in [-1,0]$.

3. Bipolar Fuzzy Filters of GNRs

This section introduces and studies the notion of bipolar fuzzy filters of GNRs and their properties.

Definition 3.1. A BFS $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ of a GNR $M_R$ is called a bipolar fuzzy filter (BFF) of $M_R$ if
(i) $\xi_{B_R}^+(\psi - \omega) \leq \max\{\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi_{B_R}^-(\psi - \omega) \geq \min\{\xi_{B_R}^-(\psi), \xi_{B_R}^-(\omega)\}, \forall \psi, \omega \in M_R$,
(iii) $\xi_{B_R}^+(\psi a\alpha) = \min\{\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$,
(iv) $\xi_{B_R}^-(\psi a\alpha) = \max\{\xi_{B_R}^-(\psi), \xi_{B_R}^-(\omega)\}, \forall \psi, \omega \in M_R, \alpha \in \Gamma$.

Example 3.1. Let $M_R$ be the real numbered set and $\Gamma = M_R$. Then $M_R$ and $\Gamma$ are additive commutative groups. Define the mapping $M_R \ast \Gamma \ast M_R \to M_R$ by $\psi a\alpha$ the usual product of $\psi, \alpha, \omega, \forall \psi, \omega \in M_R, \alpha \in \Gamma$. Then $M_R$ is a GNR with zero symmetric. Let a BFS $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ of $M_R$ defined by

\[
\xi_{B_R}^+(\psi) = \begin{cases} 
0.55, & \text{if } \psi = 0 \\
0.83, & \text{otherwise}
\end{cases}
\]
\[
\xi_{B_R}^-(\psi) = \begin{cases} 
-0.12, & \text{if } \psi = 0 \\
-0.51, & \text{otherwise}
\end{cases}
\]
Then, by routine check, $B_R$ is a BFF of $M_R$.

**Definition 3.2.** A BFS $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ of an OGNR $M_R$ is called a bipolar fuzzy filter (BFF) of $M_R$ if

(i) $\xi_{B_R}^+(\psi - \omega) \leq \max(\xi_{B_R}^+(\psi), \xi_{B_R}^-(\omega)), \forall \psi, \omega \in M_R$,

(ii) $\xi_{B_R}^-(\psi - \omega) \geq \min(\xi_{B_R}^-\psi(\psi), \xi_{B_R}^-(\omega)), \forall \psi, \omega \in M_R$,

(iii) $\xi_{B_R}^+(\psi \alpha \omega) = \min(\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)), \forall \psi, \omega \in M_R, \alpha \in \Gamma$,

(iv) $\xi_{B_R}^-(\psi \alpha \omega) = \max(\xi_{B_R}^-\psi(\psi), \xi_{B_R}^-(\omega)), \forall \psi, \omega \in M_R, \alpha \in \Gamma$,

(v) $\psi \leq \omega \Rightarrow \xi_{B_R}^+(\psi) \leq \xi_{B_R}^-(\psi), \forall \psi, \omega \in M_R$,

(vi) $\psi \leq \omega \Rightarrow \xi_{B_R}^+(\psi) \geq \xi_{B_R}^-(\omega), \forall \psi, \omega \in M_R$.

**Theorem 3.1.** A BFS $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ of a GNR $M_R$ is a BFF of $M_R$ if and only if its level subset $B_R(t_i, s_i) \neq \emptyset$ is a filter of $M_R$ for any $t_i \in [0, 1], s_i \in [-1, 0]$.

