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Mathematical Modeling for a *CHIKV* Transmission Under the Influence of Periodic Environment

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Abstract. We studied a simple mathematical model for the chikungunya virus (*CHIKV*) spread under the influence of a seasonal environment with two routes of infection. We investigated the existence and the uniqueness of a bounded positive solution, and we showed that the system admits a global attractor set. We calculated the basic reproduction number $\mathcal{R}_0$ for the both cases, the fixed and seasonal environment which permits us to characterise both, the extinction and the persistence of the disease with regard to the values of $\mathcal{R}_0$. We proved that the virus-free equilibrium point is globally asymptotically stable if $\mathcal{R}_0 \leq 1$, while the disease will persist if $\mathcal{R}_0 > 1$. Finally, we gave some numerical examples confirming the theoretical findings.

1. INTRODUCTION

Arboviruses constitute a group of viruses which are transmitted to humans or animals by bites from blood-sucking vectors (mosquitoes, ticks and sandflies). Certain viruses have had a renewed medical importance in these two recent decades, notably the Dengue virus, the Yellow Fever virus, the virus Zika disease and Chikungunya virus. The adaptation of the Chikungunya virus to new vectors (Aedes albopictus), the adaptation of these vectors to new environments, and the severe clinical forms associated with these arboviruses mean that they have become emerging and urgent issues around the world, particularly in South America and Europe. Chikungunya and Zika virus infection have had renewed medical interest following massive epidemics which started respectively in Kenya in 2004 and in the Yap Islands (Micronesia) in 2007. Chikungunya and Zika viruses are mainly transmitted to man during a blood meal of Aedes mosquitoes whose entomological surveillance (population of the environment, resistance of mosquitoes to insecticides) remains almost completely absent in Mali compared to Anopheles mosquitoes (malaria vectors). The medical

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interest of Chikungunya is in particular linked to serious forms in newborns (encephalitis, dermatological bullous lesions in particular), severe forms (hepatitis, neurological forms – uncommon), complications linked to comorbidities and finally to long-term rheumatological forms.

The mathematical modeling permits for mathematician to use a set of concepts, methods, mathematical theories that facilitate the description, the understand and the prediction of the evolution of phenomena external to mathematics which make a link between reality and mathematics. For several centuries, mathematics has not only been a tool extremely important for acting on and modifying nature, one of the main pillars of technique and technology, but also (and perhaps above all) a major instrument to understand it. In this sense, they are not only a source of utility but also of "truth". In particular, mathematical modeling is a way for studying the disease, predicting its behavior in the future, and then proposing suitable strategies. Several researchers worked on some mathematical models for several infectious diseases [1–7]. In particular, the modeling of the behavior of CHIKV dynamics was studied in several recent works [8–15].

Note that seasonality in infectious is very repetitive [16]. In particular, each year with the return of cold weather, infectious diseases spread among the population. Although they are often temporary and harmless, they can nevertheless be much more serious, particularly in the weakest people. *CHIKV* epidemics occur in a context marked by seasonal rains and tropical storms which have caused heavy flooding. Seasonal factors such as the monsoon or rainy season affect the development of an epidemic. We then talk about seasonality of *CHIKV*. Climate changes linked to global warming can interact with seasonal climatic factors, particularly through climatic anomalies (drought, floods) and be the cause of significant epidemic outbreaks Several sand simple mathematical models of infectious diseases that take into account of the seasonality were proposed [17–19]. In such mathematical models, the basic reproduction number can be calculated either using the time-averaged system (autonomous) as in [20, 21] or other definition as in [22, 23] where all these definitions are different from the one defined for time-averaged system. In [24], the authors analysed the seasonal behaviour of an SVEIR epidemic model with vaccination. Similarly, in [25–29], the authors studied the seasonal behaviour of some epidemic models related to HIV, chikungunya virus and Typhoid Fever spread.We aim in this paper to study the dynamics of *CHIKV* in relation with phages and hosts when it is considered in both, fixed and seasonal environment and with a nonlinear general incidence rate. We calculated the basic reproduction number as the spectral radius of an integral operator. We analysed the global stability of the disease-free solution where we proved that it is globally asymptotically stable if $\mathcal{R}_0 < 1$. However, $\mathcal{R}_0 > 1$, we proved that the dynamics is persistent and so the disease-present solution converges to a limit cycle. We confirmed the theoretical findings by using an intense numerical examples.

The rest of this article is organized as follows. In Section 2, we present a simple epidemic model of *CHIKV* taking into of the seasonality. In Section 3, we considered firstly the case of fixed environment, and we calculated $\mathcal{R}_0$ and we investigated the global analysis of both, the disease-free and the endemic steady states. However, in section 4, we focus on the stability of virus-free and virus-present periodic trajectories for the case of seasonal environment. Some numerical tests are given in Section 5 confirming the theoretical findings. Finally, in section 6, we give some concluding remarks.

2. *CHIKV* Epidemic Model

We consider a compartmental mathematical model for the dynamics of a *CHIKV*. Let us denote by $X_s(t)$, $X_i(t)$, $X_v(t)$ and $X_p(t)$ the quantities of susceptible hosts, infected hosts, *CHIKV* and phages, respectively. Therefore, the model is given by the fourth dimensional system of differential equations hereafter.

$$\begin{cases} \dot{X}_s(t) &=& d(t)\Theta_1(t) - \tau_1(t)X_i(t)X_s(t) - \tau_2(t)X_v(t)X_s(t) - d(t)X_s(t), \\ \dot{X}_i(t) &=& \tau_1(t)X_i(t)X_s(t) + \tau_2(t)X_v(t)X_s(t) - d(t)X_i(t), \\ \dot{X}_v(t) &=& \xi(t)X_i(t) - \mu(t)X_v(t) - \tau_3(t)X_v(t)X_p(t), \\ \dot{X}_p(t) &=& m(t)\Theta_2(t) + \varrho(t)\tau_3(t)X_v(t)X_p(t) - m(t)X_p(t). \end{cases} \tag{2.1}$$

with initial conditions given by $(X_s(0), X_i(0), X_v(0), X_p(0)) \in \mathbb{R}^4_+$. The susceptible hosts have a periodic recruited rate $d(t)\Theta_1(t)$, and a periodic death rate $d(t)$ and a periodic incidence rate $\tau_1(t)X_s(t)X_i(t) + \tau_2(t)X_s(t)X_v(t)$, where $\tau_1(t)$ and $\tau_2(t)$ are the periodic contact rates. The periodic parameters $\mu(t)$ and $m(t)$ describe the periodic death rates of the *CHIKV* and the phages, respectively. $\xi(t)$ is the periodic production rate from infected hosts to *CHIKV*. The phages have a periodic proliferation rate given by $m(t)\Theta_2(t) + \varrho(t)\tau_3(t)X_v(t)X_p(t)$. More details concerning the significance of the model parameters are given in Table 1.

Notation	**Definition**	**Notation**	**Definition**
$X_s(t)$	Concentration of susceptible hosts	Θ_1	Hosts recruitment rate
$X_i(t)$	Concentration of infected hosts	Θ_2	Phage recruitment rate
$X_v(t)$	Concentration of *CHIKV*	μ	Death rate of viruses
$X_p(t)$	Concentration of phages	d	Death rate of hosts
$\tau_1(t)$	Susceptible-infected contact rate	m	Death rate of phages
$\tau_2(t)$	Susceptible-virus contact rate	ϱ	Burst size of phages per virus
$\tau_3(t)$	Virus-phage contact rate	ξ	Proliferation rate of viruses

Table 1. Parameters and variables of system (2.1).

The model parameters satisfy the following assumption:

Assumption 2.1. *The functions* $\Theta_1(t), d(t), \xi(t), m(t), \mu(t),,$ $\tau_1(t)$, $\tau_2(t)$ *and* $\tau_3(t)$ *are non-negative continuous bounded and T-periodic.*

3. Case of Fixed Environment

In this section, we assume that all parameters are positive constant reflecting the case of fixed environment. Therefore, we obtain the the autonomous form of the dynamics (2.1).

$$\begin{cases} \dot{X}_s(t) &= d\Theta_1 - \tau_1 X_i(t)X_s(t) - \tau_2 X_v(t)X_s(t) - dX_s(t), \\ \dot{X}_i(t) &= \tau_1 X_i(t)X_s(t) + \tau_2 X_v(t)X_s(t) - dX_i(t), \\ \dot{X}_v(t) &= \xi X_i(t) - \mu X_v(t) - \tau_3 X_v(t)X_p(t), \\ \dot{X}_p(t) &= m\Theta_2 + \varrho\tau_3 X_v(t)X_p(t) - mX_p(t). \end{cases} \tag{3.1}$$

with initial conditions $(X_s(0), X_i(0), X_v(0), X_p(0)) \in \mathbb{R}^4_+$.

3.1. **Basic properties.** In this subsection, we give some classical properties for epidemiological models. Let $\sigma = \min(\mu, m)$, then we obtain the following results.

Lemma 3.1. *The bounded set* $\Gamma_1 = \{(X_s, X_i, X_v, X_p) \in \mathbb{R}^4_+ : \ X_s + X_i \le \Theta_1, \varrho X_v + X_p \le \Theta_2 + \frac{\xi\varrho}{\sigma}\Theta_1\}$ *is a positively invariant and attractor of the dynamics (3.1).*

Proof. Assume that $X_s = 0$ then $\dot{X}_s = d\Theta_1 > 0$. Assume that $X_i = 0$ then $\dot{X}_i = \tau_2 X_s X_v \ge 0$. Assume that $X_v = 0$ then $\dot{X}_p = \xi X_i \ge 0$. Assume that $X_p = 0$ then $\dot{X}_p = m\Theta_2 > 0$. Consider $T_1(t) = X_s(t) + X_i(t) - \Theta_1$ and $T_2(t) = \varrho X_v(t) + X_p(t) - \frac{\xi\varrho\Theta_1}{\sigma} - \Theta_2$. Then, one has $\dot{T}_1(t) \le d\Theta_1 - d(X_s(t) + X_i(t)) = -dT_1(t)$. Hence, $T_1(t) \le T_1(0)e^{-dt}$. Then, $T_1(t) \le 0$ if $T_1(0) \le 0$. Similarly, one has

$$\dot{T}_2(t) = \varrho\xi X_i(t) - \varrho\mu X_v(t) + m\Theta_2 - mX_p(t) \le \varrho\xi\Theta_1 - \sigma\Big(\varrho X_v(t) + X_p(t) - \Theta_2\Big) = -\sigma T_2(t).$$

Then $T_2(t) \le T_2(0)e^{-\sigma t}$. Hence, $T_2(t) \le 0$ if $T_2(0) \le 0$. Thus, Γ_1 is an invariant set for the dynamics (2.1) since all compartments are non-negative. □

3.2. **Basic reproduction number and steady states.** As our model has several compartments, the next-generation matrix method [30–32] will be used to calculate the basic reproduction number as follows.
$F = \begin{pmatrix} \tau_1\Theta_1 & \tau_2\Theta_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} d & 0 & 0 \\ -\xi & \mu + \tau_3\Theta_2 & 0 \\ 0 & 0 & m \end{pmatrix}$. Then, the next-generation matrix is given by $FV^{-1} = \begin{pmatrix} \frac{\tau_1\Theta_1}{d} + \frac{\xi\tau_2\Theta_1}{d(\mu+\tau_3\Theta_2)} & \frac{\tau_2\Theta_1}{(\mu+\tau_3\Theta_2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, the spectral radius of FV^{-1} which is the basic reproduction number is expressed by:

$$\mathcal{R}_0 = \frac{\tau_1\Theta_1}{d} + \frac{\xi\tau_2\Theta_1}{d(\mu+\tau_3\Theta_2)} = \frac{(\mu+\tau_3\Theta_2)\tau_1 + \xi\tau_2}{d(\mu+\tau_3\Theta_2)}\Theta_1. \tag{3.2}$$

Lemma 3.2.

