Single-Valued Neutrosophic Roughness via Ideals

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Abstract. In this paper, we connect the idea of single-valued neutrosophic ideal to the concept of single-valued neutrosophic approximation space to define the concept of single-valued neutrosophic ideal approximation spaces. We present the single-valued neutrosophic ideal approximation interior operator \( \text{int}_\psi \Phi \) and the single-valued neutrosophic ideal approximation closure operator \( \text{cl}_\psi \Phi \), and we present the single-valued neutrosophic ideal approximation pre-interior operator \( \text{pint}_\psi \Phi \) and the single-valued neutrosophic ideal approximation pre-closure operator \( \text{pcl}_\psi \Phi \), about this concerning single-valued neutrosophic ideal defined on the single-valued neutrosophic approximation space \( \langle \tilde{\chi}, \phi \rangle \) related with some single-valued neutrosophic set \( \psi \in \tilde{\xi} \). Also, we present single-valued neutrosophic separation axioms, single-valued neutrosophic connectedness, and single-valued neutrosophic compactness in single-valued neutrosophic approximation spaces and single-valued neutrosophic ideal approximation spaces as well, and prove the associations in between.

1. Introduction

Sometimes, it is not convenient to apply practical problems to real-life applications. Data in medical sciences, economics, weather, climate changes, etc. always involve various types of uncertainties. To exceed the difficulties in using the traditional classical methods the word neutrosophy is initiated to be a tool for handling problems involving incomplete, indeterminate, and inconsistent information. Smarandache [1] presented the idea of a neutrosophic set as an intuitionistic fuzzy set generalization. Salama et al [2] defined the neutrosophic set theory and neutrosophic crisp set. Correspondingly, Salama and Alblowi [3], introduced neutrosophic topology as they claimed a number of its characteristics. Others as Wang et al [4] defined the single-valued neutrosophic set concept. In (2020, 2021) Saber et al [5–11] introduced and studied the concepts...
of single-valued neutrosophic ideal, single-valued neutrosophic ideal open local function, connectedness in single-valued neutrosophic topological spaces \((\mathcal{E}, \tilde{T}^c, \tilde{T}^\delta)\) and compactness in single-valued neutrosophic ideal topological spaces.

Quite recently, considerable attention has been paid to the approximate operations on sets and approximate enclosure of sets introduced by Pawlak [12]. An approximation space \((\tilde{F}, \delta)\) is fashioned from a universe set of objects and an equivalence relation on these objects. The boundary among the lower approximation set \((\alpha_n)^l\) and the upper approximation set \((\alpha_n)^u\) of a set \(\delta\) in \((\tilde{F}, \delta)\) termed these rough sets. If the lower and the upper approximation sets are identical, then \(\alpha_n\) is then a thorough subset of \(\tilde{F}\), and there is no roughness. Many of their applications appear in the studies [13–19]. Irfan in [20] studied the connections between fuzzy set, rough set, and Soft set notions. Many papers studied the relationship between fuzzy rough set notions and fuzzy topologies [21, 22]. Recently, many researchers have used topological approaches in the study of rough sets and their applications. In [23], it was used the notion of ideal in soft rough ordinary topological space, and in [24], the authors introduced fuzzy soft connectedness in the sense of Chang [25].

In this article, we combined the idea of single-valued neutrosophic ideal \(\bar{h}\) with single-valued neutrosophic approximation space \((\text{SVNA-space}) (\tilde{x}, \varphi)\) related with single-valued neutrosophic set \(\psi\), and presented the concept of single-valued neutrosophic interior and single-valued neutrosophic closure operators concerning that single-valued neutrosophic ideal. The local function \(\Phi_{\psi}(\rho)\) of some \(\rho \in \tilde{x}\) concerning that single-valued neutrosophic ideal was a base in defining the associated interior and closure operators. Separation axioms in \(\text{SVNA-space} (\tilde{x}, \varphi, h)\) and in \(\text{SVNIA-space} (\tilde{x}, \varphi, h)\) have been obtained and we obtain some of their properties. Connectedness in \(\text{SVNA-space}\) and in \(\text{SVNIA-space}\) were defined and compared with examples to show the suggestions in between. Compactness in \(\text{SVNA-space} (\tilde{x}, \varphi, h)\) were defined as well.