**Proof.** Suppose $B_R$ is a BFS of $M_R$ and let $t_i \in [0, 1]$ and $s_i \in [-1, 0]$. Let $\psi \in B_R(t_i, s_i), \omega \in M_R, \alpha \in \Gamma$. Then $\xi_{B_R}^+(\psi) \geq t_i$ and $\xi_{B_R}^-\psi(\omega) \geq t_i$. Thus $\min(\xi_{B_R}^+(\psi), \xi_{B_R}^-\psi(\omega)) \geq t_i$, so $\xi_{B_R}^+(\psi \alpha \omega) \geq t_i$. Similarly, $\xi_{B_R}^-\psi(\psi) \leq s_i$ and $\xi_{B_R}^-\psi(\omega) \leq s_i$. Thus $\max(\xi_{B_R}^+(\psi), \xi_{B_R}^-\psi(\omega)) \leq s_i$, so $\xi_{B_R}^-(\psi \alpha \omega) \leq s_i$. Therefore, $\psi \alpha \omega \in B_R(t_i, s_i)$. Let $\psi \in B_R(t_i, s_i), \omega \in M_R, \alpha \in \Gamma$ be such that $\psi \leq \omega$. Then $\xi_{B_R}^+(\psi) \geq t_i$ and $\xi_{B_R}^+(\psi) \leq \xi_{B_R}^-\psi(\omega)$. Thus $t_i \leq \xi_{B_R}^-\psi(\psi) \leq \xi_{B_R}^+(\psi), \xi_{B_R}^-\psi(\omega) \geq t_i$. Similarly, $\xi_{B_R}^-\psi(\psi) \leq s_i$ and $\xi_{B_R}^-\psi(\omega) \geq \xi_{B_R}^-\psi(\omega)$. Thus $s_i \leq \xi_{B_R}^-\psi(\psi) \leq \xi_{B_R}^-\psi(\omega), \xi_{B_R}^-\psi(\omega) \leq s_i$. Therefore, $\omega \in B_R(t_i, s_i)$. Hence, $B_R(t_i, s_i)$ is a filter of $M_R$.

Conversely, suppose $B_R(t_i, s_i) \neq \emptyset$ is a filter of $M_R$ for any $t_i \in [0, 1], s_i \in [-1, 0]$. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. Suppose $t_i = \min(\xi_{B_R}^+(\psi), \xi_{B_R}^-\psi(\omega))$. Then $\xi_{B_R}^+(\psi) \geq t_i$ and $\xi_{B_R}^-\psi(\omega) \geq t_i$. Similarly, if $s_i = \max(\xi_{B_R}^-\psi(\psi), \xi_{B_R}^-\psi(\omega))$, then $\xi_{B_R}^-\psi(\psi) \leq s_i$ and $\xi_{B_R}^-\psi(\omega) \leq s_i$. Therefore, $\psi \alpha \omega \in B_R(t_i, s_i)$. Since $B_R(t_i, s_i)$ is a filter of $M_R$, $\psi \alpha \omega \in B_R(t_i, s_i)$. Thus $\xi_{B_R}^+(\psi \alpha \omega) \geq t_i, \xi_{B_R}^-\psi(\omega) \geq t_i$ and $\xi_{B_R}^-(\psi \alpha \omega) \leq s_i, \xi_{B_R}^-\psi(\omega) \leq s_i$. So, $\min(\xi_{B_R}^+(\psi), \xi_{B_R}^-(\psi \alpha \omega)) \geq t_i = \xi_{B_R}^+(\psi \alpha \omega)$ and $\max(\xi_{B_R}^-(\psi), \xi_{B_R}^-\psi(\omega)) \leq s_i = \xi_{B_R}^-\psi(\omega) \alpha \omega)$. Therefore, $\xi_{B_R}^+(\psi \alpha \omega) = \min(\xi_{B_R}^+(\psi), \xi_{B_R}^-\psi(\omega)) \alpha \omega) \alpha \omega) \leq \xi_{B_R}^-\psi(\omega) \alpha \omega)$. Hence, $B_R$ is a BFF of $M_R$.

**Theorem 3.2.** A BFS $B_R = (\xi_{B_R}^+, \xi_{B_R}^-)$ of a GNR $M_R$ is a BFPI of $M_R$ if and only if its level subset $B_R(t_i, s_i) \neq \emptyset$ is a prime ideal of $M_R$ for any $t_i \in [0, 1], s_i \in [-1, 0]$.

**Proof.** The proof is in the same way as Theorem 3.1. 

**Theorem 3.3.** Let $F$ be a non-empty subset of a GNR $M_R$. Then the characteristic set of $F$, $B_F = (\chi_{B_F}^+, \chi_{B_F}^-)$ is a BFF of $M_R$ if and only if $F$ is a filter of $M_R$.