- *If $\mathcal{R}_0 \le 1$, then (3.1) admits only $E_0 = (\Theta_1, 0, 0, \Theta_2)$ as a steady state.*

- *If* $\mathcal{R}_0 > 1$, *then the autonomous dynamics (3.1) admits two steady states;* E_0 *and an endemic steady state* $E_* = (X_s^*, X_i^*, X_v^*, X_p^*)$.

Proof. Consider $E = (X_s, X_i, X_v, X_p)$ to be a steady state then it satisfies:

$$\begin{aligned} 0 &= d\Theta_1 - \tau_1 X_s X_i - \tau_2 X_s X_v - dX_s, \\ 0 &= \tau_1 X_s X_i + \tau_2 X_s X_v - dX_i, \\ 0 &= \xi X_i - \mu X_v - \tau_3 X_p X_v, \\ 0 &= m\Theta_2 + \varrho\tau_3 X_p X_v - mX_p. \end{aligned} \tag{3.3}$$

From Eq (3.3) we obtain the *CHIKV*-free steady state $E_0 = (\Theta_1, 0, 0, \Theta_2)$. Furthermore, we have

$$\begin{cases} X_p &= \dfrac{m\Theta_2}{m - \varrho\tau_3 X_v}, \\ X_i &= \dfrac{\mu X_v + \tau_3 X_v X_p}{\xi} = \dfrac{\mu}{\xi} X_v + \dfrac{m\Theta_2 \tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)}, \\ X_s &= \Theta_1 - X_i = \Theta_1 - \dfrac{\mu}{\xi} X_v - \dfrac{m\Theta_2 \tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)}, \\ dX_i &= \tau_1 X_s X_i + \tau_2 X_s X_v. \end{cases} \tag{3.4}$$

We define the function

$$\begin{aligned} g(X_v) &= \frac{\tau_1 X_s X_i + \tau_2 X_s X_v - dX_i}{X_v} \\ &= \left(\frac{\mu\tau_1}{\xi} + \frac{m\tau_1\tau_3\Theta_2}{\xi(m - \varrho\tau_3 X_v)} + \tau_2\right)\left(\Theta_1 - \frac{\mu}{\xi} X_v - \frac{m\Theta_2\tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)}\right) - \frac{\mu d}{\xi} - \frac{md\Theta_2\tau_3}{\xi(m - \varrho\tau_3 X_v)}. \end{aligned} \tag{3.5}$$

Then, we obtain

$$\begin{aligned} \lim_{X_v \to 0^+} g(X_v) &= \left(\tau_1 \frac{\mu}{\xi} + \tau_2\right)\Theta_1 - \frac{\mu d}{\xi} - \frac{d\tau_3\Theta_2}{\xi} \\ &= \frac{\mu d}{\xi}\left(\frac{\tau_1\Theta_1}{d} + \frac{\xi\tau_2\Theta_1}{\mu d} + \frac{\tau_3\Theta_2}{\mu} - 1\right) \\ &= \frac{\mu d}{\xi}(\mathcal{R}_0 - 1) > 0 \quad \text{if} \quad \mathcal{R}_0 > 1. \end{aligned} \tag{3.6}$$

$\lim\limits_{X_v \to \left(\frac{m}{\varrho\tau_3}\right)^-} \dfrac{md\Theta_2\tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)} = -\infty$ then, $\lim\limits_{X_v \to \left(\frac{m}{\varrho\tau_3}\right)^-} g(X_v) < 0$. The derivative of the function g is given by

$$\begin{aligned} g'(X_v) &= \left[\tau_1 \frac{\left(\frac{\mu}{\xi} X_v + \frac{m^2\Theta_2\tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)^2}\right) - \left(\frac{\mu}{\xi} X_v + \frac{m\Theta_2\tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)}\right)}{X_v^2}\right] \times \left(\Theta_1 - \frac{\mu}{\xi} X_v - \frac{m\Theta_2\tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)}\right) \\ &\quad - \left(\tau_1 \frac{\left(\frac{\mu}{\xi} X_v + \frac{m\Theta_2\tau_3 X_v}{\xi(m - \varrho\tau_3 X_v)}\right)}{X_v} + \tau_2\right) \times \left(\frac{\mu}{\xi} + \frac{m^2\Theta_2\tau_3}{\xi(m - \varrho\tau_3 X_v)^2}\right) - \frac{\varrho m d\Theta_2\tau_3^2}{\xi(m - \varrho\tau_3 X_v)^2} \\ &\leq -\left(\tau_1\left(\frac{\mu}{\xi} + \frac{m\Theta_2\tau_3}{\xi(m - \varrho\tau_3 X_v)}\right) + \tau_2\right) \times \left(\frac{\mu}{\xi} + \frac{m\Theta_2\tau_3}{\xi}\frac{m}{(m - \varrho\tau_3 X_v)^2}\right) - \frac{\varrho m d\Theta_2\tau_3^2}{\xi(m - \varrho\tau_3 X_v)^2}. \end{aligned}$$

Therefore, we deduce that $g'(X_v) \leq 0$ for all $X_v \in (0, \dfrac{m}{\varrho\tau_3})$. Then, the function $g(X_v)$ admits a unique root $X_v^* \in (0, \dfrac{m}{\varrho\tau_3})$. Therefore, one obtains

$$\left\{\begin{array}{lcl} X_p^* & = & \dfrac{m\Theta_2}{m-\varrho\tau_3 X_v^*}, \\ X_i^* & = & \dfrac{\mu}{\xi}X_v^* + \dfrac{m\Theta_2\tau_3 X_v^*}{\xi(m-\varrho\tau_3 X_v^*)}, \\ X_s^* & = & \Theta_1 - \dfrac{\mu}{\xi}X_v^* + \dfrac{m\Theta_2\tau_3 X_v^*}{\xi(m-\varrho\tau_3 X_v^*)}. \end{array}\right. \tag{3.7}$$

Therefore, the infected equilibrium $E_* = (X_s^*, X_i^*, X_v^*, X_p^*)$ exists and is unique if $\mathcal{R}_0 > 1$. □

3.3. **Local analysis.** We aim, in this section, to analyse the local stability of the equilibria of the dynamics (3.1).

Theorem 3.1. *In the case where $\mathcal{R}_0 < 1$, the phage-free E_0 is locally asymptotically stable, and in the case where $\mathcal{R}_0 > 1$, E_0 is unstable.*

Proof. The linearisation of the dynamics (3.1) at the steady state E_0 is:

$$J_0 = \begin{pmatrix} -d & -\tau_1\Theta_1 & -\tau_2\Theta_1 & 0 \\ 0 & \tau_1\Theta_1 - d & \tau_2\Theta_1 & 0 \\ 0 & \xi & -(\mu+\tau_3\Theta_2) & 0 \\ 0 & 0 & \tau_3\Theta_2 & -m \end{pmatrix}.$$

J_0 admits four eigenvalues; $\Theta_1 = -d < 0$ and $\Theta_2 = -m < 0$. Θ_3 and Θ_4 are eigenvalues of the sub-matrix

$$M_0 := \begin{pmatrix} \tau_1\Theta_1 - d & \tau_2\Theta_1 \\ \xi & -(\mu+\tau_3\Theta_2) \end{pmatrix}.$$

The trace of the matrix M_0 is:

$$\begin{array}{rcl} \operatorname{tr}(M_0) & = & \tau_1\Theta_1 - d - (\mu+\tau_3\Theta_2) \\ & \leq & -(\mu+\tau_3\Theta_2) - d\Big(1 - \dfrac{\tau_1\Theta_1}{d} - \dfrac{\tau_2\Theta_1}{d(\mu+\tau_3\Theta_2)}\Big) \\ & = & -(\mu+\tau_3\Theta_2) - d\Big(1-\mathcal{R}_0\Big) \end{array}$$

and the determinant of M_0 is:

$$\begin{array}{rcl} \det(M_0) & = & -(\tau_1\Theta_1 - d)\Big(\mu+\tau_3\Theta_2\Big) - \xi\tau_2\Theta_1 \\ & = & -d\Big(\mu+\tau_3\Theta_2\Big)\Big(\dfrac{\tau_1\Theta_1}{d} - 1 + \dfrac{\xi\tau_2\Theta_1}{d(\mu+\tau_3\Theta_2)}\Big) \\ & = & -d\Big(\mu+\tau_3\Theta_2\Big)\Big(\mathcal{R}_0 - 1\Big). \end{array}$$

Then, E_0 is locally asymptotically stable once $\mathcal{R}_0 < 1$, however, it is unstable once $\mathcal{R}_0 > 1$. □

Theorem 3.2. *If $\mathcal{R}_0 > 1$, therefore E_* is locally asymptotically stable.*

Proof. The linearisation of the dynamics (3.1) at the steady state $E_* = (X_s^*, X_i^*, X_v^*, X_p^*)$ is:

$$J^* = \begin{pmatrix} -d-\tau_1X_i^*-\tau_2X_v^* & -\tau_1X_s^* & -\tau_2X_s^* & 0 \\ \tau_1X_i^*+\tau_2X_v^* & \tau_1X_s^*-d & \tau_2X_s^* & 0 \\ 0 & \xi & -(\mu+\tau_3X_p^*) & -\tau_3X_v^* \\ 0 & 0 & \varrho\tau_3X_p^* & \varrho\tau_3X_v^*-m \end{pmatrix}.$$

The characteristic polynomial is then given by:

$$\begin{aligned} Q(\lambda) &= \begin{vmatrix} -\lambda-d-\tau_1X_i^*-\tau_2X_v^* & -\tau_1X_s^* & -\tau_2X_s^* & 0 \\ \tau_1X_i^*+\tau_2X_v^* & -\lambda+\tau_1X_s^*-d & \tau_2X_s^* & 0 \\ 0 & \xi & -\lambda-(\mu+\tau_3X_p^*) & -\tau_3X_v^* \\ 0 & 0 & \varrho\tau_3X_p^* & -\lambda+\varrho\tau_3X_v^*-m \end{vmatrix} \\ &= \begin{vmatrix} -(\lambda+d) & -(\lambda+d) & 0 & 0 \\ \tau_1X_i^*+\tau_2X_v^* & -\lambda+\tau_1X_s^*-d & \tau_2X_s^* & 0 \\ 0 & \xi & -\lambda-(\mu+\tau_3X_p^*) & -\tau_3X_v^* \\ 0 & 0 & \varrho\tau_3X_p^* & -\lambda+\varrho\tau_3X_v^*-m \end{vmatrix} \\ &= -(\lambda+d)\begin{vmatrix} -\lambda+\tau_1X_s^*-d & \tau_2X_s^* & 0 \\ \xi & -\lambda-(\mu+\tau_3X_p^*) & -\tau_3X_v^* \\ 0 & \varrho\tau_3X_p^* & -\lambda+\varrho\tau_3X_v^*-m \end{vmatrix} \\ &+(\lambda+d)\begin{vmatrix} \tau_1X_i^*+\tau_2X_v^* & \tau_2X_s^* & 0 \\ 0 & -\lambda-(\mu+\tau_3X_p^*) & -\tau_3X_v^* \\ 0 & \varrho\tau_3X_p^* & -\lambda+\varrho\tau_3X_v^*-m \end{vmatrix} \\ &= -(\lambda+d)\Big[(-\lambda+\tau_1X_s^*-d)\Big((-\lambda-(\mu+\tau_3X_p^*))(-\lambda+\varrho\tau_3X_v^*-m) \\ &+\varrho\tau_3X_p^*\tau_3X_v^*\Big)-\xi\tau_2X_s^*(-\lambda+\varrho\tau_3X_v^*-m)\Big]+(\lambda+d)(\tau_1X_i^*+\tau_2X_v^*) \\ &\Big((-\lambda-(\mu+\tau_3X_p^*))(-\lambda+\varrho\tau_3X_v^*-m)+\varrho\tau_3X_p^*\tau_3X_v^*\Big). \end{aligned}$$