**Definition 1.1.** [1] Let \(\tilde{x}\) be a non-empty set. A neutrosophic set (briefly, \(\text{NS}\)) in \(\tilde{x}\) is an object having the form

\[
\psi = \{\langle \kappa, \tilde{\eta}_\psi, \tilde{\gamma}_\psi, \tilde{\delta}_\psi \rangle : \omega \in \tilde{x}\}
\]

where

\[
\tilde{\eta} : \tilde{x} \rightarrow [-0, 1^+] \land \tilde{\gamma} : \tilde{x} \rightarrow [-0, 1^+] \land \tilde{\delta} : \tilde{x} \rightarrow [-0, 1^+]
\]

and

\[-0 \leq \tilde{\eta}_\psi(\kappa) + \tilde{\gamma}_\psi(\kappa) + \tilde{\delta}_\psi(\kappa) \leq 3^+\]

represent the degree of membership (\(\tilde{\eta}_\psi\)), the degree of indeterminacy (\(\tilde{\gamma}_\psi\)), and the degree of non-membership (\(\tilde{\delta}_\psi\)) respectively of any \(\kappa \in \tilde{x}\) to the set \(\psi\).
Definition 1.2. [4] Let $\mathfrak{X}$ be a space of points (objects), with a generic element in $\mathfrak{X}$ denoted by $\kappa$. Then $\psi$ is called a single-valued neutrosophic set (briefly, SVNS) in $\mathfrak{X}$, if $\psi$ has the form

$$\psi = \langle \kappa, \tilde{\eta}_\psi, \tilde{\gamma}_\psi, \tilde{\delta}_\psi \rangle : \kappa \in \mathfrak{X},$$

where, $\tilde{\eta}_\psi, \tilde{\gamma}_\psi, \tilde{\delta}_\psi : \mathfrak{X} \to [0, 1]$. In this case, $\tilde{\eta}_\psi, \tilde{\gamma}_\psi, \tilde{\delta}_\psi$ are called the truth of membership function, indeterminacy membership function, and falsity membership function, respectively.

For conformist motives and as there is no ambiguity, we denote an SVNS merely as a neutrosophic set throughout this article; we too paraphrase the definition, in order to view it clearly as a function from a non-empty set $\mathfrak{X}$ to $\xi = [0, 1]^3$, in the next method.

Let $\mathbb{X}$ be a nonempty set and $\xi = [0, 1]$. A NS on $\mathfrak{X}$ is a mapping defined as $\psi = \langle \eta_\psi, \tilde{\gamma}_\psi, \delta_\psi \rangle : \mathfrak{X} \to \xi$, where $\xi = I^3$ and $\eta_\psi, \tilde{\gamma}_\psi, \delta_\psi : \mathfrak{X} \to \xi$ such that $0 \leq \eta_\psi + \tilde{\gamma}_\psi + \delta_\psi \leq 3$.

We denote the set of all neutrosophic sets of $\mathfrak{X}$ by $\xi^\mathfrak{X}$ and the neutrosophic sets $\langle 0, 1, 1 \rangle$ and $\langle 1, 0, 0 \rangle$ by $\tilde{\theta}$ and $\tilde{\bar{\theta}}$ respectively.

Definition 1.3. [4,26,27] Let $\mathfrak{X}$ be a non-empty set and let $\psi, \rho \in \xi^\mathfrak{X}$ be given by $\psi = \langle \eta_\psi, \tilde{\gamma}_\psi, \delta_\psi \rangle$ and $\rho = \langle \eta_\rho, \tilde{\gamma}_\rho, \delta_\rho \rangle$. Then

1. The complement of $\psi$ (briefly, $\psi^c$) is given by

   $$\bar{\eta}_\psi(\kappa) = \delta_\psi(\kappa), \quad \bar{\gamma}_\psi(\kappa) = [\bar{\gamma}_\psi](\kappa), \quad \bar{\delta}_\psi(\kappa) = \tilde{\eta}_\psi(\kappa).$$

2. We say that $\psi \subseteq \rho$ for every $\kappa \in \mathfrak{X}$ if

   $$\eta_\psi(\kappa) \leq \eta_\rho(\kappa), \quad \tilde{\gamma}_\psi(\kappa) \geq \tilde{\gamma}_\rho(\kappa), \quad \delta_\psi(\kappa) \geq \delta_\rho(\kappa).$$

3. The union of $\psi$ and $\rho$ (briefly, $\psi \cup \rho$) is an SVNS in $\mathfrak{X}$ is given by,

   $$\psi \cup \rho = \langle (\eta_\psi \vee \eta_\rho)(\kappa), (\tilde{\gamma}_\psi \wedge \tilde{\gamma}_\rho)(\kappa), (\delta_\psi \wedge \delta_\rho)(\kappa) \rangle.$$

