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Optimal Quadrature Formula of Hermite Type in the Space of Differentiable Functions

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ABSTRACT. In this research work, a new derived optimal quadrature formula is discussed, which includes the sum of the values of the function and its first and second order derivatives at the points located at the same distance on the interval [0,1] in the $L_2^{(m)}(0,1)$ space. we first obtain an analytical representation of the error function norm, and a system of equations of the Wiener-Hopf type construct using the method of Lagrange unknown multipliers for finding the conditional extremum of multivariable functions. Optimal coefficients found by solving the system. Using the exact form of the optimal coefficients, the norm of the error functional of the optimal quadrature formula for $m=3$ and $m=4$ calculate and the order of approximation was shown to be $O(h^m)$. The obtain theoretical conclusions confirmed by numerical experiments.

1. Introduction: Statement of the Problem

It is well known that numerical integration formulas or quadrature formulas are a method of approximate estimation of definite integrals. They are used when the initial functions of the functions under the integral cannot be expressed by elementary functions, when the integral exists only at discrete points, or when some special types of integrals with the property of singularity are approximated, for example:

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$$\int_0^1 \frac{e^{-3x}}{x^{1/5}}dx, \qquad \int_0^1 \frac{e^{x^4}}{\sqrt{1-x^3}}dx, \qquad \int_1^\infty \cos x^3 dx.$$

The effectiveness of quadrature formulas is usually classified according to its degree of accuracy and the order of approximation of the error. Most problems of applied sciences and mathematical physics are brought to the calculation of integrals. In particular, in the numerical-analytical solution of integral equations, in the calculation of the center of mass, the moment of inertia or various properties of physical systems, in the calculation of integrals related to image and signal analysis and filtering, the construction of quadrature formulas and the evaluation of their errors is one of the targeted scientific studies.

We consider the following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_0[\beta]\varphi[\beta] + \frac{h^2}{12}(\varphi'(0)-\varphi'(1)) + \sum_{\beta=0}^{N} C_1[\beta]\varphi''[\beta] \tag{1.1}$$

with error functional

$$\ell_N(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0[\beta]\delta(x-[\beta]) + \frac{h^2}{12}(\delta'(x)-\delta'(x-1)) - \sum_{\beta=0}^{N} C_1[\beta]\delta''(x-[\beta]) \tag{1.2}$$

where $\varepsilon_{[0,1]}(x)$ is the indicator of the interval [0,1], $\delta(x)$ is the Dirac's delta-function, $C_0[\beta] = \begin{cases} \frac{h}{2}, & \beta = 0, N, \\ h, & \beta = \overline{1, N-1}, \end{cases}$ are known coefficients, $C_1[\beta]$, $\beta = \overline{0,N}$ are unknown coefficients of the quadrature formula (1.1), $[\beta] = h\beta$, $\varphi(x)$ is an element of the space $L_2^{(m)}(0,1)$.

The norm in space $L_2^{(m)}(0,1)$ is defined by the following form:

$$\|\varphi\|_{L_2^{(m)}} = \left\{ \int_0^1 \left(\frac{d^m(\varphi)}{dx^m} \right)^2 dx \right\}^{\frac{1}{2}}.$$

In addition, the error functional (1.2) is required to satisfy the following conditions ([1,2])

$$(\ell_N(x), x^\alpha) = 0, \qquad \alpha = 0,1,2,...,m-1. \tag{1.3}$$

So, for the existence of a quadrature formula of the form (1.1), condition $N \geq m-2$ must be fulfilled, that is, starting from $m=3$, unknown coefficients of the quadrature formula $C_1[\beta]$ can be found.

From the definition of the functional norm

$$\left\|\ell_N\right\|_{L_2^{(m)*}} = \sup_{\|\varphi\|_{L_2^{(m)}} \neq 0} \frac{|(\ell_N, \varphi)|}{\|\varphi\|_{L_2^{(m)}}},$$

from this equality we get the Cauchy-Schwarz inequality

$$|(\ell_N, \varphi)| \leq \|\varphi\|_{L_2^{(m)}} \cdot \|\ell_N\|_{L_2^{(m)*}}.$$

It can be seen from this inequality that the error of the quadrature formula (1.1) is estimated by the norm of the error functional $\ell_N(x)$ obtained from the conjunction space $L_2^{(m)*}(0,1)$ from above. Therefore, the estimation of the error of the quadrature formula (1.1) is related to the minimization of the norm of the error functional $\ell_N(x)$. In computational mathematics, quadrature formulas are constructed mainly in three directions: the spline method, the method of $\varphi-$ functions, and the Sobolev method is based on using discrete analog of the linear differential operator.

I. J. Shoenberg ([3,4]) constructed a quadrature formula in the $L_2^{(m)}(0,n)$ space by spline method. From among the following formulas:

$$\int_0^1 \varphi(x)dx \cong \sum_{\gamma=0}^{N}\sum_{t=0}^{\alpha} K_{\gamma t}\varphi^{(t)}(x_\gamma)$$

a) In the work of S.A. Michelli [5] it was shown that $W_2^{(m)}$ is the best formula, when m is an odd natural number;

b) A.A. Jensikbayev [6] proved this formula is optimal in the $L_2^{(m)}(0,1)$ space;

c) T. Catinash, G.T. Koman [7] constructed an optimal quadrature formula using $\varphi-$ function method in the $L_2^{(2)}(0,1)$ space;

d) Kh.M. Shadimetov [8] constructed the optimal quadrature formula in the $L_2^{(m)}(0,1)$ space when $\alpha = 0$ and calculated the norm of the error function;

e) Kh.M.Shadimetov, A.R.Hayotov, F.A.Nuraliev [9] constructed the optimal quadrature formula for $\alpha = 1$ and estimated its error.

Article [10] presents new and effective quadrature formulas, which merge function and first derivative estimation at equally-spaced data points, with a particular emphasis on improving computational efficiency in terms of both cost and time. The objective of the research presented in work [11] is to simplify the computation of the components involved in the integral transformation, denoted as F^m and $m \geq 0$. The analytical expressions for these components encompass definite integrals. Instead of the Newton-Cotes formulas, it is proposed to use nontrivial quadrature formulas with unevenly distributed integration points. The quadrature method is essential in the approximate solution of integral equations. In [12], the trapezoidal numerical integration formula is used to solve the Fredholm-Hammerstein integral equations. In [13], the perturbed Milne quadrature rule was derived for n-fold differentiable functions.

Thus, in order to construct an optimal quadrature formula of the form (1.1), we need to solve the following problems.