The characteristic polynomial $Q(\lambda) = 0$ if, and only if

$$\begin{aligned} \Big[(\lambda+d)(\tau_1X_i^*+\tau_2X_v^*)-(\lambda+d)(\lambda+d-\tau_1X_s^*)\Big]\Big((\lambda+(\mu+\tau_3X_p^*))(\mathrm{X}+m-\varrho\tau_3X_v^*)+\varrho\tau_3X_p^*\tau_3X_v^*\Big) \\ = \xi\tau_2X_s^*(\lambda+d)(\lambda+m-\varrho\tau_3X_v^*) \end{aligned}$$

or if

$$\Big[(\lambda+d)(\tau_1X_i^*+\tau_2X_v^*)+(\lambda+d)(\lambda+d-\tau_1X_s^*)\Big] = \frac{\xi\tau_2X_s^*(\lambda+d)(\lambda+m-\varrho\tau_3X_v^*)}{\Big((\lambda+(\mu+\tau_3X_p^*))(\lambda+m-\varrho\tau_3X_v^*)+\varrho\tau_3X_p^*\tau_3X_v^*\Big)}.$$

Suppose that the eigenvalue λ is with positive real part. Therefore, since $d-\tau_1X_s^* = \dfrac{\tau_2X_s^*X_v^*}{X_i^*}$ and $\dfrac{X_v^*}{X_i^*} \geq \dfrac{\xi}{(\mu+\tau_3X_p^*)}$, then, by considering the left-hand side, we obtain

$$\begin{aligned}\Big|(\lambda+d)(\tau_1X_i^*+\tau_2X_v^*)+(\lambda+d)(X+d-\tau_1X_s^*)\Big| > (d-\tau_1X_s^*)|\lambda+d| \quad &\geq \quad \frac{\tau_2X_s^*X_v^*}{X_i^*}|\lambda+d| \\ &\geq \quad \frac{\xi\tau_2X_s^*}{(\mu+\tau_3X_p^*)}|\lambda+d|\end{aligned}$$

however, by considering the right-hand side, we obtain

$$\begin{aligned}\Big|\frac{\xi\tau_2X_s^*(\lambda+d)(\lambda+m-\varrho\tau_3X_v^*)}{\Big((\lambda+(\mu+\tau_3X_p^*))(\lambda+m-\varrho\tau_3X_v^*)+\varrho\tau_3X_p^*\tau_3X_v^*\Big)}\Big| \quad &< \quad \Big|\frac{\xi\tau_2X_s^*(\lambda+d)(\lambda+m-\varrho\tau_3X_v^*)}{\Big((\lambda+(\mu+\tau_3X_p^*))(\lambda+m-\varrho\tau_3X_v^*)}\Big| \\ &= \quad \xi\tau_2X_s^*\Big|\frac{(\lambda+d)}{\Big((\lambda+(\mu+\tau_3X_p^*))}\Big| \qquad (3.8)\\ &\leq \quad \frac{\xi\tau_2X_s^*}{(\mu+\tau_3X_p^*)}|\lambda+d|.\end{aligned}$$

This is a contradiction and then λ has non-positive real-part and then the endemic equilibrium point E_* should be locally asymptotically stable. □

3.4. **Global analysis.** Our aim, in this section, is to prove the global stability of the equilibria of the dynamics (3.1). Consider the function $G(x)=x-1-\ln x$ that we will use is this section.

Theorem 3.3. *E_0 is a globally asymptotically stable steady state if $\mathcal{R}_0 \leq 1$.*

Proof. Let us define the Lyapunov function $F_0(X_s,X_i,X_v,X_p)$ given by:

$$F_0(X_s,X_i,X_v,X_p)=X_s-\Theta_1-\int_{\Theta_1}^{X_s}\frac{\Theta_1}{v}dv+X_i+\frac{\tau_2\Theta_1}{\mu+\tau_3\Theta_2}\Big(X_v+\frac{\Theta_2}{\varrho}G\Big(\frac{X_p}{\Theta_2}\Big)\Big).$$

Note that $F_0(X_s,X_i,X_v,X_p)>0$ for all $X_s,X_i,X_v,X_p>0$ and $F_0(\Theta_1,0,0,\Theta_2)=0$. Furthermore, we have

$$\begin{aligned}\dot{F}_0 \quad &= \quad \Big(1-\frac{\Theta_1}{X_s}\Big)\Big(d\Theta_1-dX_s-\tau_1X_sX_i-\tau_2X_sX_v\Big)+\tau_1X_sX_i+\tau_2X_sX_v-dX_i \\ &\quad +\frac{\tau_2\Theta_1}{\mu+\tau_3\Theta_2}\Big(\xi X_i-\mu X_v-\tau_3X_vX_p+\frac{1}{\varrho}(1-\frac{\Theta_2}{X_p})(m\Theta_2+\varrho\tau_3X_vX_p-mX_p)\Big) \\ &= \quad \Big(1-\frac{\Theta_1}{X_s}\Big)(d\Theta_1-dX_s)+\tau_1\Theta_1X_i+\tau_2\Theta_1X_v-dX_i \\ &\quad +\frac{\tau_2\Theta_1}{\mu+\tau_3\Theta_2}\Big(\xi X_i+\frac{1}{\varrho}(1-\frac{\Theta_2}{X_p})(m\Theta_2-mX_p)-\mu X_v-\tau_3X_v\Theta_2\Big) \\ &= \quad \Big(1-\frac{\Theta_1}{X_s}\Big)(d\Theta_1-dX_s)+\tau_1\Theta_1X_i+\tau_2\Theta_1X_v-dX_i \\ &\quad +\frac{\tau_2\Theta_1}{\mu+\tau_3\Theta_2}\Big(\xi X_i+\frac{1}{\varrho}(1-\frac{\Theta_2}{X_p})(m\Theta_2-mX_p)\Big)-\frac{\tau_2\Theta_1\tau_3X_v\Theta_2}{\mu+\tau_3\Theta_2} \\ &\leq \quad -d\frac{(X_s-\Theta_1)^2}{X_s}-\frac{\tau_2\Theta_1}{\mu+\tau_3\Theta_2}\frac{m}{\varrho}\frac{(X_p-\Theta_2)^2}{X_p}+d(\mathcal{R}_0-1)X_i.\end{aligned}$$

If $\mathcal{R}_0 \leq 1$, thus $\dot{F}_0 \leq 0$, ∀ $X_i,X_i,X_v,X_p>0$. Let $W_0=\{(X_s,X_i,X_v,X_p):\dot{F}_0=0\}=\{E_0\}$. Using LaSalle's invariance principle [33], one can deduces that E_0 is globally asymptotically stable if $\mathcal{R}_0 \leq 1$. □

Theorem 3.4. *E_* is globally asymptotically stable for the dynamics (3.1) once $\mathcal{R}_0 > 1$.*

Proof. Let us define the Lyapunov function $F_*(X_s, X_i, X_v, X_p)$ given by:

$$F_*(X_s, X_i, X_v, X_p) = X_s - X_s^* - \int_{X_s^*}^{S} \frac{X_s^*}{v} dv + X_i^* G\Big(\frac{X_i}{X_i^*}\Big) + \frac{\tau_1 X_s^*}{\xi} X_v^* G\Big(\frac{X_v}{X_v^*}\Big) + \frac{\tau_1 X_s^*}{\varrho\xi} X_p^* G\Big(\frac{X_p}{X_p^*}\Big).$$

Clearly, $F_*(X_s, X_i, X_v, X_p) > 0$ for all variables $X_s, X_i, X_v, X_p > 0$ and $F_*(X_s^*, X_i^*, X_v^*, X_p^*) = 0$. The derivative of F_* with respect to time is given by:

$$\begin{aligned}
\dot{F}_* \;=\;& \Big(1 - \frac{X_s^*}{X_s}\Big)\Big(d\Theta_1 - \tau_1 X_s X_i - \tau_2 X_v X_s - dX_s\Big) + \Big(1 - \frac{X_i^*}{X_i}\Big)\Big(\tau_1 X_s X_i + \tau_2 X_v X_s - dX_i\Big) \\
&+ \frac{\tau_1 X_s^*}{\xi}\Big(1 - \frac{X_v^*}{X_v}\Big)\Big(\xi X_i - \mu X_v - \tau_3 X_v X_p\Big) + \frac{\tau_1 X_s^*}{\varrho\xi}\Big(1 - \frac{X_p^*}{X_p}\Big)\Big(m\Theta_2 + \varrho\tau_3 X_v X_p - mX_p\Big) \\
=\;& \Big(1 - \frac{X_s^*}{X_s}\Big)(d\Theta_1 - dX_s) + \tau_1 X_i X_s^* + \tau_2 X_v X_s^* - \tau_1 X_i^* X_s - \tau_2 \frac{X_i^*}{X_i} X_s X_v - dX_i + dX_i^* \\
&+ \tau_1 X_s^* X_i - \frac{\tau_1 \mu}{\xi} X_s^* X_v - \tau_1 \frac{X_v^*}{X_v} X_i X_s^* + \frac{\tau_1 \mu}{\xi} X_v^* X_s^* + \frac{\tau_3 \tau_1}{\xi} X_s^* X_v^* X_p - \frac{\tau_1 \tau_3}{\xi} X_s^* X_v X_p^* \\
&+ \frac{\tau_1 X_s^*}{\varrho\xi X_i^*}\Big(1 - \frac{X_p^*}{X_p}\Big)\Big(m\Theta_2 - mX_p\Big)
\end{aligned}$$

Since the steady state E_* satisfies $d\Theta_1 = \tau_1 X_i^* X_s^* + \tau_2 X_v^* X_s^* + dX_s^*$, $\tau_1 X_i^* X_s^* + \tau_2 X_v^* X_s^* = dX_i^*$, $\mu X_v^* = \xi X_i^* - \tau_3 X_v^* X_p^*$, and $m\Theta_2 = mX_p^* - \varrho\tau_3 X_v^* X_p^*$,then we obtain

$$\begin{aligned}
\dot{F}_* =\;& -d\frac{(X_s - X_s^*)^2}{X_s} + \tau_1 X_i^* X_s^* + \tau_2 X_v^* X_s^* - \tau_1 X_i^* X_s^* \frac{X_s^*}{X_s} - \tau_2 X_v^* X_s^* \frac{X_s^*}{X_s} + \tau_1 X_i X_s^* + \tau_2 X_v X_s^* - \tau_1 X_i^* X_s \\
& -\tau_2 X_v \frac{X_i^*}{X_i} X_s - \tau_1 X_i^* X_s^* \frac{X_i}{X_i^*} - \tau_2 X_v^* X_s^* \frac{X_i}{X_i^*} + \tau_1 X_i^* X_s^* + \tau_2 X_v^* X_s^* + \tau_1 X_s^* X_i^* X_i - \tau_1 X_s^* X_i^* \frac{X_v}{X_v^*} \\
& -\frac{\tau_1 \tau_3}{\xi} X_s^* X_v X_p^* - \tau_1 X_s^* X_i^* X_i \frac{X_v^*}{X_v} + \tau_1 X_s^* X_i^* + \frac{\tau_1 \tau_3}{\xi} X_s^* X_v^* X_p^* + \frac{\tau_1 \tau_3}{\xi} X_s^* X_v^* X_p - \frac{\tau_1 \tau_3}{\xi} X_s^* X_v X_p^* \\
& -\frac{m\tau_1 X_s^*}{\varrho\xi} \frac{(X_p - X_p^*)^2}{X_p} - \frac{\tau_1 \tau_3}{\xi} X_s^* X_v^* X_p^* + \frac{\tau_1 \tau_3}{\xi} \frac{X_p^*}{X_p} X_s^* X_v^* X_p^* \\
\leq\;& -d\frac{(X_s - X_s^*)^2}{X_s} - \frac{m\tau_1}{\varrho\xi} \frac{(X_p - X_p^*)^2}{X_p} X_s^* + \tau_1 X_i^* X_s^* \Big(5 - \frac{X_s^*}{X_s} - \frac{X_s}{X_s^*} - \frac{X_v}{X_v^*} - \frac{X_i X_v^*}{X_v X_i^*} - \frac{X_i^*}{X_i}\Big) \\
& +\tau_2 X_v^* X_s^* \Big(4 - \frac{X_s^*}{X_s} - \frac{X_i^*}{X_i} \frac{X_v X_s}{X_v^* X_s^*} - \frac{X_i}{X_i^*} - \frac{X_v^*}{X_v}\Big).
\end{aligned}$$

Using the rule that $\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}$, we get $\frac{1}{5}\Big(\frac{X_s^*}{X_s} + \frac{X_s}{X_s^*} + \frac{X_v}{X_v^*} + \frac{X_i X_v^*}{X_v X_i^*} + \frac{X_i^*}{X_i}\Big) \geq 1$ and $\frac{1}{4}\Big(\frac{X_s^*}{X_s} + \frac{X_i^*}{X_i} \frac{X_v X_s}{X_v^* X_s^*} + \frac{X_i}{X_i^*} + \frac{X_v^*}{X_v}\Big) \geq 1$. Thus, $\dot{F}_* \leq 0$, $\forall\ X_s, X_i, X_v, X_p > 0$ and $\dot{F}_* = 0$ if and only if $X_s = X_s^*, X_i = X_i^*, X_v = X_v^*$ and $X_p = X_p^*$. One can deduce easily that E_* is globally stable by using the LaSalle's invariance principle [33]. □

4. Influence of Periodic Environment

Let return to the main dynamics (2.1) for a seasonal environment. For any continuous, positive T-periodic function $g(t)$, we define $g^u = \max_{t \in [0,T)} g(t)$ and $g^l = \min_{t \in [0,T)} g(t)$.