**Proof.** Let $B_F = (\chi_{B_F}^+, \chi_{B_F}^-)$ is a BFF of $M_R$.

Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$ be such that $\psi \alpha \omega \in F$. Since $B_F$ is a BFF of $M_R$, we have $\chi_{B_F}^+(\psi \alpha \omega) = \min(\chi_{B_F}^+(\psi), \chi_{B_F}^+(\omega)) \alpha \omega) = 1$. Thus $\chi_{B_F}^+(\psi) = 1$ and $\chi_{B_F}^+(\omega) = 1$. Also, $\chi_{B_F}^-(\psi \alpha \omega) = \max(\chi_{B_F}^-(\psi), \chi_{B_F}^-(\omega)) \alpha \omega) \alpha \omega) = -1$. Thus $\chi_{B_F}^-(\psi) = -1$ and $\chi_{B_F}^-(\omega) = -1$. Therefore, $\psi, \omega \in F$. Hence $F$ is a filter of $M_R$. 

Conversely, suppose that $F$ is a filter of $M_R$. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. (i) Let $\psi \omega \notin F$. Then $\psi \notin F$ or $\omega \notin F$. Thus $\chi^+_B (\psi \omega) = 0$ and $\chi^+_B (\psi) = 0$ or $\chi^+_B (\omega) = 0$. So, $\chi^+_B (\psi \omega) = \min (\chi^+_B (\psi), \chi^+_B (\omega))$.

Also, $\chi^-_B (\psi \omega) = 0$ and $\chi^-_B (\psi) = 0$ or $\chi^-_B (\omega) = 0$. Thus $\chi^-_B (\psi \omega) = \max (\chi^-_B (\psi), \chi^-_B (\omega))$. (ii) Let $\psi \omega \in F$. Then $\psi, \omega \in F$. Thus $\chi^+_B (\psi \omega) = 1, \chi^+_B (\psi) = 1, \chi^+_B (\omega) = 1$. So, $\chi^+_B (\psi \omega) = \min (\chi^+_B (\psi), \chi^+_B (\omega))$. Also, $\chi^-_B (\psi \omega) = -1, \chi^-_B (\psi) = -1, \chi^-_B (\omega) = -1$. Thus $\chi^-_B (\psi \omega) = \max (\chi^-_B (\psi), \chi^-_B (\omega))$. Hence, $B_F = (\chi^+_B, \chi^-_B)$ is a BFF of $M_R$. □

**Theorem 3.4.** Let $S$ be a non-empty subset of a GNR $M_R$. Then the characteristic set of $S$, $B_S = (\chi^+_B, \chi^-_B)$ is a BFPI of $M_R$ if and only if $S$ is a prime ideal of $M_R$.

**Proof.** The proof is in the same way as Theorem 3.3. □

**Theorem 3.5.** Let $A_R = (\xi^+_A, \xi^-_A)$ and $B_R = (\xi^+_B, \xi^-_B)$ be BFFs of an OGNR $M_R$. Then $A_R \cap B_R$ is a BFF of $M_R$.

**Proof.** Let $A_R = (\xi^+_A, \xi^-_A)$ and $B_R = (\xi^+_B, \xi^-_B)$ be BFFs of $M_R$. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. Then

\[(\xi^+_A \cap \xi^+_B) (\psi - \omega) = \min [\xi^+_A (\psi - \omega), \xi^+_B (\psi - \omega)] \leq \min [\max [\xi^+_A (\psi), \xi^+_B (\omega)], \max [\xi^+_A (\psi), \xi^+_B (\omega)]] = \max [\min [\xi^+_A (\psi), \xi^+_B (\psi)], \min [\xi^+_A (\omega), \xi^+_B (\omega)]] \]