4. The intersection of $\psi$ and $\rho$ (briefly, $\psi \cap \rho$) is an SVNS in $\mathfrak{X}$ is given by,

   $$\psi \cap \rho = \langle (\eta_\psi \wedge \eta_\rho)(\kappa), (\tilde{\gamma}_\psi \vee \tilde{\gamma}_\rho)(\kappa), (\delta_\psi \vee \delta_\rho)(\kappa) \rangle.$$

For any arbitrary collection $\{\psi_i\}_{i \in J} \in \xi^\mathfrak{X}$ of SVNS the union and intersection are given by

5. $\bigvee_{i \in J} \psi_i = \langle \bigwedge_{i \in J} \eta_\psi_i(\kappa), \bigvee_{i \in J} \tilde{\gamma}_\psi_i(\kappa), \bigwedge_{i \in J} \delta_\psi_i(\kappa) \rangle$.

6. $\bigwedge_{i \in J} \psi_i = \langle \bigvee_{i \in J} \eta_\psi_i(\kappa), \bigwedge_{i \in J} \tilde{\gamma}_\psi_i(\kappa), \bigvee_{i \in J} \delta_\psi_i(\kappa) \rangle$.

Suppose that single valued neutrosophic relation (for short, SVN) $\varphi$ is defined as:

1. $\eta_\varphi(\kappa, \kappa) = 1$, $\tilde{\gamma}_\varphi(\kappa, \kappa) = 0$, $\delta_\varphi(\kappa, \kappa) = 0 \forall \kappa \in \mathfrak{X}$,

2. $\eta_\varphi(\kappa, \nu) = \tilde{\gamma}_\varphi(\nu, \kappa), \tilde{\gamma}_\varphi(\kappa, \nu) = \gamma_\varphi(\nu, \kappa), \delta_\varphi(\kappa, \nu) = \delta_\varphi(\nu, \kappa) \forall \kappa, \nu \in \mathfrak{X}$,

3. $\eta_\varphi(\kappa, \nu) \geq \left( \left( \eta_\varphi(\kappa, \omega) \wedge \eta_\varphi(\omega, \nu) \right), \tilde{\gamma}_\varphi(\kappa, \nu) \leq \left( \eta_\varphi(\kappa, \omega) \vee \tilde{\gamma}_\varphi(\omega, \nu) \right) \right)$ and $\delta_\varphi(\kappa, \nu) \leq \left( \gamma_\varphi(\kappa, \omega) \vee \tilde{\gamma}_\varphi(\omega, \nu) \right) \forall \kappa, \nu, \omega \in \mathfrak{X}$. 
The pair \((\bar{\chi}, \varphi)\) is said to be single valued neutrosophic approximation space ((for short, SVNA-space) created on single valued neutrosophic equivalence relation (briefly, SVNER) \(\varphi\) on \(\bar{\chi}\).

**Definition 1.4.** For each \(\kappa \in \bar{\chi}\), define a single valued neutrosophic coset (briefly, SVN-coset) by:

\[
\hat{\eta}_{\kappa}(v) = \bar{\eta}_{\varphi}(\kappa, v), \quad \bar{\gamma}_{\kappa}(v) = \bar{\gamma}_{\varphi}(\kappa, v), \quad \delta_{\kappa}(v) = \delta_{\varphi}(\kappa, v) \quad \forall v \in \bar{\chi},
\]