4.1. **Preliminary.** Let $A(t)$ to be a T-periodic $m \times m$ matrix continuous function that it is irreducible and cooperative. Let $\beta_A(t)$ to be the fundamental matrix with positive entries, solution of

$$\dot{w}(t) = A(t)w(t). \tag{4.1}$$

Let us denote the spectral radius of the matrix $\beta_A(T)$ by $r(\beta_A(T))$. By using the Perron-Frobenius theorem, one can define $r(\beta_A(T))$ to be the principal eigenvalue of $\beta_A(T)$. According to [34], we have:

Lemma 4.1. *[34]. (4.1) admits a positive T-periodic function $x(t)$ such that $w(t) = x(t)e^{at}$ with $a = \dfrac{1}{T}\ln(r(\beta_A(T)))$.*

In order to define the disease-free periodic trajectory of model (2.1), let us consider the subsystem

$$\begin{cases} \dot{X}_s(t) &= d(t)\Theta_1(t) - d(t)X_s(t), \\ \dot{X}_p(t) &= m(t)\Theta_2(t) - m(t)X_p(t). \end{cases} \tag{4.2}$$

with the initial condition $(X_s(0), X_p(0)) \in \mathbb{R}^2_+$. The dynamics (4.2) has a unique T-periodic trajectory $(X^*_s(t), X^*_p(t))$ such that $X^*_s(t) > 0$ and $X^*_p(t) > 0$. This solution is globally attractive in $\mathbb{R}^2_+$; therefore, the dynamics (2.1) admits a unique disease-free periodic trajectory $(X^*_s(t), 0, 0, X^*_p(t))$.

Let us define $\sigma(t) = \min\limits_{t\geq 0}(\mu(t), m(t))$ and then we have

Proposition 4.1. *The compact set*

$$\Gamma_2 = \left\{(X_s, X_i, X_v, X_p) \in \mathbb{R}^4_+ \ / \ X_s + X_i \leq \Theta^u_1; \varrho X_v + X_p \leq \frac{\varrho^u \xi^u}{\sigma^l}\Theta^u_1 + \frac{m^u}{\sigma^l}\Theta^u_2\right\}$$

is a positively invariant and attractor of trajectories of dynamics (2.1) with

$$\begin{aligned} &\lim_{t\to\infty} X_s(t) + X_i(t) - X^*_s(t) = 0, \\ &\lim_{t\to\infty} \varrho(t)X_v(t) + X_p(t) - X^*_p(t) = 0. \end{aligned} \tag{4.3}$$

Proof. Using the dynamics (2.1), we obtain

$$\begin{aligned} \dot{X}_s(t) + \dot{X}_i(t) &= d(t)\Theta_1(t) - d(t)(X_s(t) + X_i(t)) \\ &\leq d(t)\Big(\Theta^u_1 - (X_s(t) + X_i(t))\Big) \leq 0, \text{ if } X_s(t) + X_i(t) \geq \Theta^u_1, \end{aligned}$$

and

$$\begin{aligned} \varrho(t)\dot{X}_v(t) + \dot{X}_p(t) =& \varrho(t)\xi(t)X_i(t) - \varrho(t)\mu(t)X_v(t) + m(t)\Theta_2(t) - m(t)X_p(t) \\ &\leq \varrho^u\xi^u\Theta^u_1 + m^u\Theta^u_2 - \varrho(t)\sigma(t)X_v(t) - \sigma(t)X_p(t) \\ &\leq \varrho^u\xi^u\Theta^u_1 + m^u\Theta^u_2 - \sigma^l(\varrho(t)X_v(t) + X_p(t)). \end{aligned}$$

Let $Z_1(t) = X_s(t) + X_i(t)$ and $Z_2(t) = \varrho(t)X_v(t) + X_p(t)$. For $x_1(t) = Z_1(t) - X_s^*(t), t \geq 0$, it follows that $\dot{x}_1(t) = -d(t)x_1(t)$, and thus $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty}(Z_1(t) - X_s^*(t)) = 0$. By the same way, let $x_2(t) = Z_2(t) - X_p^*(t), t \geq 0$, then $\dot{x}_2(t) \leq -\sigma(t)x_2(t)$, and thus $\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty}(Z_2(t) - X_p^*(t)) = 0$. □

In section 4.2, we aim to define the basic reproduction number; $\mathcal{R}_0$, the disease-free and then its global stability for $\mathcal{R}_0 \leq 1$. Later, in section 4.3, we aim to prove that compartments $X_i(t)$ and $X_v(t)$ persists if $\mathcal{R}_0 > 1$.

4.2. **Disease-free trajectory.** By using the definition of $\mathcal{R}_0$ given by the theory in [23]. For $Y = (X_i, X_v, X_s, X_p)$, let

$$\mathcal{F}(t,Y) = \begin{pmatrix} \tau_1(t)X_i(t)X_s(t) + \tau_2(t)X_v(t)X_s(t) \\ \xi(t)X_i(t) \\ 0 \\ 0 \end{pmatrix}, \mathcal{V}^+(t,Y) = \begin{pmatrix} 0 \\ 0 \\ d(t)\Theta_1(t) \\ m(t)\Theta_2(t) + \varrho(t)\tau_3(t)X_v(t)X_p(t) \end{pmatrix}$$

and $$\mathcal{V}^-(t,Y) = \begin{pmatrix} d(t)X_i(t) \\ \mu(t)X_v(t) + \tau_3(t)X_v(t)X_p(t) \\ \tau_1(t)X_i(t)X_s(t) + \tau_2(t)X_v(t)X_s(t) + d(t)X_s(t) \\ m(t)X_p(t) \end{pmatrix}.$$

Our aim is to satisfy conditions (A1)–(A7) in [23, Section 1]. The dynamics (2.1) can take the form hereafter:

$$\dot{Y} = \mathcal{F}(t,Y) - \mathcal{V}(t,Y) = \mathcal{F}(t,Y) - \mathcal{V}^-(t,Y) + \mathcal{V}^+(t,Y). \tag{4.4}$$

Thus, the first five conditions (A1)–(A5) are satisfied.

The dynamics (4.4) has a disease-free periodic solution $Y^*(t) = (0, 0, X_s^*(t), X_p^*(t))$. Let $f(t,Y(t)) = \mathcal{F}(t,Y) - \mathcal{V}^-(t,Y) + \mathcal{V}^+(t,Y)$ and $M(t) = \left(\dfrac{\partial f_i(t,Y^*(t))}{\partial Y_j}\right)_{3\leq i,j\leq 4}$ where $f_i(t,Y(t))$ and Y_i are the i-th components of $f(t,Y(t))$ and Y, respectively. A simple calculation give us $M(t) = \begin{pmatrix} -d(t) & 0 \\ 0 & -m(t) \end{pmatrix}$ and thus $r(\beta_M(T)) < 1$. Therefore, the trajectory $Y^*(t)$ is linearly asymptotically stable in $\Omega_s = \left\{(0,0,X_s,X_p) \in R_+^4\right\}$. Therefore, the condition (A6) in [23, Section 1] is also fulfilled.

Let us define $\mathbf{F}(t)$ and $\mathbf{V}(t)$ to be two matrices defined by $\mathbf{F}(t) = \left(\dfrac{\partial \mathcal{F}_i(t,Y^*(t))}{\partial Y_j}\right)_{1\leq i,j\leq 2}$ and $\mathbf{V}(t) = \left(\dfrac{\partial \mathcal{V}_i(t,Y^*(t))}{\partial Y_j}\right)_{1\leq i,j\leq 2}$ where $\mathcal{F}_i(t,Y)$ and $\mathcal{V}_i(t,Y)$ are the i-th components of $\mathcal{F}(t,Y)$ and $\mathcal{V}(t,Y)$, respectively. A simple calculation by using (4.4) give us the expressions of matrices $\mathbf{F}(t)$ and $\mathbf{V}(t)$ as the following:

$$\mathbf{F}(t) = \begin{pmatrix} \tau_1(t)X_s^*(t) & \tau_2(t)X_s^*(t) \\ \xi(t) & 0 \end{pmatrix}, \mathbf{V}(t) = \begin{pmatrix} d(t) & 0 \\ 0 & \mu(t) + \tau_3(t)X_p^*(t) \end{pmatrix}.$$

Consider $Z(t_1,t_2)$ to be the two by two matrix solution of the system $\frac{d}{dt}Z(t_1,t_2) = -\mathbf{V}(t_1)Z(t_1,t_2)$ for any $t_1 \geq t_2$, with $Z(t_1,t_1) = I_2$, i.e., the 2×2 identity matrix. Therefore, condition (A7) is also fulfilled.

Denote by C_T the ordered Banach space of T-periodic functions that are defined on $\mathbb{R} \mapsto \mathbb{R}^2$, with the maximum norm $\|.\|_\infty$ and the positive cone $C_T^+ = \{\psi \in C_T : \psi(s) \geq 0, \text{ for any } s \in \mathbb{R}\}$. Consider the linear operator $K : C_T \to C_T$ given by

$$(K\phi)(\omega) = \int_0^\infty Z(\omega,\omega-z)\mathbf{F}(\omega-z)\phi(\omega-z)dz, \quad \forall\omega \in \mathbb{R}, \phi \in C_T \tag{4.5}$$

Therefore, the basic reproduction number, $\mathcal{R}_0$, of dynamics (2.1) is given by $\mathcal{R}_0 = r(K)$.

Thus, the local stability of the disease-free periodic trajectory, $\mathcal{E}_0(t) = (X_s^*(t),0,0,X_p^*(t))$, of the dynamics (2.1) with respect to $\mathcal{R}_0$ is given hereafter.

Theorem 4.1. *[23, Theorem 2.2] The following statements are satisfied:* $\mathcal{R}_0 < 1 \Leftrightarrow r(\beta_{F-V}(T)) < 1$, $\mathcal{R}_0 = 1 \Leftrightarrow r(\beta_{F-V}(T)) = 1$ *and* $\mathcal{R}_0 > 1 \Leftrightarrow r(\beta_{F-V}(T)) > 1$.

Then, $\mathcal{E}_0(t)$ is asymptotically stable if $\mathcal{R}_0 < 1$, however, it is unstable if $\mathcal{R}_0 > 1$.

Theorem 4.2. $\mathcal{E}_0(t)$ *is globally asymptotically stable if* $\mathcal{R}_0 < 1$. *It is unstable if* $\mathcal{R}_0 > 1$.