Thus $\xi^-_A (\psi - \omega) = \min [\xi^+_A (\psi, \xi^+_B (\psi)], \xi^+_B (\psi, \xi^+_B (\psi))]

and $\xi^-_B (\psi - \omega) = \min [\xi^-_A (\psi), \xi^-_B (\omega), \min [\xi^-_A (\psi), \xi^-_B (\omega)]]

if $\psi \leq \omega$, then $\xi^-_A (\psi) \leq \xi^-_A (\omega)$ and $\xi^-_B (\psi) \leq \xi^-_B (\omega)$. Thus

\[(\xi^+_A \cap \xi^+_B) (\psi) = \min [\xi^+_A (\psi), \xi^+_B (\psi)] \]

and

\[(\xi^-_A \cap \xi^-_B) (\psi) = \min [\xi^-_A (\psi), \xi^-_B (\psi)] \]

Similarly, we can prove that

\[(\xi^-_A \cap \xi^-_B) (\psi - \omega) \geq \min [\xi^-_A (\psi - \omega), \xi^-_B (\psi - \omega)] \]

\[\psi \leq \omega \Rightarrow (\xi^-_A \cap \xi^-_B) (\psi) \geq (\xi^-_A \cap \xi^-_B) (\omega).\]

Hence, $A_R \cap B_R$ is a BFF of $M_R$. □

**Theorem 3.6.** A BFS $B_R = (\xi^+_B, \xi^-_B)$ of an OGNR $M_R$ is a BFF of $M_R$ if and only if its level subset $B_R(t,s) \neq \emptyset$ is a filter of $M_R$ for any $t_1 \in [0,1], s_1 \in [-1,0]$. 


Proof. The proof is in the same way as Theorem 3.1. ∎

Let \( M_R \) be an OGNR, \( a_l \in M_R \), and \( B_R = (\xi_{B_R}^+, \xi_{B_R}^-) \) be a BFF of \( M_R \). Then the set \( \{ \psi \in M_R \mid \xi_{B_R}^+(a_l) \leq \xi_{B_R}^+(\psi) \text{ and } \xi_{B_R}^-(a_l) \geq \xi_{B_R}^-(\psi) \} \) is denoted by \( F_{B(a_l)} \).

**Theorem 3.7.** Let \( B_R = (\xi_{B_R}^+, \xi_{B_R}^-) \) be a BFF of an OGNR \( M_R \). Then the set \( F_{B(a_l)} \) is a filter of \( M_R \).

Proof. Let \( B_R = (\xi_{B_R}^+, \xi_{B_R}^-) \) be a BFF of \( M_R \). Let \( \psi, \omega \in M_R \) and \( \alpha \in \Gamma \) be such that \( \psi \rho \omega \in F_{B(a_l)} \). Then

\[
\xi_{B_R}^+(a_l) \leq \xi_{B_R}^+(\psi \rho \omega) \implies \xi_{B_R}^+(a_l) \leq \min\{\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)\}
\]

\[
\implies \xi_{B_R}^+(a_l) \leq \xi_{B_R}^+(\psi), \xi_{B_R}^+(a_l) \leq \xi_{B_R}^+(\omega),
\]

Thus \( \psi, \omega \in F_{B(a_l)} \). Let \( \omega \in M_R, \psi \in F_{B(a_l)} \) be such that \( \psi \leq \omega \). Then

\[
\xi_{B_R}^+(a_l) \leq \xi_{B_R}^+(\psi), \xi_{B_R}^+(\psi) \leq \xi_{B_R}^+(\omega) \implies \xi_{B_R}^+(a_l) \leq \xi_{B_R}^+(\omega),
\]

Thus \( \omega \in F_{B(a_l)} \). Hence, \( F_{B(a_l)} \) is a filter of \( M_R \). ∎

**Definition 3.3.** Let a function \( \phi : M_R \to N_R \) be a homomorphism of OGNRs \( M_R \) and \( N_R \). A BFS \( B_R = (\xi_{B_R}^+, \xi_{B_R}^-) \) of \( M_R \) is said to be a \( \phi \) homomorphism invariant if \( \phi(\psi) \leq \phi(\omega) \), then \( \xi_{B_R}^+(\psi) \leq \xi_{B_R}^+(\omega) \) and \( \xi_{B_R}^-(\psi) \geq \xi_{B_R}^-(\omega) \) for any \( \psi, \omega \in M_R \).