Problem 1. Find the norm of the error functional (1.2) of the quadrature formula (1.1).

Problem 2. Finding the coefficients $C_1[\beta], \quad \beta=\overline{0,N}$, which give the smallest value to the norm $\|\ell_N\|_{L_2^{(m)*}}$ of the error functional (1.2), that is, calculating the value

$$\left\|\overset{\circ}{\ell}_N\right\|_{L_2^{(m)*}} = \inf_{C_1[\beta]} \|\ell_N\|_{L_2^{(m)*}}. \tag{1.4}$$

It is important to mention that, the coefficients $C_1[\beta], \quad \beta=\overline{0,N}$ satisfying the quantity (1.4) of the error functional (1.2), i.e., specifying the minimum norm $\|\ell_N\|_{L_2^{(m)*}}$. If such coefficients exist, these are called optimal coefficients, denoted as $\overset{\circ}{C}_1[\beta]$.

2. Known Definitions and Theorems

In this section, we provide definitions and formulas necessary to prove the main results. Assume that φ and ψ are real-valued functions of real variable and are defined in real line $\mathbb{R}$.

Definition 2.1. Function $\varphi(h\beta)$ is a function of a discrete argument if it is defined for a set of integer values of β.

Definition 2.2. The inner product of two discrete functions $\varphi(h\beta)$ and $\phi(h\beta)$ is defined following

$$[\varphi,\phi] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta)\cdot\phi(h\beta)$$

Definition 2.3. The convolution of two discrete functions $\varphi(h\beta)$ and $\phi(h\beta)$ is defined following

$$\varphi(h\beta) * \phi(h\beta) = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma)\cdot\phi(h\beta-h\gamma)$$

The Euler-Frobenius polynomials $E_k(x)$, $k=1,2,...$ are defined by the following formula [5]

$$E_k(x) = \frac{(1-x)^{k+2}}{x}\left(x\frac{d}{dx}\right)^k \frac{x}{(1-x)^2}, \qquad E_0(x)=1. \tag{2.1}$$

Theorem 2.1. Polynomial $Q_k(x)$ which is defined by the formula [2]

$$Q_k(x) = (x-1)^{k+1} \sum_{i=0}^{k+1} \frac{\Delta^i 0^{k+1}}{(x-1)^i} \tag{2.2}$$

is the Euler-Frobenius polynomial (2.1) of degree k, i.e., $Q_k(x) = E_k(x)$, where $\Delta^i 0^k = \sum_{l=1}^{i} (-1)^{i-l} C_i^l l^k$.

When calculating sums, we use the following equations derived from the sum formula of geometric progression [14]

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q} \right)^i \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q} \right)^i \Delta^i \gamma^k \,|_{\gamma=n}, \tag{2.3}$$

where $\Delta^i \gamma^k$ is the finite difference of order i of γ^k.

We use the following formula to calculate some sums [15]

$$\sum_{\beta=0}^{\gamma-1} \gamma^\alpha = \sum_{i=1}^{\alpha+1} \frac{\alpha! B_{\alpha+1-i}}{i!(\alpha+1-i)!} \gamma^i, \tag{2.4}$$

here $B_{\alpha+1-i}$ are Bernoulli numbers

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu} C_l^p \Delta^\alpha 0^p x^{\nu-p}.$$

For any continuous functions, the operation of convolution is defined as follows

$$\varphi(x) * \phi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\phi(y)dy = \int_{-\infty}^{\infty} \varphi(y)\phi(x-y)dy.$$

Lemma 2.1.
$$\sum_{k=1}^{m-3} \sum_{t=0}^{\alpha} \frac{a_k q_k + (-1)^{t+1} b_k q_k^{N+t}}{(q_k-1)^{t+1}} \Delta^t 0^\alpha = (-1)^{\alpha+1} \sum_{k=1}^{m-3} \sum_{t=0}^{\alpha} \frac{a_k q_k^t + (-1)^{t+1} b_k q_k^{N+1}}{(1-q_k)^{t+1}} \Delta^t 0^\alpha$$

where q_k are the roots of the Euler-Frobenius polynomial $E_{2m-6}(q)$, $\Delta^t 0^\alpha = \Delta^t \gamma^\alpha \,|_{\gamma=0}$ is given by in (2.2), $N \in \mathbb{N}$, $\alpha \in \mathbb{Z}^+$.

The proof of this Lemma is given in [9].

3. The Expression of the Error Functional Norm

In this section, we find general representation of the norm of the (1.2). We utilize the extremal function for this purpose ([1,2]). The function ψ_ℓ is called an extremal function of the error functional (1.2) if the following equality holds

$$\left(\ell_N, U_\ell \right) = \left\| \ell_N \right\|_{L_2^{(m)*}} \left\| U_\ell \right\|_{L_2^{(m)}} \tag{3.1}$$

In the $L_2^{(m)}(0,1)$ space it is defined as follows

$$U_\ell(x) = (-1)^m \ell_N(x) * G_m(x) + P_{m-1}(x) \tag{3.2}$$

where

$$G_m(x) = \frac{|x|^{2m-1}}{2\cdot(2m-1)!} \tag{3.3}$$

is a solution of the equation

$$\frac{d^{2m}}{dx^{2m}} G_m(x) = \delta(x), \tag{3.4}$$

$P_{m-1}(x) = a_m x^{m-1} + a_{m-1}x^{m-2} + \ldots + a_2 x + a_1$.

In addition, the extremal function satisfies the following relationships. (Riesz theorem) [16]

$$\|\ell_N\|_{L_2^{(m)*}(0,1)} = \|U_\ell\|_{L_2^{(m)}(0,1)} \quad \text{and} \quad (\ell_N, U_\ell) = \|\ell_N\|^2_{L_2^{(m)*}(0,1)} \tag{3.5}$$

Based on equations (3.2) and (3.5), the square of the norm of any linear continuous $\ell_N(x)$ error function in the $L_2^{(m)*}(0,1)$ space can be written as follows

$$\|\ell_N\|^2_{L_2^{(m)*}} = (\ell_N, U_\ell) = \left(\ell_N(x), (-1)^m \ell_N(x) * G_m(x) + P_{m-1}(x)\right) = \left(\ell(x), (-1)^m \ell(x) * G_m(x)\right)$$

$$= \int_{-\infty}^{\infty} \ell_N(x) \left((-1)^m \int_{-\infty}^{\infty} \ell_N(y) G_m(x-y)\,dy \right) dx$$