Proof. By Theorem 4.1, one has $\mathcal{E}_0(t)$ is locally stable if $\mathcal{R}_0 < 1$ however it is unstable if $\mathcal{R}_0 > 1$. Therefore, it remains to satisfy the global attractivity of $\mathcal{E}_0(t)$ once $\mathcal{R}_0 < 1$. Using (4.3) in Proposition 4.1, for any $m_1 > 0$, $\exists\, T_1 > 0$ such that $X_s(t) + X_i(t) \leq X_s^*(t) + m_1$ and $\varrho(t)X_v(t) + X_p(t) \leq X_p^*(t) + m_1$ for $t > T_1$. Therefore, $X_s(t) \leq X_s^*(t) + m_1$ and $X_p(t) \leq X_p^*(t) + m_1$; and

$$\begin{cases} \dot{X}_i(t) & \leq & \tau_1(t)X_i(t)(X_s^*(t)+m_1) + \tau_2(t)X_v(t)(X_s^*(t)+m_1) - d(t)X_i(t), \\ \dot{X}_v(t) & \leq & \xi(t)X_i(t) - \mu(t)X_v(t) - \tau_3(t)X_v(t)(X_p^*(t)+m_1) \end{cases} \tag{4.6}$$

for $t > T_1$. Let $M_2(t)$ be the two by two matrix function given hereafter

$$M_2(t) = \begin{pmatrix} \tau_2(t)(X_s^*(t)+m_1) & \tau_1(t)(X_s^*(t)+m_1) \\ \xi(t) & 0 \end{pmatrix}. \tag{4.7}$$

using the equivalences in Theorem 4.1, one has $r(\varphi_{F-V}(T)) < 1$. By choosing $m_1 > 0$ satisfying $r(\varphi_{F-V+m_1M_2}(T)) < 1$ and we consider the dynamics hereafter,

$$\begin{cases} \dot{\bar{X}}_i(t) & = & \tau_1(t)\bar{X}_i(t)(X_s^*(t)+m_1) + \tau_2(t)\bar{X}_v(t)(X_s^*(t)+m_1) - d(t)\bar{X}_i(t), \\ \dot{\bar{X}}_v(t) & = & \xi(t)\bar{X}_i(t) - \mu(t)\bar{X}_v(t) - \tau_3(t)\bar{X}_v(t)(X_p^*(t)+m_1). \end{cases} \tag{4.8}$$

Using Lemma 4.1, there exists a positive T-periodic function $x_1(t)$ such that $w(t) \leq x_1(t)e^{a_1 t}$ with $w(t) = \begin{pmatrix} X_i(t) \\ X_v(t) \end{pmatrix}$ and $a_1 = \frac{1}{T}\ln\left(r(\varphi_{F-V+m_1M_2}(T))\right) < 0$. Thus, $\lim_{t\to\infty} X_i(t) = 0$ and $\lim_{t\to\infty} X_v(t) = 0$. Furthermore, we have that $\lim_{t\to\infty} X_s(t) - X_s^*(t) = \lim_{t\to\infty} Z_1(t) - X_i(t) - X_s^*(t) = 0$ and $\lim_{t\to\infty} X_p(t) - X_p^*(t) = \lim_{t\to\infty} Z_2(t) - \varrho(t)X_v(t) - X_p^*(t) = 0$. Then, we deduce that the disease-free periodic trajectory $\mathcal{E}_0(t)$ is globally attractive. □

4.3. **Endemic trajectory .** Note that the dynamics (2.1) admits Γ_2 as an invariant compact set. Let $Y_0 = (X_s(0), X_i(0), X_v(0), X_p(0))$ and $Y_1 = (X_s^*(0), 0, 0, X_p^*(0))$. Define $\mathcal{P} : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ to be the Poincaré map related to the dynamics (2.1) with $Y_0 \mapsto u(T, Y^0)$, where $u(t, Y^0)$ is the unique solution of dynamics (2.1) and initial condition $u(0, Y^0) = Y^0 \in \mathbb{R}_+^4$. Let us define

$$\Omega = \left\{(X_s, X_i, X_v, X_p) \in \mathbb{R}_+^4\right\},\ \Omega_0 = Int(\mathbb{R}_+^4) \text{ and } \partial\Omega_0 = \Omega \setminus \Omega_0.$$

Ω and Ω_0 are both positively invariant. $\mathcal{P}$ is point dissipative. Define

$$M_\partial = \left\{(Y_0) \in \partial\Omega_0 : \mathcal{P}^k(Y_0) \in \partial\Omega_0, \text{ for any } k \geq 0\right\}.$$

By using the persistence theory given in [35] (also in [34, Theorem 2.3]), we have

$$M_\partial = \left\{(X_s, 0, 0, X_p),\ X_s \geq 0, X_p \geq 0\right\}. \tag{4.9}$$

It is easy to see that $M_\partial \supseteq \left\{(X_s, 0, 0, X_p),\ X_s \geq 0, X_p \geq 0\right\}$. To prove that $M_\partial \setminus \left\{(X_s, 0, 0, X_p),\ X_s \geq 0, X_p \geq 0\right\} = \emptyset$, consider $(Y_0) \in M_\partial \setminus \left\{(X_s, 0, 0, X_p),\ X_s \geq 0, X_p \geq 0\right\}$. If $X_v(0) = 0$ and $0 < X_i(0)$, then $X_i(t) > 0$ for all $t > 0$. Then $\dot{X}_v(t)_{|t=0} = m(0)X_i(0) > 0$. If $X_v(0) > 0$ and $X_i(0) = 0$, then $X_v(t) > 0$ and $X_s(t) > 0$ for all $t > 0$. Thus, for all $t > 0$, we obtain

$$X_i(t) = \left[X_i(0) + \int_0^t (\tau_1(\omega)S(\omega)X_i(\omega) + \tau_2 S(\omega)X_v(\omega))e^{\int_0^\omega d(u)du} d\omega\right] e^{-\int_0^t d(u)du} > 0$$

for all $t > 0$. This means that $Y(t) \notin \partial\Omega_0$ for $0 < t \ll 1$. Therefore, Ω_0 is positively invariant from which we deduce (4.9). Using the previous discussion, we deduce that there exists one fixed point Y_1 of $\mathcal{P}$ in M_∂. We deduce, therefore, the uniform persistence of the disease as follows.

Theorem 4.3. *Assume that* $\mathcal{R}_0 > 1$. *The dynamics (2.1) admits at least one periodic solution such that there exists* $\varepsilon > 0$ *that satisfies* $\forall\ Y_0 \in \mathbb{R}_+ \times Int(\mathbb{R}_+^2) \times \mathbb{R}_+$ *and* $\liminf_{t\to\infty} X_i(t) \geq \varepsilon > 0$.

Proof. We aim to prove that $\mathcal{P}$ is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$ which permits to prove that the solution of the dynamics (2.1) is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$ by using [35, Theorem 3.1.1]. From Theorem 4.1, we have $r(\varphi_{F-V}(T)) > 1$. Therefore, there exists $\xi > 0$ such that $r(\varphi_{F-V-\xi M_2}(T)) > 1$. Define the system of equations:

$$\begin{cases} \dot{X}_{s\alpha}(t) &= d(t)\Theta_1(t) - d(t)X_{s\alpha}(t) - (\tau_1(t)\alpha + \tau_2(t)\alpha)X_{s\alpha}(t), \\ \dot{X}_{p\alpha}(t) &= m(t)\Theta_2(t) - m(t)X_{p\alpha}(t) + \varrho(t)\tau_3(t)\alpha X_{p\alpha}(t). \end{cases} \tag{4.10}$$

$\mathcal{P}$ associated with the dynamics (4.10) admits a unique fixed point $(\bar{X}_{s\alpha}^0, \bar{X}_{p\alpha}^0)$ which is globally attractive in $\mathbb{R}_+^2$. By using the implicit function theorem, $\alpha \mapsto (\bar{X}_{s\alpha}^0, \bar{X}_{p\alpha}^0)$ is continuous. Thus, $\alpha > 0$ can be chosen small enough such that $\bar{X}_{s\alpha}(t) > \bar{X}_s(t) - \xi$, and $\bar{X}_{p\alpha}(t) > \bar{X}_p(t) - \xi$, $\forall\ t > 0$. Using the continuity property of the solution with respect to the initial condition, $\exists\alpha^*$ such that $Y_0 \in \Omega_0$ with $\|Y_0 - u(t, Y_1)\| \leq \alpha^*$; then $\|u(t, Y_0) - u(t, Y_1)\| < \alpha$ for $0 \leq t \leq T$. We prove by contradiction that

$$\limsup_{k\to\infty}\ d(\mathcal{P}^k(Y_0), Y_1) \geq \alpha^*\ \forall\ Y_0 \in \Omega_0. \tag{4.11}$$

Suppose that $\limsup_{k\to\infty} d(\mathcal{P}^k(Y_0), Y_1) < \alpha^*$ for some $Y_0 \in \Omega_0$. We can assume that $d(\mathcal{P}^k(Y_0), Y_1) < \alpha^*$ for all $k > 0$. Therefore $\|u(t, \mathcal{P}^k(Y_0)) - u(t, Y_1)\| < \alpha \ \forall\ k > 0$ and $0 \le t \le T$.
For $t \ge 0$, let $t = kT + t_1$, where $t_1 \in [0, T)$ and $k = \lfloor \frac{t}{T} \rfloor$. Therefore

$$\|u(t, Y_0) - u(t, Y_1)\| = \|u(t_1, \mathcal{P}^k(Y_0)) - u(t_1, Y_1)\| < \alpha \text{ for all } t \ge 0.$$

Set $(X_s(t), X_i(t), X_v(t), X_p(t)) = u(t, Y_0)$. Therefore $0 \le X_i(t), X_v(t) \le \alpha, t \ge 0$ and

$$\left\{ \begin{array}{lcl} \dot{X}_s(t) & \ge & d(t)\Theta_1(t) - d(t)X_s(t) - (\tau_1(t)\alpha + \tau_2(t)\alpha)X_s(t), \\ \dot{X}_p(t) & \ge & m(t)\Theta_2(t) - m(t)X_p(t). \end{array} \right. \tag{4.12}$$

$\mathcal{P}$ applied to the dynamics (4.10) admits a fixed point $\bar{X}^0_{s\alpha}$ that it is globally attractive with $\bar{X}_{s\alpha}(t) > \bar{X}_s(t) - \xi$, and $\bar{X}_{p\alpha}(t) > \bar{X}_p(t) - \xi$; then, $\exists\ T_2 > 0$ such that $X_s(t) > \bar{X}_s(t) - \xi$ and $X_p(t) > \bar{X}_p(t) - \xi$ for $t > T_2$. Then, for $t > T_2$, we have

$$\left\{ \begin{array}{lcl} \dot{X}_i(t) & \ge & \tau_1(t)X_i(t)(\bar{X}_s(t) - \xi) + \tau_2(t)X_v(t)(\bar{X}_s(t) - \xi) - d(t)X_i(t), \\ \dot{X}_v(t) & = & \xi(t)X_i(t) - \mu(t)X_v(t) - \tau_3(t)X_v(t)X_p(t). \end{array} \right. \tag{4.13}$$