**Definition 3.4.** Let \( M_R \) and \( N_R \) be two OGNRs and \( \phi : M_R \to N_R \) be an onto homomorphism. If \( f = (\xi_f^+, \xi_f^-) \) is a BFS of \( M_R \), then the image of \( f \) under \( \phi \), denoted by \( \phi(f) = (\phi(\xi_f^+), \phi(\xi_f^-)) \), is the BFS of \( N_R \), defined by \( \phi(\xi_f^+) = \inf_{t \in \phi^{-1}(\psi)} \xi_f^+(t) \) and \( \phi(\xi_f^-) = \sup_{t \in \phi^{-1}(\psi)} \xi_f^-(t) \) for all \( \psi \in N_R \).

**Theorem 3.8.** Let \( \phi : M_R \to N_R \) be an onto homomorphism of OGNRs \( M_R \) and \( N_R \). If \( f = (\xi_f^+, \xi_f^-) \) is a \( \phi \) homomorphism invariant BFS of \( M_R \), then \( \phi(f) \) is a BFS of \( N_R \).

Proof. Let \( f = (\xi_f^+, \xi_f^-) \) be a \( \phi \) homomorphism invariant BFF of \( M_R \). Suppose \( \psi \in N_R, t_i \in \phi^{-1}(\psi) \), and \( \psi = \phi(a_l) \). Then \( a_l \in \phi^{-1}(\psi) \), so \( \phi(t_i) = \psi = \phi(a_l) \). Thus \( \phi(a_l b_l) = \phi(a_l) \phi(b_l) = \psi \rho \omega \). So, \( \phi(\xi_f^+)(\psi \rho \omega) = \xi_f^+(a_l b_l) = \min\{\xi_f^+(a_l), \xi_f^+(b_i)\} = \min\{\phi(\xi_f^+)\}(\psi), \psi(\xi_f^+(\omega)) \} \) and \( \phi(\xi_f^-)(\psi \rho \omega) = \xi_f^-(a_l b_l) = \max\{\xi_f^-(a_l), \xi_f^-(b_i)\} = \max\{\phi(\xi_f^-)\}(\psi), \psi(\xi_f^-(\omega)) \} \). Since \( f \) is a \( \phi \) homomorphism invariant, we have

\[
\xi_f^+(t_i) = \xi_f^+(a_l) \implies \phi(\xi_f^+) = \inf_{t_i \in \phi^{-1}(\psi)} \xi_f^+(t_i) = \xi_f^+(a_l)
\]

\[
\phi(\xi_f^+) = \xi_f^+(a_l),
\]

\[
\xi_f^-(t_i) = \xi_f^-(a_l) \implies \phi(\xi_f^-) = \sup_{t_i \in \phi^{-1}(\psi)} \xi_f^-(t_i) = \xi_f^-(a_l)
\]

\[
\phi(\xi_f^-) = \xi_f(a_l).
\]
Let $\psi, \omega \in N_R$. Then there exist $a_l, b_l \in M_R$ such that $\phi(a_l) = \psi, \phi(b_l) = \omega$, so $\phi(a_l - b_l) = \psi - \omega$. Thus

$$\phi(\xi^+_f)(\psi - \omega) = \xi^+_f(a_l - b_l)$$
$$\leq \max\{\xi^+_f(a_l), \xi^+_f(b_l)\}$$
$$= \max\{\phi(\xi^+_f)(\psi), \phi(\xi^+_f)(\omega)\},$$

$$\phi(\xi^-_f)(\psi - \omega) = \xi^-_f(a_l - b_l)$$
$$\geq \min\{\xi^-_f(a_l), \xi^-_f(b_l)\}$$
$$= \min\{\phi(\xi^-_f)(\psi), \phi(\xi^-_f)(\omega)\}.$$