Using this equation, we get a general representation of the square of the norm of the error functional (1.2)

$$\begin{aligned}
\|\ell_N\|^2_{L_2^{(m)*}} = (-1)^m \Bigg[& \sum_{\beta=0}^{N} C_1[\beta] \sum_{\gamma=0}^{N} C_1[\gamma] \frac{|[\beta-\gamma]|^{2m-5}}{2(2m-5)!} - 2\sum_{\beta=0}^{N} C_1[\beta] \int_0^1 \frac{|x-[\beta]|^{2m-3}}{2(2m-3)!} dx \\
& - \frac{h^2}{6} \sum_{\beta=0}^{N} C_1[\beta] \left[\frac{[\beta]^{2m-4} + ([\beta]-1)^{2m-4}}{2(2m-4)!} \right] \\
& + 2\sum_{\beta=0}^{N} C_0[\beta] \sum_{\gamma=0}^{N} C_1[\gamma] \frac{|[\beta-\gamma]|^{2m-3}}{2(2m-3)!} - \frac{h^2}{6} \sum_{\beta=0}^{N} C_0[\beta] \left[\frac{[\beta]^{2m-2} + ([\beta]-1)^{2m-2}}{2(2m-2)!} \right] \\
& - 2\sum_{\beta=0}^{N} C_0[\beta] \int_0^1 \frac{|x-[\beta]|^{2m-1}}{2(2m-1)!} dx + \sum_{\beta=0}^{N} C_0[\beta] \sum_{\gamma=0}^{N} C_0[\gamma] \frac{|[\beta-\gamma]|^{2m-1}}{2(2m-1)!} \\
& + \frac{h^2}{6(2m-1)!} + \frac{1}{(2m+1)!} + \frac{h^4}{144(2m-3)!} \Bigg].
\end{aligned} \tag{3.6}$$

Thus, Problem 1 was resolved.

4. Optimal Coefficients of the Optimal Quadrature Formula Form (1)

In this section, we will consider the problem of finding the minimum of the equation (3.6) under the conditions (1.3) for coefficients $C_1[\beta],\ \beta = 0,1,...,N$. Using the method of Lagrange unknown multipliers, we find the conditional extremum of multivariable functions. Therefore, we construct the Lagrange function

$$\Psi = \|\ell_N\|^2 - 2\cdot(-1)^m \sum_{\alpha=0}^{m-1} \lambda_\alpha (\ell_N, x^\alpha),$$

where λ_α unknown multipliers. The Ψ function is a multivariable function with respect to the coefficients of $C_1[\beta]$ and λ_α. By equalizing the derivatives of the function Ψ with respect to $C_1[\beta]$ and λ_α to zero, we obtain the following system of equations

$$\sum_{\gamma=0}^{N} C_1[\gamma] G_{m-2}[\beta-\gamma] + P_{m-3}[\beta] = F_m[\beta], \qquad \beta = \overline{0,N}, \tag{4.1}$$

$$\sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^\alpha = -\sum_{j=1}^{\alpha} \frac{\alpha!\, B_{\alpha+3-j}}{j!(\alpha+3-j)!} h^{\alpha+3-j}, \qquad \alpha = \overline{0,m-3}, \tag{4.2}$$

where

$$F_m[\beta] = \sum_{i=0}^{2m-5} \frac{[\beta]^{2m-5-i}}{(2m-5-i)!}\left[-\frac{B_{i+3}h^{i+3}}{(i+3)!} + \sum_{j=1}^{i} \frac{(-1)^i B_{i+3-j}}{2j!(i+3-j)!} h^{i+3-j}\right] + \frac{B_{2m-2}h^{2m-2}}{(2m-2)!} \tag{4.3}$$

$P_{m-3}(h\beta)$ is a polynomial of degree $m-3$, $G_{m-2}(x) = \frac{|x|^{2m-5}}{2(2m-5)!}$, B_{i+3-j} are Bernoulli numbers.

The system of equations (4.1)-(4.2) is the discrete Wiener-Hopf system used to find optimal coefficients. This system has a unique solution, and this solution gives a minimum value to $\|\ell_N\|^2$. To solve this system, we use an approach based on the $D_{m-2}(h\beta)$ discrete analogue of the d^{2m-4}/dx^{2m-4} operator, which constracted in [17].

To do this, we rewrite equation (4.1) in the form of convolution, taking into $C_1[\beta] = 0$ for $\beta < 0$ and $\beta > N$.

$$G_{m-2}(h\beta) * C_1[\beta] + P_{m-3}(h\beta) = F_m(h\beta), \quad \beta = 0,1,...,N. \tag{4.4}$$

Also, instead of the left side of equation (4.4), we introduce functions

$$\vartheta(h\beta) = G_{m-2}(h\beta) * C_1[\beta] \tag{4.5}$$

$$u(h\beta) = v(h\beta) + P_{m-3}(h\beta). \tag{4.6}$$

First of all, $C_1[\beta]$ coefficients should be expressed by $u(h\beta)$ function. For this we need the $D_{m-2}(h\beta)$ operator satisfying the equality $hD_{m-2}(h\beta)*G_{m-2}(h\beta)=\delta(h\beta)$.

The following theorems are valid for the d^{2m-4}/dx^{2m-4} discrete analogue of the $D_{m-2}(h\beta)$ operator.

Theorem 4.1. The discrete analogue of the differential operator d^{2m-4}/dx^{2m-4} has the form

$$D_{m-2}(h\beta)=\frac{(2m-5)!}{h^{2m-4}}\begin{cases}\sum\limits_{k=1}^{m-3}\frac{(1-q_k)^{2m-3}q_k^{|\beta|}}{q_kE_{2m-5}(q_k)}, & |\beta|\geq 2,\\ 1+\sum\limits_{k=1}^{m-3}\frac{(1-q_k)^{2m-3}}{E_{2m-5}(q_k)}, & |\beta|=1,\\ -2^{2m-5}+\sum\limits_{k=1}^{m-3}\frac{(1-q_k)^{2m-3}}{q_kE_{2m-5}(q_k)}, & \beta=0,\end{cases} \quad (4.7)$$

where $E_{2m-5}(q)$ is the Euler-Frobenius polynomial of degree $2m-5$, q_k are the roots of the Euler-Frobenius polynomial $E_{2m-6}(q)$, $|q_k|<1$.