Since $r(\varphi_{F-V-\xi M_2}(T)) > 1$, then by using Lemma 4.1, there exists a positive T-periodic function $x_2(t)$ such that $J(t) \ge e^{a_2 t} x_2(t)$ where $a_2 = \frac{1}{T} \ln\ r\left(\varphi_{F-V-\xi M_2}(T)\right) > 0$, then $\lim_{t\to\infty} X_i(t) = \infty$ which contradicts the boundedness of the solution. Therefore, (4.11) is satisfied and $\mathcal{P}$ is weakly uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$. By applying Proposition 4.1, $\mathcal{P}$ has a global attractor. We deduce that Y_1 is an isolated invariant set inside Ω and that $W^s(Y_1) \cap \Omega_0 = \emptyset$. All trajectories inside M_∂ converges to Y_1 which is acyclic in M_∂. Applying [35, Theorem 1.3.1 and Remark 1.3.1], we deduce that $\mathcal{P}$ is uniformly persistent with respect to $(\Omega_0, \partial\Omega_0)$. Moreover, by using [35, Theorem 1.3.6], $\mathcal{P}$ has a fixed point $\tilde{Y}_0 = (\tilde{X}^0_s, \tilde{X}^0_i, \tilde{X}^0_v, \tilde{X}^0_p) \in \Omega_0$ with $\tilde{Y}_0 \in R_+ \times Int(R^2_+) \times R_+$. Suppose that $\tilde{X}^0_s = 0$. From the first equation of the dynamics (2.1), $\tilde{X}_s(t)$ satisfies

$$\dot{\tilde{X}}_s(t) = d(t)\Theta_1(t) - \tau_1(t)\tilde{X}_i(t)\tilde{X}_s(t) - \tau_2(t)\tilde{X}_v(t)\tilde{X}_s(t) - d(t)\tilde{X}_s(t), \tag{4.14}$$

where $\tilde{X}^0_s = \tilde{X}_s(nT) = 0, n = 1, 2, 3, \cdots$. By using Proposition 4.1, $\forall\ m_3 > 0$, $\exists\ T_3 > 0$ such that $\tilde{X}_i(t) \le \Theta^u_1 + m_3$ and $\tilde{X}_v(t) \le \frac{\varrho^u \xi^u}{\varrho^l \sigma^l}\Theta^u_1 + \frac{m^u}{\varrho^l \sigma^l}\Theta^u_2 + m_3$ for $t > T_3$. Then, we obtain

$$\dot{\tilde{X}}_s(t) \ge d(t)\Theta_1(t) - d(t)\tilde{X}_s(t) - \Big(\tau_1(t)(\Theta^u_1 + m_3) + \tau_2(t)\Big(\frac{\Theta^u_2}{k^l} + \frac{m^u k^u \Theta^u_1}{k^l m^l_a} + m_3\Big)\Big)\tilde{X}_s(t), \text{ for } t \ge T_3. \tag{4.15}$$

$\exists\ \bar{n}$ such that $nT > T_3$ for all $n > \bar{n}$. Therefore

$$\tilde{X}_s(nT) \ge \Big[\tilde{X}^0_s + \int_0^{nT} d(\omega)\Theta_1(\omega) e^{\int_0^{\omega} \left(\tau_1(u)(\Theta^u_1 + m_3) + \tau_2(u)\left(\frac{\Theta^u_2}{k^l} + \frac{m^u k^u \Theta^u_1}{k^l m^l_a} + m_3\right) + d(u)\right)du} d\omega\Big]$$
$$\times e^{-\int_0^{nT} \left(\tau_1(u)(\Theta^u_1 + m_3) + \tau_2(u)\left(\frac{\Theta^u_2}{k^l} + \frac{m^u k^u \Theta^u_1}{k^l m^l_a} + m_3\right) + d(u)\right)du}$$

for all $n > \bar{n}$ which contradicts the fact that $\tilde{X}_s(nT) = 0$. Then, $\tilde{S}^0 > 0$ and $\tilde{Y}_0$ is a positive T-periodic solution of the dynamics (2.1). □

5. Numerical Examples

The periodic functions are given by

$$\left\{\begin{array}{lcllcllcl} d(t) & = & d^0(1+d^1\cos(2\pi t)), & \mu(t) & = & \mu^0(1+\mu^1\cos(2\pi t)), & \xi(t) & = & \xi^0(1+\xi^1\cos(2\pi t)), \\ m(t) & = & m^0(1+m^1\cos(2\pi t)), & \Theta_1(t) & = & \Theta_1^0(1+\Theta_1^1\cos(2\pi t)), & \Theta_2(t) & = & \Theta_2^0(1+\Theta_2^1\cos(2\pi t)), \\ \tau_1(t) & = & \tau_1^0(1+\tau_1^1\cos(2\pi t)), & \tau_2(t) & = & \tau_2^0(1+\tau_2^1\cos(2\pi t)), & \tau_3(t) & = & \tau_3^0(1+\tau_3^1\cos(2\pi t)), \\ \rho(t) & = & \rho^0(1+\rho^1\cos(2\pi t)). & & & & & & \end{array}\right.$$

with $|d^1|$, $|\xi^1|$, $|m^1|$, $|\Theta_1^1|$, $|\Theta_2^1|$, $|\tau_1^1|$, $|\tau_2^1|$, $|\tau_3^1|$, $|\mu^1|$ and $|\rho^1|$ describe the seasonal cycles frequencies. The numerical values of all used parameters are given in Table 2. Three cases were consider here. The first one was dedicated to the case of fixed environment. However, the second was concentrated on the case where only the contact rates are seasonal. Finally, the last case were allocated to the case where all parameters are periodic.

Table 2. Used values for the numerical simulations.

d^0	ξ^0	m^0	Θ_1^0	Θ_2^0	μ^0	ρ^0	d^1	ξ^1	m^1	Θ_1^1	Θ_2^1	τ_1^1	τ_2^1	τ_3^1	μ^1	ρ^1
1.3	1.7	1.9	5	2.5	1.2	0.2	−0.6	−0.8	0.3	0.8	−0.5	0.2	0.6	−0.7	−0.7	0.5

5.1. **Case of fixed environment.** Let us start by the simple case where there is no influence of the seasonality on the dynamics. Thus, we restrict our attention on the autonomous dynamics (3.1), i.e., all parameters are positive constants.

$$\left\{\begin{array}{lcl} \dot{X}_s(t) & = & d^0\Theta_1^0 - \tau_1^0 X_i(t)X_s(t) - \tau_2^0 X_v(t)X_s(t) - d^0 X_s(t), \\ \dot{X}_i(t) & = & \tau_1^0 X_i(t)X_s(t) + \tau_2^0 X_v(t)X_s(t) - d^0 X_i(t), \\ \dot{X}_v(t) & = & \xi^0 X_i(t) - \mu^0 X_v(t) - \tau_3^0 X_v(t)X_p(t), \\ \dot{X}_p(t) & = & m^0\Theta_2^0 + \varrho^0\tau_3^0 X_v(t)X_p(t) - m^0 X_p(t). \end{array}\right. \tag{5.1}$$

with an initial condition $(S^0, I^0, V^0, P^0) \in \mathbb{R}_+^4$. In Figures 1 and 2, the trajectories of dynamics (5.1) converge asymptotically to E_* if $\mathcal{R}_0 > 1$. However, in Figures 3 and 4, the trajectories of the dynamics (5.1) converge to the disease-free steady state E_0, then confirming the global asymptotic stability of E_0 if $\mathcal{R}_0 \leq 1$.

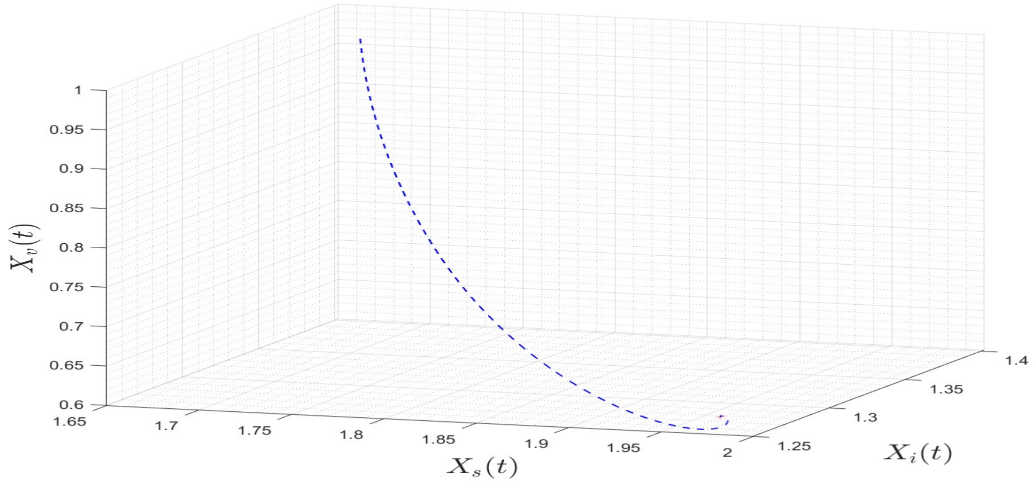

FIGURE 1. Behavior of the dynamics (2.1) for $\tau_1^0 = 1.2$, $\tau_2^0 = 0.8$ and $\tau_3^0 = 0.9$ then $\mathcal{R}_0 \approx 2.57 > 1$.

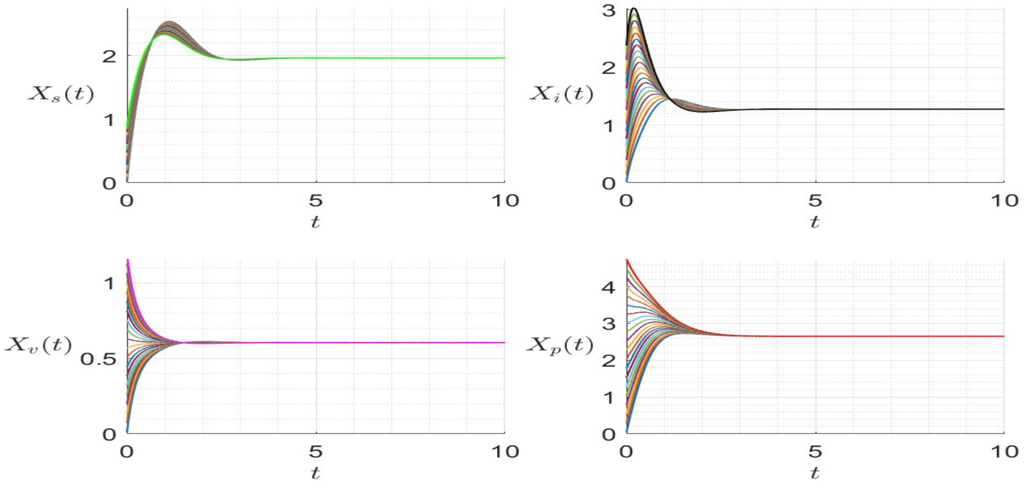

FIGURE 2. Behavior of the dynamics (2.1) for $\tau_1^0 = 1.2$, $\tau_2^0 = 0.8$ and $\tau_3^0 = 0.9$ then $\mathcal{R}_0 \approx 2.57 > 1$.

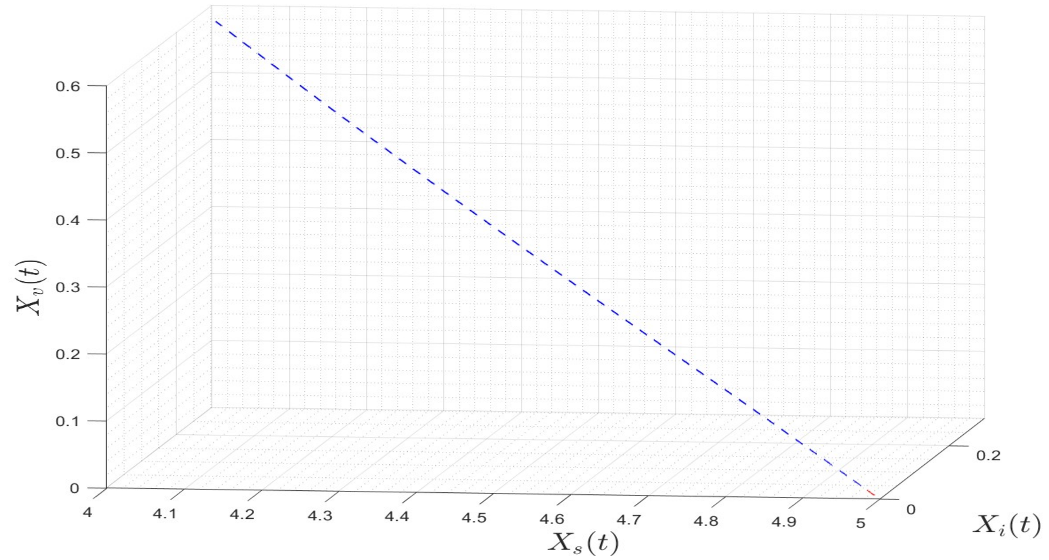

Figure 3. Behavior of the dynamics (2.1) for $\tau_1^0 = 0.3$, $\tau_2^0 = 0.2$ and $\tau_3^0 = 0.1$ then $\mathcal{R}_0 \approx 0.86 < 1$.