Let $\psi, \omega \in N_R$ and $\psi \leq \omega$. Then there exist $a_l, b_l \in M_R$ such that $\phi(a_l) = \psi, \phi(b_l) = \omega$. Thus $\phi(\xi^+_f)(\psi) = \xi^+_f(a_l), \phi(\xi^+_f)(\omega) = \xi^+_f(b_l), \phi(\xi^-_f)(\psi) = \xi^-_f(a_l)$, and $\phi(\xi^-_f)(\omega) = \xi^-_f(b_l)$. Thus

$$\psi \leq \omega \Rightarrow \phi(\psi) \leq \phi(\omega)$$
$$\Rightarrow \xi^+_f(a_l) \leq \xi^+_f(b_l)$$
$$\Rightarrow \phi(\xi^+_f)(\psi) \leq \phi(\xi^+_f)(\omega),$$

$$\psi \leq \omega \Rightarrow \phi(\psi) \leq \phi(\omega)$$
$$\Rightarrow \xi^-_f(a_l) \geq \xi^-_f(b_l)$$
$$\Rightarrow \phi(\xi^-_f)(\psi) \geq \phi(\xi^-_f)(\omega).$$

Hence, $\phi(f)$ is a BFF of $N_R$.

**Theorem 3.9.** Let $f : M_R \to N_R$ be a homomorphism of OGNRs $M_R$ and $N_R$ and $A_R = (v^+_A, v^-_A)$ be a BFF of $N_R$. If $A_R \circ f = B_R$ such that $B_R = (\xi^+_B, \xi^-_B)$, then $B_R$ is a BFF of $M_R$.

**Proof.** Assume that $A_R \circ f = B_R$ such that $B_R = (\xi^+_B, \xi^-_B)$. Let $\psi, \omega \in M_R$ and $\alpha \in \Gamma$. Then

$$\xi^+_B(\psi - \omega) = v^+_A(f(\psi) - f(\omega))$$
$$= v^+_A(f(\psi) - f(\omega))$$
$$\leq \max\{v^+_A(f(\psi)), v^+_A(f(\omega))\}$$
$$= \max\{\xi^+_B(\psi), \xi^+_B(\omega)\}.$$

$$\xi^-_B(\psi - \omega) = v^-_A(f(\psi) - f(\omega))$$
$$= v^-_A(f(\psi) - f(\omega))$$
$$\geq \min\{v^-_A(f(\psi)), v^-_A(f(\omega))\}$$
$$= \min\{\xi^-_B(\psi), \xi^-_B(\omega)\},$$
\[
\xi_{B_R}^+(\psi \alpha \omega) = v_{A_R}^+(f(\psi \alpha \omega)) \\
= v_{A_R}^+(f(\psi) \alpha f(\omega)) \\
= \min\{v_{A_R}^+(f(\psi)), v_{A_R}^+(f(\omega))\} \\
= \min\{\xi_{B_R}^+(\psi), \xi_{B_R}^+(\omega)\},
\]

\[
\xi_{B_R}^-(\psi \alpha \omega) = v_{A_R}^-(f(\psi \alpha \omega)) \\
= v_{A_R}^-(f(\psi) \alpha f(\omega)) \\
= \max\{v_{A_R}^-(f(\psi)), v_{A_R}^-(f(\omega))\} \\
= \max\{\xi_{B_R}^-(\psi), \xi_{B_R}^-(\omega)\}.
\]

Let \(\psi, \omega \in M_R\) be such that \(\psi \leq \omega\). Since \(f : M_R \to N_R\) is a homomorphism, we have \(f(\psi) \leq f(\omega)\). Thus \(v_{A_R}^+(f(\psi)) \leq v_{A_R}^+(f(\omega))\), so \(\xi_{B_R}^+(\psi \alpha \omega) \leq \xi_{B_R}^+(\omega)\). Also, \(v_{A_R}^-(f(\psi)) \geq v_{A_R}^-(f(\omega))\), so \(\xi_{B_R}^-(\psi \alpha \omega) \geq \xi_{B_R}^-(\omega)\). Hence, \(B_R\) is a BFF of \(M_R\).