Theorem 4.2. The monomials $(h\beta)^k$ have the following relation with the discrete operator $D_{m-2}(h\beta)$

$$\sum_{\beta=-\infty}^{\infty}D_{m-2}(h\beta)(h\beta)^k=\begin{cases}0 & \text{for} \quad 0\leq k\leq 2m-5,\\ (2m-4)! & \text{for} \quad k=2m-6.\end{cases} \quad (4.8)$$

Considering these theorems, we obtain the following equality for the $C_1[\beta]$ optimal coefficients

$$D_{m-2}(h\beta)*u(h\beta)=D_{m-2}(h\beta)*\left(G_{m-2}(h\beta)*C_1[\beta]+P_{m-3}(h\beta)\right)$$
$$=C_1[\beta]*\left(D_{m-2}(h\beta)*G_{m-2}(h\beta)\right)=C_1[\beta]*\delta(h\beta)=C_1[\beta] \quad (4.9)$$

So, we need overview of the function $u(h\beta)$ all integer values of β to calculate the (4.5) convolution. If $h\beta\in[0,1]$, then $u(h\beta)=F_m(h\beta)$.

We find the $u(h\beta)$ overview of the function when $\beta<0$ and $\beta>N$.

Suppose $\beta<0$, then considering (4.2), we have

$$\vartheta(h\beta)=C_1[\beta]*G_{m-2}(h\beta)=C_1[\beta]*\frac{|h\beta|^{2m-5}}{2(2m-5)!}=\sum_{\gamma=0}^{N}C_1[\gamma]\frac{|h\beta-h\gamma|^{2m-5}}{2(2m-5)!}$$

$$=\sum_{i=0}^{m-3}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}}{j!(i+3-j)!}h^{i+3-j}-\sum_{i=m-2}^{2m-5}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^i$$

for $\beta>N$

$$\vartheta(h\beta) = -\sum_{i=0}^{m-3} \frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{j=1}^{i} \frac{i!B_{i+3-j}}{j!(i+3-j)!} h^{i+3-j} + \sum_{i=m-2}^{2m-5} \frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{\gamma=0}^{N} C_1[\gamma](h\gamma)^i$$

we enter the following function

$$R_{2m-5}(h\beta) = \sum_{i=0}^{m-3} \frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{j=1}^{i} \frac{i!B_{i+3-j}}{j!(i+3-j)!} h^{i+3-j},$$

$$Q_{m-3}(h\beta) = \sum_{i=m-2}^{2m-5} \frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!} \sum_{\gamma=0}^{N} C_1[\gamma](h\gamma)^i \tag{4.10}$$

Then,

$$\vartheta(h\beta) = \begin{cases} R_{2m-5}(h\beta) - Q_{m-3}(h\beta), & \beta < 0, \\ -R_{2m-5}(h\beta) + Q_{m-3}(h\beta), & \beta > N. \end{cases} \tag{4.11}$$

So, the general representation of the $u(h\beta)$ function is as follows

$$u(h\beta) = \begin{cases} R_{2m-5}(h\beta) + Q^{-}_{m-3}(h\beta), & \beta < 0, \\ f_m(h\beta), & 0 \le \beta \le N, \\ -R_{2m-5}(h\beta) + Q^{+}_{m-3}(h\beta), & \beta < 0 \end{cases} \tag{4.12}$$

where

$$\begin{aligned} Q^{-}_{m-3}(h\beta) &= P_{m-3}(h\beta) - Q_{m-3}(h\beta), \\ Q^{+}_{m-3}(h\beta) &= P_{m-3}(h\beta) + Q_{m-3}(h\beta) \end{aligned} \tag{4.13}$$

And $Q^{-}_{m-3}(h\beta)$, $Q^{+}_{m-3}(h\beta)$ are polynomial of degree $m-3$.

By finding $Q^{-}_{m-3}(h\beta)$ and $Q^{+}_{m-3}(h\beta)$, we obtain from (4.13)

$$P_{m-3}(h\beta) = \frac{1}{2}\left(Q^{+}_{m-3}(h\beta) + Q^{-}_{m-3}(h\beta)\right),$$

$$Q_{m-3}(h\beta) = \frac{1}{2}\left(Q^{+}_{m-3}(h\beta) - Q^{(-)}_{m-3}(h\beta)\right).$$

Now we find the $C_1[\beta]$ optimal coefficients when $\beta = 1, 2, ..., N-1$ using the form (4.7) and (4.12) of functions with discrete arguments $D_{m-2}(h\beta)$ and $u(h\beta)$.

We introduce the following equalities

$$a_k = \frac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-4} q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} \left(R_{2m-5}(-h\gamma) + Q^{-}_{m-3}(-h\gamma) - f_m(-h\gamma)\right),$$

$$b_k = \frac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-4} q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} \left(-R_{2m-5}(1+h\gamma) + Q^{+}_{m-3}(1+h\gamma) - f_m(1+h\gamma)\right) \tag{4.14}$$

Theorem 4.3. The coefficients $C_1[\beta],\ \beta=1,2,...,N-1$ of the optimal quadrature formulas of the form (1.1) for $m\geq 3$ in the space $L_2^{(m)}(0,1)$ have the following form

$$C_1[\beta]=h^3\sum_{k=1}^{m-3}\left(d_k q_k^{\beta}+p_k q_k^{N-\beta}\right),\ \beta=1,2,...,N-1, \tag{4.15}$$

where a_k, b_k are defined by (4.14), q_k are given in Theorem 4.2.

Proof. When $\beta=1,2,...,N-1$, using equations (4.3), (4.7), (4.9) and (4.12), we can write the following for $C_1[\beta]$

$$C_1[\beta]=hD_{m-2}(h\beta)*u(h\beta)=h\sum_{\gamma=-\infty}^{\infty}D_{m-2}(h\beta-h\gamma)u(h\gamma)$$

$$=h\left(\sum_{\gamma=-\infty}^{-1}D_{m-2}(h\beta-h\gamma)\left[R_{2m-5}(h\gamma)+Q_{m-2}^{-}(h\gamma)\right]+\sum_{\gamma=0}^{N}D_{m-2}(h\beta-h\gamma)f_m(h\gamma)\right.$$

$$\left.+\sum_{\gamma=N+1}^{\infty}D_{m-2}(h\beta-h\gamma)\left[-R_{2m-5}(h\gamma)+Q_{m-3}^{+}(h\gamma)\right]\right).$$

After some simplifications, we get the following

$$C_1[\beta]=h\{D_{m-2}(h\beta)*f_m(h\beta)$$

$$+\sum_{k=1}^{m-3}q_k^{\beta}\frac{(2m-5)!}{h^{2m-4}}\frac{(1-q_k)^{2m-3}}{q_kE_{2m-5}(q_k)}\sum_{\gamma=1}^{\infty}q_k^{\gamma}[R_{2m-5}(-h\gamma)+Q_{m-3}^{-}(-h\gamma)-f_m(-h\gamma)]$$

$$+\sum_{k=1}^{m-3}q_k^{N-\beta}\frac{(2m-5)!}{h^{2m-4}}\frac{(1-q_k)^{2m-3}}{q_kE_{2m-5}(q_k)}\sum_{\gamma=1}^{\infty}q_k^{\gamma}[-R_{2m-5}(1+h\gamma)+Q_{m-3}^{+}(1+h\gamma)-f_m(1+h\gamma)]\}$$

Considering Theorem 4.2 and $D_{m-2}(h\beta)*f_m(h\beta)=0$, we completely proved the Theorem 4.3.