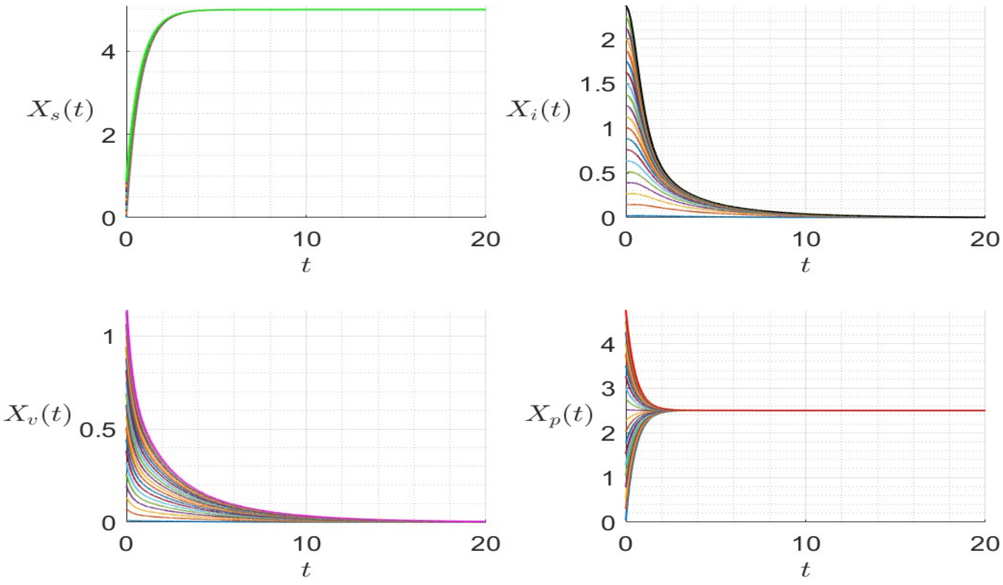

Figure 4. Behavior of the dynamics (2.1) for $\tau_1^0 = 0.3$, $\tau_2^0 = 0.2$ and $\tau_3^0 = 0.1$ then $\mathcal{R}_0 \approx 0.86 < 1$.

5.2. **Case of seasonal contact.** The second was allocated to the case where only the contact rates, τ_1, τ_2 and τ_3 are seasonal functions. All the rest of parameters are fixed. We obtain the following system.

$$\left\{\begin{array}{lcl} \dot{X}_s(t) & = & d^0\Theta_1^0 - \tau_1(t)X_i(t)X_s(t) - \tau_2(t)X_v(t)X_s(t) - d^0X_s(t), \\ \dot{X}_i(t) & = & \tau_1(t)X_i(t)X_s(t) + \tau_2(t)X_v(t)X_s(t) - d^0X_i(t), \\ \dot{X}_v(t) & = & \xi^0X_i(t) - \mu^0X_v(t) - \tau_3(t)X_v(t)X_p(t), \\ \dot{X}_p(t) & = & m^0\Theta_2^0 + \varrho^0\tau_3(t)X_v(t)X_p(t) - m^0X_p(t). \end{array}\right. \tag{5.2}$$

with the positive initial condition $(S^0, I^0, V^0, P^0) \in \mathbb{R}_+^4$.

We give the results of some numerical simulations confirming the stability of the steady states of system (5.2). The approximation of the basic reproduction number $\mathcal{R}_0$ was performed using the time-averaged system.

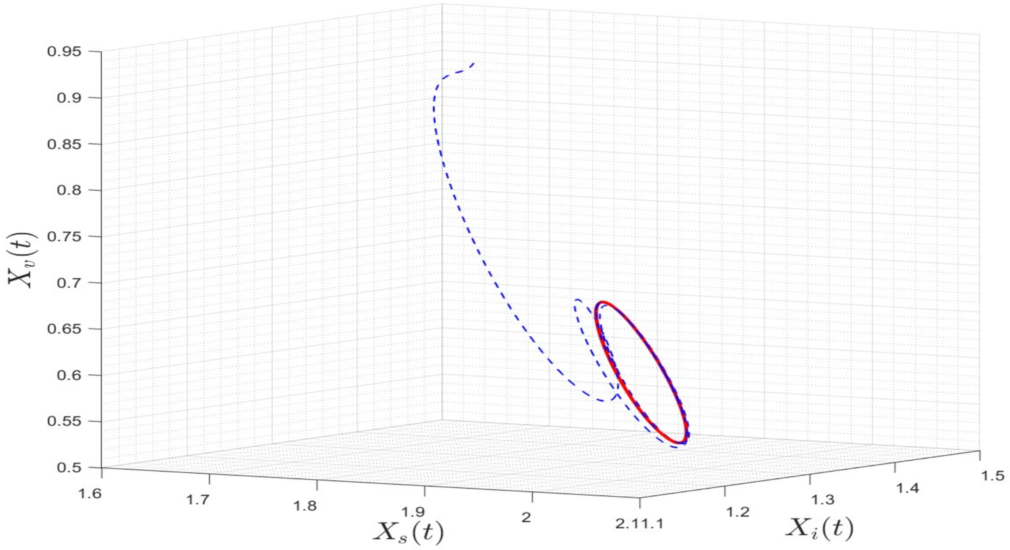

Figure 5. Behavior of the dynamics (2.1) for $\tau_1^0 = 1.2$, $\tau_2^0 = 0.8$ and $\tau_3^0 = 0.9$ then $\mathcal{R}_0 \approx 2.57 > 1$.

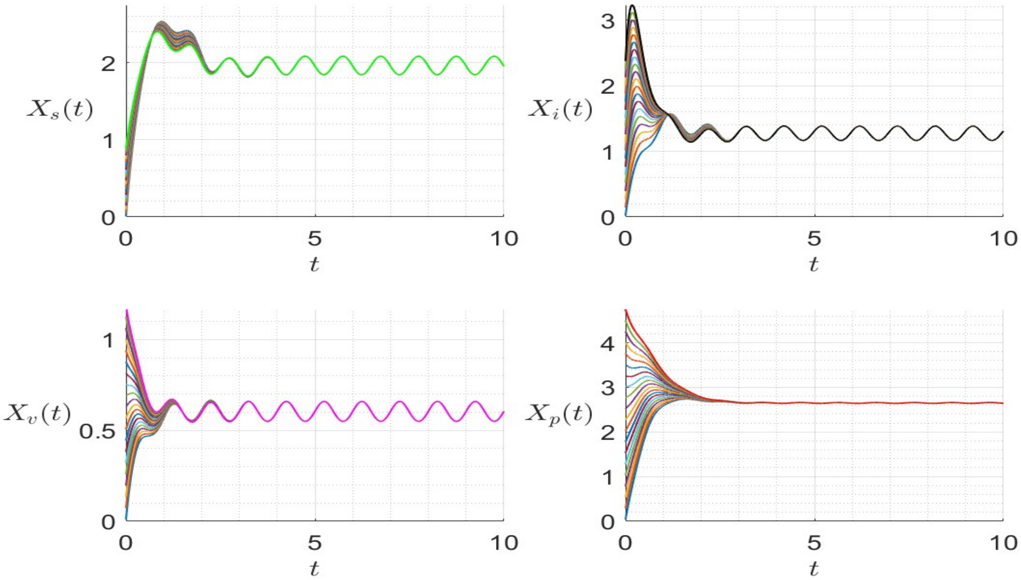

FIGURE 6. Behavior of the dynamics (2.1) for $\tau_1^0 = 1.2$, $\tau_2^0 = 0.8$ and $\tau_3^0 = 0.9$ then $\mathcal{R}_0 \approx 2.57 > 1$.

In Figures 5 and 6, the trajectories of the dynamics (5.2) converge asymptotically to the periodic solution corresponding to the disease-persistence. In Figures 7 and 8, the trajectories of the dynamics (5.2) converge to the disease-free trajectory if $\mathcal{R}_0 < 1$.

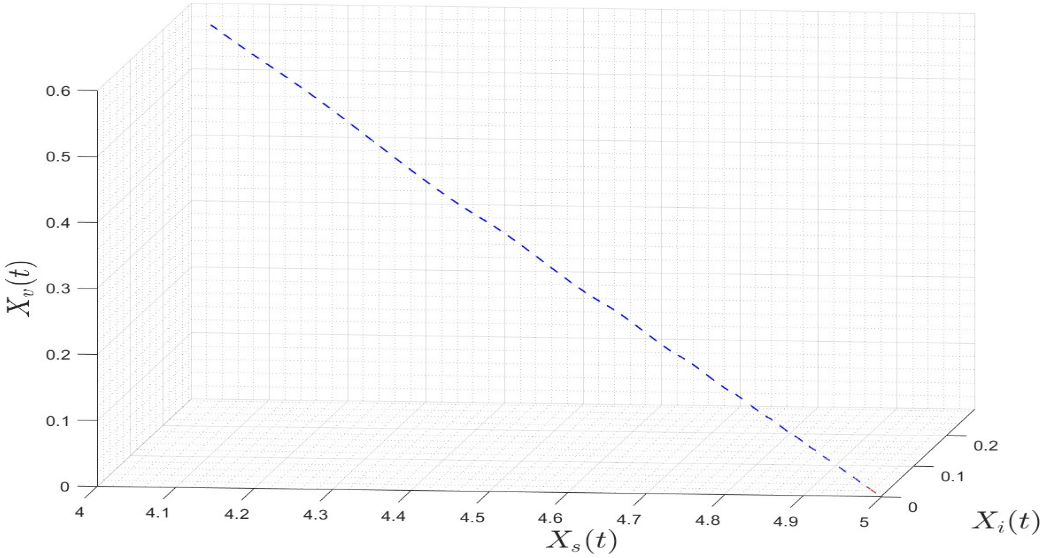

FIGURE 7. Behavior of the dynamics (2.1) for $\tau_1^0 = 0.3$, $\tau_2^0 = 0.2$ and $\tau_3^0 = 0.1$ then $\mathcal{R}_0 \approx 0.86 < 1$.

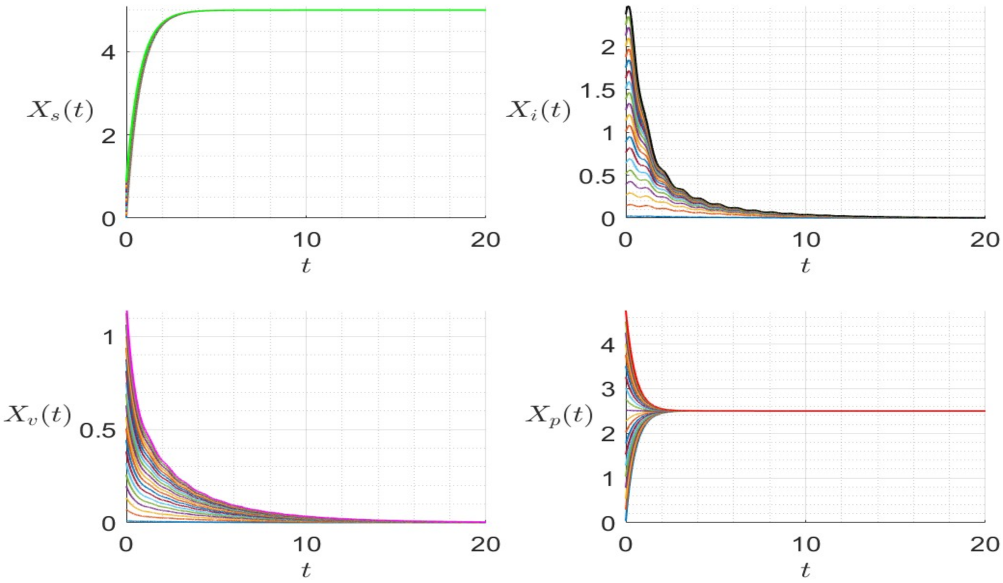

FIGURE 8. Behavior of the dynamics (2.1) for $\tau_1^0 = 0.3$, $\tau_2^0 = 0.2$ and $\tau_3^0 = 0.1$ then $\mathcal{R}_0 \approx 0.86 < 1$.