**Definition 3.5.** Let \(M_R\) and \(N_R\) be two OGNRs and \(f : M_R \to N_R\) be a function. If \(B_R = (\xi_{B_R}^+, \xi_{B_R}^-)\) is a BFS of \(N_R\), then the pre-image of \(B_R\) under \(f\), denoted by \(f^{-1}(B_R) = (f^{-1}(\xi_{B_R}^+), f^{-1}(\xi_{B_R}^-))\), is the BFS of \(M_R\), defined by \(f^{-1}(\xi_{B_R}^+) = \xi_{B_R}^+(f(\psi))\) and \(f^{-1}(\xi_{B_R}^-) = \xi_{B_R}^-(f(\psi))\) for all \(\psi \in M_R\).

**Theorem 3.10.** Let \(f : M_R \to N_R\) be an onto homomorphism of OGNRs \(M_R\) and \(N_R\). If \(B_R = (\xi_{B_R}^+, \xi_{B_R}^-)\) is a BFF of \(N_R\), then \(f^{-1}(B_R)\) is a BFF of \(M_R\).

**Proof.** Let \(B_R = (\xi_{B_R}^+, \xi_{B_R}^-)\) be a BFF of \(N_R\). Let \(\psi, \omega \in M_R\) and \(\alpha \in \Gamma\). Then

\[
f^{-1}(\xi_{B_R}^+)(\psi - \omega) = \xi_{B_R}^+(f(\psi - \omega)) \\
= \xi_{B_R}^+(f(\psi) - f(\omega)) \\
\leq \max\{\xi_{B_R}^+(f(\psi)), \xi_{B_R}^+(f(\omega))\} \\
= \max\{f^{-1}(\xi_{B_R}^+)(\psi), f^{-1}(\xi_{B_R}^+)(\omega)\},
\]

\[
f^{-1}(\xi_{B_R}^-)(\psi - \omega) = \xi_{B_R}^-(f(\psi - \omega)) \\
= \xi_{B_R}^-(f(\psi) - f(\omega)) \\
\geq \min\{\xi_{B_R}^-(f(\psi)), \xi_{B_R}^-(f(\omega))\} \\
= \min\{f^{-1}(\xi_{B_R}^-)(\psi), f^{-1}(\xi_{B_R}^-)(\omega)\},
\]

\[
f^{-1}(\xi_{B_R}^+)(\psi \alpha \omega) = \xi_{B_R}^+(f(\psi \alpha \omega)) \\
= \xi_{B_R}^+(f(\psi) \alpha f(\omega)) \\
= \min\{\xi_{B_R}^+(f(\psi)), \xi_{B_R}^+(f(\omega))\} \\
= \min\{f^{-1}(\xi_{B_R}^+)(\psi), f^{-1}(\xi_{B_R}^+)(\omega)\},
\]
\[ f^{-1}(\xi_{BR}^+)(\psi\alpha\omega) = \xi_{BR}^-(f(\psi\alpha\omega)) = \xi_{BR}^-(f(\psi)\alpha f(\omega)) = \max\{\xi_{BR}^-(f(\psi)), \xi_{BR}^-(f(\omega))\} = \max\{f^{-1}(\xi_{BR}^-(\psi)), f^{-1}(\xi_{BR}^-(\omega))\}. \]

Let \( \psi, \omega \in M_R \) be such that \( \psi \leq \omega \). Since \( f : M_R \to N_R \) is a homomorphism, we have \( f(\psi) \leq f(\omega) \). Thus \( \xi_{BR}^+(f(\psi)) \leq \xi_{BR}^+(f(\omega)) \), so \( f^{-1}(\xi_{BR}^+(\psi)) \leq f^{-1}(\xi_{BR}^+(\omega)) \). Also, \( \xi_{BR}^-(f(\psi)) \geq \xi_{BR}^-(f(\omega)) \), so \( f^{-1}(\xi_{BR}^-(\psi)) \geq f^{-1}(\xi_{BR}^-(\omega)) \). Hence, \( f^{-1}(BR) \) is a BFF of \( M_R \).

4. Conclusion

In this paper, we inspected the notations of BFFs and BFPIs of GNRs and OGNRs and studied their properties and relations among them. Further, we extended our study to the homomorphic image and pre-image of BFFs of OGNRs.

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