Theorem 4.4. The coefficients of the optimal quadrature formula of the form (1.1) for $m\geq 3$ in $L_2^{(m)}(0,1)$ space are determined as follows

$$C_1[0]=h^3\sum_{k=1}^{m-3}a_k\frac{q_k^N-q_k}{1-q_k}, \tag{4.16}$$

$$C_1[\beta]=h^3\sum_{k=1}^{m-3}a_k\left(q_k^{\beta}+q_k^{N-\beta}\right),\qquad \beta=\overline{1,N-1}, \tag{4.17}$$

$$C_1[N]=h^3\sum_{k=1}^{m-3}a_k\frac{q_k^N-q_k}{1-q_k} \tag{4.18}$$

where a_k satisfy the following system of $m-3$ linear equations

$$\sum_{k=1}^{m-3}a_k\sum_{i=1}^{\alpha}\frac{q_k^{N+i}+(-1)^{i+1}q_k}{(1-q_k)^{i+1}}\Delta^i 0^{\alpha}=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)},\qquad \alpha=\overline{1,m-3}. \tag{4.19}$$

Proof. First we use equation (4.2) to find the representation of the coefficients $C_1[0]$ and $C_1[N]$ for $\alpha=0$ and $\alpha=1$

$$C_1[0]=\sum_{\beta=1}^{N-1}C_1[\beta](h\beta)-\sum_{\beta=1}^{N-1}C_1[\beta], \tag{4.20}$$

$$C_1[N]=-\sum_{\beta=1}^{N-1}C_1[\beta](h\beta). \tag{4.21}$$

Equations (4.20) and (4.21) show that $C_1[0]$ and $C_1[N]$ coefficients depend on $C_1[\beta],\ \beta=\overline{1,N-1}$ coefficients. $C_1[\beta],\ \beta=\overline{1,N-1}$ is represented by coefficients a_k and b_k $(k=\overline{1,m-3})$. So we need to determine the unknowns a_k and b_k to find the optimal coefficients $C_1[\beta],\ \beta=\overline{0,N}$.

We first calculate the sum of (4.1)

$$S=\sum_{\gamma=0}^{N}C_1[\gamma]G_m^{(IV)}(h\beta-h\gamma)=\sum_{\gamma=0}^{N}C_1[\gamma]\frac{(h\beta-h\gamma)^{2m-5}\operatorname{sign}(h\beta-h\gamma)}{2(2m-5)!}$$
$$=\sum_{\gamma=0}^{\beta}C_1[\gamma]\frac{(h\beta-h\gamma)^{2m-5}}{(2m-5)!}-\sum_{\gamma=0}^{N}C_1[\gamma]\frac{(h\beta-h\gamma)^{2m-5}}{2(2m-5)!}$$

From equalities (2.2), (4.2) and (4.15) and taking into account that q_k is the root of the Euler-Frobenius polynomial of degree $(2m-6)$, we get the following for S

$$S=\frac{(h\beta)^{2m-5}}{(2m-5)!}C_1[0]-\sum_{i=0}^{2m-5}\frac{(h\beta)^{2m-5-i}h^{i+3}}{(2m-5-i)!\cdot i!}\left[\sum_{k=1}^{m-3}d_k\frac{q_k}{q_k-1}\sum_{\alpha=0}^{i}\left(\frac{1}{q_k-1}\right)^{\alpha}\Delta^{\alpha}0^{i}\right.$$
$$\left.+\sum_{k=1}^{m-3}p_k\frac{q_k^N}{1-q_k}\sum_{\alpha=0}^{i}\left(\frac{q_k}{1-q_k}\right)^{\alpha}\Delta^{\alpha}0^{i}\right]+\sum_{i=0}^{m-3}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}}{j!(i+3-j)!}h^{i+3-j} \tag{4.22}$$
$$-\sum_{i=m-2}^{2m-5}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^i.$$

Substituting the expression (4.22) into (4.1), we get the following equation with respect to $h\beta$

$$S+P_{m-3}(h\beta)=f_m(h\beta) \tag{4.23}$$

From (4.23) we get the system of equations to find the coefficient $C_1[0]$ and the unknowns a_k and b_k by equating the respective levels of $(h\beta)^{2m-5-i}$ when $i=0,1,...,m-3$ and $(h\beta)^{2m-5}$

$$C_1[0]=h^3\sum_{k=1}^{m-3}a_k\frac{q_k^N-q_k}{1-q_k}, \tag{4.24}$$

$$\sum_{k=1}^{m-3}\sum_{i=1}^{\alpha}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{\alpha}=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)},\qquad \alpha=\overline{1,m-3}. \tag{4.25}$$

By equating the corresponding degrees of $(h\beta)^{2m-5-i}$ when $i = m-2, m, ..., 2m-5$, we find the exact form of the polynomial $P_{m-3}(h\beta)$

$$P_{m-3}(h\beta) = \sum_{i=m-2}^{2m-5} \frac{(h\beta)^{2m-5-i}}{(2m-5-i)!}\left[\frac{h^{i+3}}{i!}\sum_{k=1}^{m-3}\sum_{\alpha=1}^{i}\frac{d_k q_k + p_k q_k^{N+\alpha}(-1)^{\alpha+1}}{(q_k-1)^{\alpha+1}}\Delta^{\alpha}0^{i}\right.$$

$$\left.+\frac{(-1)^i}{2i!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^i - \frac{B_{i+3}h^{i+3}}{(i+3)!} + \sum_{j=1}^{i}\frac{(-1)^i B_{i+3-j}}{2j!(i+3-j)!}\right] + \frac{B_{2m-2}h^{2m-2}}{(2m-2)!}. \tag{4.26}$$