5.3. **Case of periodic parameters.** In the third step, we performed numerical simulations for the system (2.1) where all parameters were set as T-periodic functions. Thus the model is given by

$$\begin{cases} \dot{X}_s(t) &= d(t)\Theta_1(t) - \tau_1(t)X_i(t)X_s(t) - \tau_2(t)X_v(t)X_s(t) - d(t)X_s(t), \\ \dot{X}_i(t) &= \tau_1(t)X_i(t)X_s(t) + \tau_2(t)X_v(t)X_s(t) - d(t)X_i(t), \\ \dot{X}_v(t) &= \xi(t)X_i(t) - \mu(t)X_v(t) - \tau_3(t)X_v(t)X_p(t), \\ \dot{X}_p(t) &= m(t)\Theta_2(t) + \varrho(t)\tau_3(t)X_v(t)X_p(t) - m(t)X_p(t). \end{cases} \tag{5.3}$$

with the positive initial condition $(S^0, I^0, V^0, P^0) \in \mathbb{R}_+^4$.

We give the results of some numerical simulations confirming the stability of the steady states of system (5.3). The basic reproduction number $\mathcal{R}_0$ was approximated by using the time-averaged system.

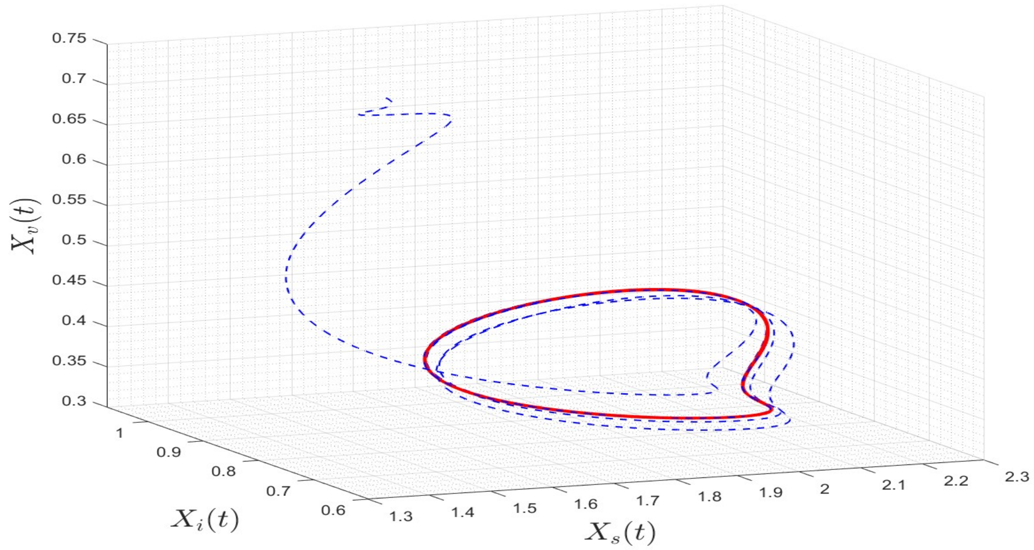

FIGURE 9. Behavior of the dynamics (2.1) for $\tau_1^0 = 1.2$, $\tau_2^0 = 0.8$ and $\tau_3^0 = 0.9$ then $\mathcal{R}_0 \approx 2.57 > 1$.

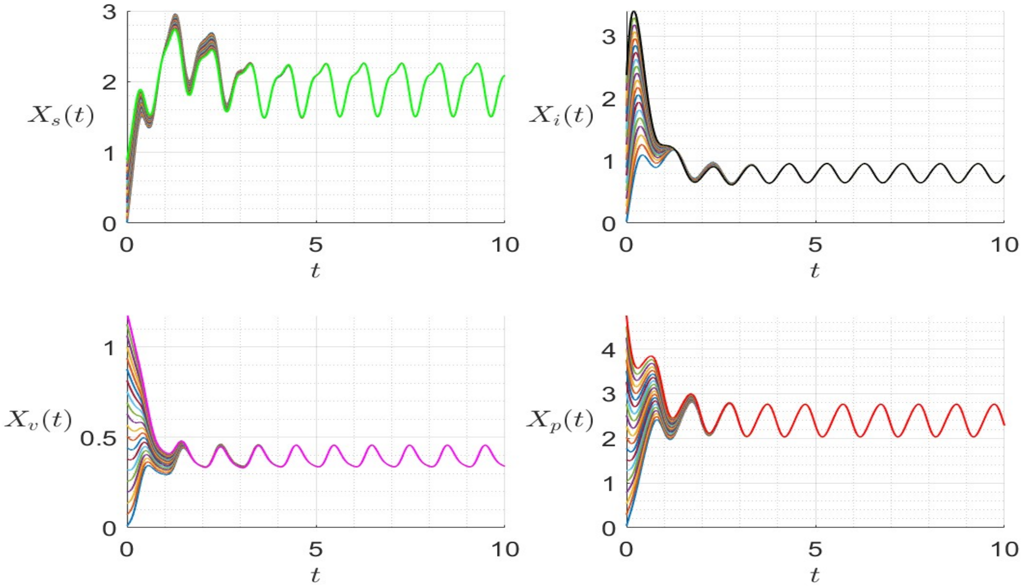

FIGURE 10. Behavior of the dynamics (2.1) for $\tau_1^0 = 1.2$, $\tau_2^0 = 0.8$ and $\tau_3^0 = 0.9$ then $\mathcal{R}_0 \approx 2.57 > 1$.

In Figures 9 and 10, the trajectories of the dynamics (5.3) converge asymptotically to the periodic solution corresponding to the disease persistence if $\mathcal{R}_0 > 1$. In Figures 11 and 12, the trajectories of the dynamics (5.3) converge to the disease-free periodic solution $\mathcal{E}_0(t) = (X_s^*(t), 0, 0, X_p^*(t))$ for the case where $\mathcal{R}_0 \leq 1$.

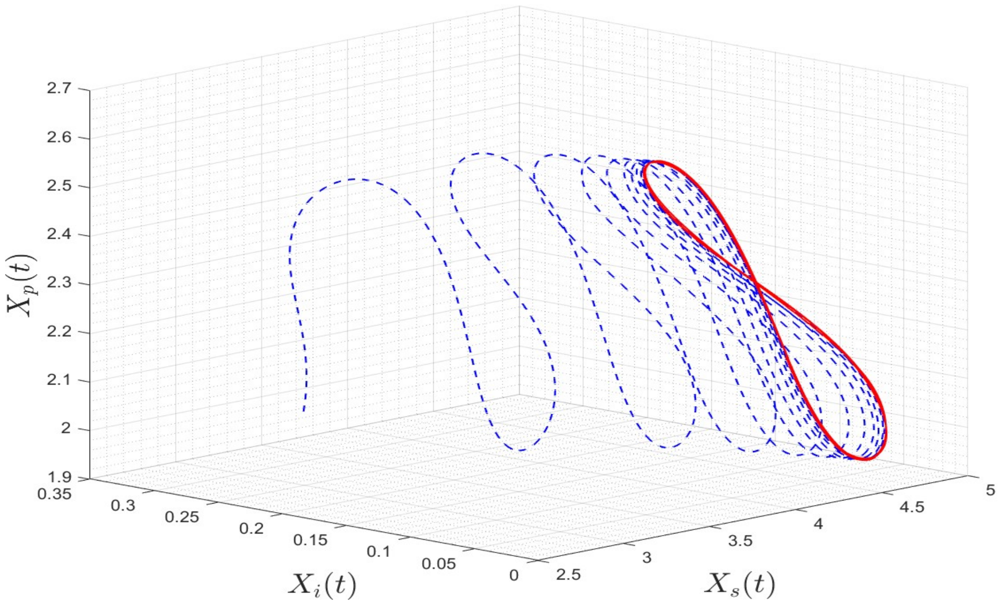

FIGURE 11. Behavior of the dynamics (2.1) for $\tau_1^0 = 0.3$, $\tau_2^0 = 0.2$ and $\tau_3^0 = 0.1$ then $\mathcal{R}_0 \approx 0.86 < 1$.

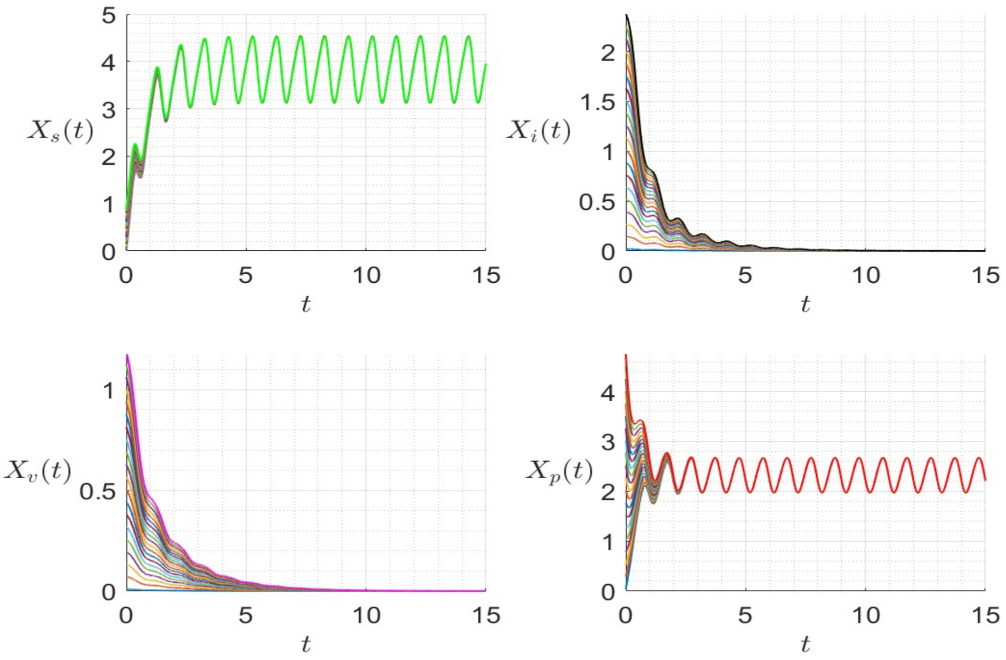

FIGURE 12. Behavior of the dynamics (2.1) for $\tau_1^0 = 0.3$, $\tau_2^0 = 0.2$ and $\tau_3^0 = 0.1$ then $\mathcal{R}_0 \approx 0.86 < 1$.

6. Conclusions

In order to more understand the *CHIKV* dynamics when describing the contamination of uninfected hosts, an important way is to take into account of both, contact with *CHIKV* (*CHIKV*-to-host transmission) and contact with infected hosts (host-to-host transmission). The marked seasonality of *CHIKV*, impose the consideration of this property when modelling its dynamics. In this article, we proposed and analysed a mathematical model for *CHIKV* dynamics reflecting the seasonality observed in real life. The basic reproduction number was defined and the steady states of the dynamics were calculated for the first step when considering the autonomous dynamics. We characterised the existence and uniqueness of the steady states. We characterised also the stability conditions for these steady states. Later, we concentrated on the non-autonomous dynamics and we defined the basic reproduction number, $\mathcal{R}_0$ by using an integral operator. It is proved that once $\mathcal{R}_0 \leq 1$, all solution of the dynamics converge to the disease-free periodic trajectory and that the disease persists if $\mathcal{R}_0 > 1$. We performed the theoretical findings by some numerical examples using explicit Runge-Kutta formulas of orders 4 and 5 under Matlab for three cases, the autonomous dynamics, the seasonal contact dynamics and the fully seasonal dynamics. As it is seen in the numerical simulations and proved theoretically that for the first case, the solution converge to one of the equilibria of the dynamics (5.1) regarding Theorems 3.3 and 3.4. However, for the second and third cases, the solutions converge to a limit cycle regarding Theorems 4.2 and 4.3.

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Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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