From (4.2), we get the following for $\alpha = 0$

$$\sum_{\gamma=0}^{N} C_1[\gamma] = 0.$$

Using (4.17) and (4.24), we find $C_1[N]$

$$C_1[N] = h^3 \sum_{k=1}^{m-3} a_k \frac{q_k^N - q_k}{1-q_k}.$$

Thus, to find the unknowns a_k and b_k $(k = 1, 2, ..., m-3)$ from equation (4.23), we obtained the system of equations (4.25), and we find the remaining $m-3$ using equation (4.2)

$$\sum_{\beta=0}^{N} C_1[\beta](h\beta)^{\alpha} = -\sum_{j=1}^{\alpha}\frac{\alpha!B_{\alpha+3-j}}{j!(\alpha+3-j)!}h^{\alpha+3-j}, \qquad \alpha = \overline{1, m-3} \tag{4.27}$$

Using (2.2) and (4.15) for the left side of equation (4.27), we get the following

$$\sum_{\beta=0}^{N} C_1[\beta](h\beta)^{\alpha} = \sum_{k=1}^{m-3}\sum_{\beta=1}^{N-1} h^3\left(a_k q_k^{\beta} + b_k q_k^{N-\beta}\right)(h\beta)^{\alpha} + C_1[N] = C_1[N]$$

$$+h^3\sum_{k=1}^{m-3}\left[a_k\sum_{\beta=1}^{N-1}q_k^{\beta}\beta^{\alpha} + b_k\sum_{\beta=1}^{N-1}q_k^{N-\beta}\beta^{\alpha}\right]\cdot h^{\alpha} = h^{\alpha+3}\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_k q_k^i + b_k q_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i 0^{\alpha}$$

$$-\sum_{t=0}^{\alpha}\frac{\alpha! h^{t+3}}{t!(\alpha-t)!}\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_k q_k^{N+i} + (-1)^{i+1}b_k q_k}{(1-q_k)^{i+1}}\Delta^i 0^t + C_1[N], \qquad \alpha = \overline{1, m-3}. \tag{4.28}$$

Taking equality (4.28) into account, we subtract the left and right sides of (4.27)

$$h^{\alpha+3}\sum_{k=1}^{m-2}\sum_{i=0}^{\alpha}\frac{a_k q_k^i + b_k q_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i 0^{\alpha} - \sum_{t=0}^{\alpha}\frac{\alpha! h^{t+3}}{t!(\alpha-t)!}\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_k q_k^{N+i} + (-1)^{i+1}b_k q_k}{(1-q_k)^{i+1}}\Delta^i 0^t$$

$$+C_1[N] + \sum_{j=1}^{\alpha}\frac{\alpha!B_{\alpha+3-j}}{j!(\alpha+3-j)!}h^{\alpha+3-j} = \sum_{j=3}^{\alpha+3}a_j h^j = 0, \qquad \alpha = \overline{1, m-3} \tag{4.29}$$

Equating the corresponding degrees of h from (4.29), we get the following system

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_k q_k^{N+i}+(-1)^{i+1}b_k q_k}{(1-q_k)^{i+1}}\Delta^i 0^\alpha=\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_k q_k^{i}+(-1)^{i+1}b_k q_k^{N+1}}{(1-q_k)^{i+1}}\Delta^i 0^\alpha,\quad \alpha=\overline{1,m-3}, \tag{4.30}$$

$$\sum_{k=1}^{m-3}\sum_{i=0}^{j}\frac{a_k q_k^{N+i}+(-1)^{i+1}b_k q_k}{(1-q_k)^{i+1}}\Delta^i 0^j=\frac{B_{j+3}}{(j+1)(j+2)(j+3)},\qquad j=\overline{1,\alpha-1},\ \ \alpha=\overline{1,m-3}. \tag{4.31}$$

It can be seen that (4.31) is a part of system (4.25). Thus, using (4.30), (4.25) and Lemma 2.1 , we obtain the new system for the unknowns a_k and b_k

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_k q_k^{N+i}+(-1)^{i+1}b_k q_k}{(1-q_k)^{i+1}}\Delta^i 0^\alpha=(-1)^{\alpha+1}\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)},\qquad \alpha=\overline{1,m-3} \tag{4.32}$$

$B_{\alpha+3}=0$ when α is even numbers, from this the equation (4.32) can be written as follows

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_k q_k^{N+i}+(-1)^{i+1}b_k q_k}{(1-q_k)^{i+1}}\Delta^i 0^\alpha=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)},\quad \alpha=\overline{1,m-3} \tag{4.33}$$

If we subtract (4.33) from (4.25), we get the following system of equations

$$\sum_{k=1}^{m-3}(a_k-b_k)\sum_{i=1}^{\alpha}\frac{q_k^{N+i}-q_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i 0^\alpha=0,\qquad \alpha=\overline{1,m-3}. \tag{4.34}$$

(4.34) from equality $a_k=b_k,\quad k=1,2,...,m-3$.

So, from (4.25) we will have a system of equations

$$\sum_{k=1}^{m-3}a_k\sum_{i=1}^{\alpha}\frac{q_k^{N+i}+(-1)^{i+1}q_k}{(1-q_k)^{i+1}}\Delta^i 0^\alpha=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)},\qquad \alpha=\overline{1,m-3}, \tag{4.35}$$

to find a_k unknowns.

Theorem 4.4 is proved. From Theorem 4.4, we get the following results:

Result 4.1. The coefficients of the optimal quadrature formula (1.1) in the $L_2^{(3)}(0,1)$ space are determined as follows

$$C_1[\beta]=0,\qquad \beta=\overline{0,N}$$

Result 4.2. The coefficients of the optimal quadrature formula (1.1) in the $L_2^{(4)}(0,1)$ space are determined as follows

$$C_1[0]=h^3\frac{a\left(q-q^N\right)}{q-1},$$

$$C_1[\beta]=ah^3\left(q^\beta+q^{N-\beta}\right),\qquad \beta=\overline{1,N-1},$$

$$C_1[N]=h^3\frac{a\left(q-q^N\right)}{q-1},$$

where $a=\dfrac{1}{120(1+q^N)}$, $q=\sqrt{3}-2$.

5. High Estimate of the Error of the Optimal Quadrature Formula

In this section, we calculate the square of the norm of the error function.

Theorem 5.1. The squared norm of the error functional (1.3) of the optimal quadrature formula (1.1) in $L_2^{(3)}(0,1)$ space has the following form

$$\left\|\ell_N\right\|^2_{L_2^{(3)*}}=\frac{h^6}{30240}. \tag{5.1}$$

Theorem 5.2. The squared norm of the error functional (1.3) of the optimal quadrature formula (1.1) in $L_2^{(4)}(0,1)$ space has the following form

$$\left\|\ell_N\right\|^2_{L_2^{(4)*}}=\frac{h^8}{1209600}+\frac{h^9(3q+1)(q^N-1)}{518400(5q+1)(q^N+1)}, \tag{5.2}$$

here $q=\sqrt{3}-2$.

Now we give the proof of Theorem 5.2.

Proof. To do this, we simplify the expression (3.6)

$$\left\|\ell_N\left|L_2^{(m)*}(0,1)\right.\right\|^2=(-1)^m\left[\sum_{\beta=0}^{N}C_1[\beta]\left\{-P_{m-3}[\beta]-F_m[\beta]\right\}\right.$$
$$+\sum_{\beta=0}^{N}C_0[\beta]\left\{\sum_{\gamma=0}^{N}C_0[\gamma]\frac{\left|[\beta-\gamma]\right|^{2m-1}}{2(2m-1)!}-\frac{h^2}{6}\frac{[\beta]^{2m-2}+([\beta]-1)^{2m-2}}{2(2m-2)!}+\int_0^1\frac{\left|x-[\beta]\right|^{2m-1}}{(2m-1)!}dx\right\}$$
$$\left.+\frac{h^2}{6(2m-1)!}+\frac{1}{(2m+1)!}+\frac{h^4}{144(2m-3)!}\right] \tag{5.3}$$

We calculate the following sums

$$S_1=\sum_{\gamma=0}^{\beta}C_0[\gamma]\frac{\left|[\beta-\gamma]\right|^{2m-1}}{(2m-1)!}-\sum_{\gamma=0}^{N}C_0[\gamma]\frac{\left|[\beta-\gamma]\right|^{2m-1}}{2(2m-1)!}$$
$$=\frac{[\beta]^{2m}}{(2m)!}-\frac{[\beta]^{2m-1}}{2(2m-1)!}+\frac{[\beta]^{2m-2}}{4(2m-2)!}+\frac{h^2}{12}\frac{[\beta]^{2m-2}}{(2m-2)!}-\frac{B_{2m}h^{2m}}{(2m)!} \tag{5.4}$$
$$+\sum_{i=0}^{2m-3}\frac{[\beta]^{2m-3-i}}{(2m-3-i)!}\left[\frac{B_{i+3}h^{i+3}}{(i+3)!}-\frac{(-1)^i}{2(i+3)!}-\frac{(-1)^i}{2}\sum_{j=1}^{i+1}\frac{B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!}\right]$$

$$S_2=\frac{[\beta]^{2m-2}+([\beta]-1)^{2m-2}}{2(2m-2)!}=\frac{[\beta]^{2m-2}}{(2m-2)!}+\sum_{i=0}^{2m-3}\frac{[\beta]^{2m-3-i}(-1)^{i+1}}{2(2m-3-i)!(i+1)!} \tag{5.5}$$

$$S_3=\int_0^1\frac{\left|x-[\beta]\right|^{2m-1}}{2(2m-1)!}dx=\frac{[\beta]^{2m}}{(2m)!}-\frac{[\beta]^{2m-1}}{2(2m-1)!}+\frac{[\beta]^{2m-2}}{4(2m-2)!}+\sum_{i=0}^{2m-3}\frac{[\beta]^{2m-3-i}(-1)^{i+3}}{2(2m-3-i)!(i+3)!} \tag{5.6}$$

Putting (5.4), (5.5) and (5.6) into (5.3), we get the following

$$\begin{aligned}\|\ell_N\|^2_{L_2^{(m)*}}=(-1)^m\Bigg[&\sum_{\beta=0}^{N}C_1[\beta]\{-P_{m-3}[\beta]-F_m[\beta]\}\\
&+\sum_{\beta=0}^{N}C_0[\beta]\Bigg\{-\frac{[\beta]^{2m}}{(2m)!}+\frac{[\beta]^{2m-1}}{2(2m-1)!}-\frac{[\beta]^{2m-2}}{4(2m-2)!}-\frac{h^2}{12}\frac{[\beta]^{2m-2}}{(2m-2)!}-\frac{B_{2m}h^{2m}}{(2m)!}\\
&+\sum_{i=0}^{2m-3}\frac{[\beta]^{2m-3-i}}{(2m-3-i)!}\left[\frac{B_{i+3}h^{i+3}}{(i+3)!}+\frac{(-1)^i}{2(i+3)!}-\sum_{j=1}^{i+1}\frac{(-1)^iB_{i+3-j}h^{i+3-j}}{2j!(i+3-j)!}-\frac{h^2}{12}\frac{(-1)^{i+1}}{(i+1)!}\right]\Bigg\}\\
&+\frac{h^2}{6(2m-1)!}+\frac{1}{(2m+1)!}+\frac{h^4}{144(2m-3)!}\Bigg]\end{aligned} \tag{5.7}$$

Now we simplify the equation (5.7) for $m=4$

$$\begin{aligned}\|\ell_N\|^2_{L_2^{(4)*}}=\Bigg[&\sum_{\beta=0}^{N}C_1[\beta]\{-P_1[\beta]-F_4[\beta]\}\\
&+\sum_{\beta=0}^{N}C_0[\beta]\Bigg\{-\frac{[\beta]^8}{40320}+\frac{[\beta]^7}{10080}-\frac{[\beta]^6}{2880}-\frac{h^2[\beta]^6}{8640}+\frac{h^8}{1209600}\\
&+\sum_{i=0}^{5}\frac{[\beta]^{5-i}}{(5-i)!}\left[\frac{B_{i+3}h^{i+3}}{(i+3)!}+\frac{(-1)^i}{2(i+3)!}-\sum_{j=1}^{i+1}\frac{(-1)^iB_{i+3-j}h^{i+3-j}}{2j!(i+3-j)!}-\frac{h^2}{12}\frac{(-1)^{i+1}}{(i+1)!}\right]\Bigg\}\\
&+\frac{h^2}{30240}+\frac{1}{362880}+\frac{h^4}{17280}\Bigg]\end{aligned} \tag{5.8}$$

From (4.2) we get the following for $\alpha=0$ and $\alpha=1$

$$\sum_{\gamma=0}^{N}C_1[\gamma]=0,$$

$$\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)=0$$

taking above equations into account, we simplify (5.8) the

$$\|\ell_N\|^2_{L_2^{(4)*}}=-\frac{h^4}{1440}\sum_{\beta=0}^{N}C_1[\beta][\beta]^2+\frac{h^8}{362880} \tag{5.9}$$

We calculate the optimal coefficients found in Result 4.2. by putting them in (5.9)

$$\|\ell_N\|^2_{L_2^{(4)*}} = -\frac{h^7}{172800(q^N+1)}\left[h^2\sum_{\beta=1}^{N-1} C_1[\beta]\beta^2 + C_1[\beta]\right] + \frac{h^8}{362880}$$

$$= -\frac{h^7}{172800(q^N+1)}\left[h^2\sum_{\beta=1}^{N-1}(q^\beta + q^{N-\beta})\beta^2 + \frac{q-q^N}{q-1}\right] + \frac{h^8}{362880}$$

$$= -\frac{h^7}{172800(q^N+1)}\left[\frac{q(N+1)(q^N-q)+(1-N)(q^{N+2}-q)}{3N^2(5q+1)}\right] + \frac{h^8}{362880}$$

$$= -\frac{h^7}{172800}\left[\frac{1}{3N} - \frac{(3q+1)(q^N-1)}{3N^2(5q+1)(q^N+1)}\right] + \frac{h^8}{362880}$$

$$= \frac{h^8}{1209600} + \frac{h^9(3q+1)(q^N-1)}{518400(5q+1)(q^N+1)}$$

Theorem 5.2 is completely proved.The proof of the Theorem 5.1 is similarly.

6. Numerical Results

We numerically analyze the analytical results obtained in this section and compare them with other works.

Determining the absolute value of $\|\ell\|_{L_2^{(4)*}}$ which constructed in the $L_2^{(4)}(0,1)$ space by $|R_N(\varphi)|$, we obtain the following from the Cauchy-Schwarz inequality

$$|R_N(\varphi)| \le \|\varphi\|_{L_2^{(4)}} \cdot \|\ell_N\|_{L_2^{(4)*}} .$$

The square of the $\left\|\overset{\circ}{\ell}_N\right\|_{L_2^{(4)*}}$ error functional norm of the optimal quadrature formula considered in the $L_2^{(4)}(0,1)$ space and the square of the $\|\ell\|_{L_2^{(4)*}}$ error functional norm of the quadrature formula constructed in the $L_2^{(4)}(0,1)$ space in [9] are numerically analyzed in TABLE 1 below.

TABLE 1. Squared norm of error functional of optimal quadrature formula

	N=10	**N=50**	**N=100**	**N=500**	**N=1000**
$\|\ell\|^2_{L_2^{(4)*}}$	9.93*10^(-15)	2.19*10^(-20)	8.42*10^(-23)	2.12*10^(-28)	2.29*10^(-31)
$\left\|\overset{\circ}{\ell}_N\right\|^2_{L_2^{(4)*}}$	9.38*10^(-15)	2.17*10^(-20)	8.37*10^(-23)	2.02*10^(-28)	8.27*10^(-31)

When the nodes are $N=10;50;100;150$, we calculate the integral value of the functions $\varphi(x)=x^5+\ln\left(x^2+1\right)$ (TABLE 2) and $f(x)=\cos\left(x^2\right)+e^{-2x}$ (TABLE 3) using the Hermite-type optimal quadrature formula built in the $L_2^{(4)}(0,1)$ space and denote it as O_2 EM.

At the same time, we calculate the integral value of the functions $\varphi(x)=x^5+\ln\left(x^2+1\right)$ (TABLE 2) and $f(x)=\cos\left(x^2\right)+e^{-2x}$ (TABLE 3) using the optimal quadrature formula [9] built in the $L_2^{(4)}(0,1)$ space when $N=10;50;100;150$ is present and denote it as O_1 EM.

The exact value of the integral $I_1=\int_0^1\varphi(x)dx$ and $I_2=\int_0^1 f(x)dx$.

TABLE 2. Error of optimal quadrature formula

N	I_1 (Exact value)	I_1- O_1 EM	I_1- O_2 EM
10	0.430601017	6.40*10^(-7)	4.45*10^(-7)
50	0.430601017	2.05*10^(-10)	1.42*10^(-10)
100	0.430601017	6.41*10^(-12)	4.45*10^(-12)
150	0.430601017	8.55*10^(-13)	5.95*10^(-13)

TABLE 3. Error of optimal quadrature formula

N	I_2 (Exact value)	I_2- O_1 EM	I_2- O_2 EM
10	1.33685640	2.57*10^(-7)	1.88*10^(-7)
50	1.33685640	8.84*10^(-11)	6.20*10^(-11)
100	1.33685640	2.79*10^(-12)	1.93*10^(-12)
150	1.33685640	3.72*10^(-13)	2.38*10^(-13)

Result 4.2 in the $L_2^{(4)}(0,1)$ space and the error of the derivative formula constructed in the work [10] are numerically analyzed in **TABLE 4** using the following 5 functions.

Example 1: $f(x)=e^{-x}$

Example 2: $f(x)=\dfrac{1}{1+x}$

Example 3: $f(x)=\sqrt{1+x^2}$

Example 4: $f(x)=\dfrac{\ln(1+x)}{1+x^2}$

Example 5: $f(x)=e^{-(x+\cos x)}$

where $I=\int_0^1 f(x)dx$

TABLE 4. Error of optimal quadrature formula

Error	Example 1	Example 2	Example 3	Example 4	Example 5
I - MSONC4	0.00096	0.00796	0.001277	0.01425	0.00198
I - O_2 EM	0.00085	0.00564	0.000387	0.00753	0.00151

7. Conclusion

In this research work, the derivative optimal quadrature formula was built using the values of the function up to the second derivative at the nodal points for the approximate calculation of the exact integrals. We found the representation of the error functional corresponding to the difference between the quadrature sum and the exact integral. The error functional $C_1[\beta]$ is a multivariate function with respect to the coefficients. To find the conditional extremum of a multivariable function, we constructed the Lagrange function and obtained the system of equations. By solving the system of equations, we found the analytical representation of the coefficients. Using the optimal coefficients, we calculated the norm of the error function and numerically analyzed the order of its approximation. We proved that this quadrature formula accurately integrates polynomials of $m-1$ degree. We analyzed the error of the proposed quadrature formula in numerical experiments using degree, exponential and logarithmic functions.

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