All elements \(v \in \bar{\chi}\) with SVN value \(\bar{\eta}_{\varphi}(\kappa, v) > 0, \bar{\gamma}_{\varphi}(\kappa, v) \leq 1, \delta_{\varphi}(\kappa, v) \leq 1\) are points having a membership value in the SVN-coset \([\kappa]\), and any point \(v \in \bar{\chi}\) with \(\bar{\eta}_{\varphi}(\kappa, v) = 0, \bar{\gamma}_{\varphi}(\kappa, v) = 1\) and \(\delta_{\varphi}(\kappa, v) = 1\) is not included in the SVN-coset \([\kappa]\). Any SVN-coset \([\kappa]\) confidiently contained within the point \(\kappa \in \bar{\chi}\), and So, \(\eta_{\varphi}\mid_{w\in\bar{\chi}}(\omega) = 1, \gamma_{\varphi}\mid_{w\in\bar{\chi}}(\omega) = 0, \delta_{\varphi}\mid_{w\in\bar{\chi}}(\omega) = 0, \forall \kappa \in \bar{\chi}\). Also, \(\eta_{\varphi}\mid_{w\in\bar{\chi}}(\omega) = 1, \gamma_{\varphi}\mid_{w\in\bar{\chi}}(\omega) = 0, \delta_{\varphi}\mid_{w\in\bar{\chi}}(\omega) = 0, \forall v \in \bar{\chi} [i.e. \bigvee_{\omega \in \bar{\chi}}(\omega)] = (0, 1, 1)].\n
Clearly, if \(\hat{\eta}_{\varphi}(\kappa, v) > 0, \bar{\gamma}_{\varphi}(\kappa, v) \leq 1, \delta_{\varphi}(\kappa, v) \leq 1\), then the SVN-cosets \([\kappa], [v]\) (as SVN-cosets) are having the same points of \(\bar{\chi}\) with some non zero membership values, and also, if \(\eta_{\varphi}\mid_{v}(\omega) = 0, \gamma_{\varphi}\mid_{v}(\omega) = 1\) and \(\delta_{\varphi}\mid_{v}(\omega) = 1\), then \(\hat{\eta}_{\kappa}(\omega) = 0, \hat{\gamma}_{\kappa}(\omega) = 1\) and \(\delta_{\kappa}(\omega) = 1\) whenever \(\hat{\eta}_{\varphi}(\kappa, v) > 0, \hat{\gamma}_{\varphi}(\kappa, v) \leq 1, \delta_{\varphi}(\kappa, v) \leq 1\). That is very two SVN-cosets are either two SVN-coset containing the same points of \(\bar{\chi}\) with some non-zero membership values or encompassing totally different points of \(\bar{\chi}\) with some non-zero membership values.

Let us define the difference between two SVN-cosets as follows:

\[
\hat{\eta}_{\psi}\Delta\hat{\rho}_{\kappa}(\kappa) = \begin{cases} 0, & \text{if } \hat{\eta}_{\psi}(\kappa) \leq \hat{\rho}_{\kappa}(\kappa), \\ (\hat{\eta}_{\psi} \land \hat{\rho}_{\kappa})(\kappa), & \text{otherwise.} \end{cases}
\]

\[
(\hat{\gamma}_{\psi}\lor\hat{\gamma}_{\kappa}(\kappa) = \begin{cases} 1, & \text{if } \hat{\gamma}_{\psi}(\kappa) \geq \hat{\gamma}_{\kappa}(\kappa), \\ (\hat{\gamma}_{\psi} \lor \hat{\gamma}_{\kappa})(\kappa), & \text{otherwise.} \end{cases}
\]

\[
\hat{\delta}_{\psi}\lor\hat{\delta}_{\kappa}(\kappa) = \begin{cases} 1, & \text{if } \hat{\delta}_{\psi}(\kappa) \geq \hat{\delta}_{\kappa}(\kappa), \\ (\hat{\delta}_{\psi} \lor \hat{\delta}_{\kappa})(\kappa), & \text{otherwise.} \end{cases}
\]

**Definition 1.5.** Let \(\psi \in \mathcal{E}\) and \(\varphi\) a SVN on \(\bar{\chi}\) and the SVN-cosets [are explained as in (*)]. Therefore, the single-valued neutrosophic lower (briefly, SVN-L) \(\psi_{\varphi}\) set, the single-valued neutrosophic upper (briefly, SVN-U) \(\psi_{\varphi}\) set and the single-valued neutrosophic boundary region (briefly, SVNBR) \(\psi_{\varphi}\) set can be defined as follows:

\[
\hat{\eta}_{\psi_{\varphi}}(\kappa) = \hat{\eta}_{\psi}(\kappa) \lor \bigwedge_{\psi_{\varphi}(\omega) > 0, \omega \neq \kappa} \hat{\delta}_{\kappa}(\omega),
\]

\[
\hat{\gamma}_{\psi_{\varphi}}(\kappa) = \hat{\gamma}_{\psi}(\kappa) \land \bigvee_{\psi_{\varphi}(\omega) > 0, \omega \neq \kappa} (1 - \hat{\gamma}_{\kappa}(\omega)),
\]

\[
\hat{\delta}_{\psi_{\varphi}}(\kappa) = \hat{\delta}_{\psi}(\kappa) \land \bigvee_{\psi_{\varphi}(\omega) > 0, \omega \neq \kappa} \hat{\eta}_{\kappa}(\omega),
\]

\[
\hat{\eta}_{\psi_{\varphi}}(\kappa) = \hat{\eta}_{\psi}(\kappa) \land \bigvee_{\psi(\omega) > 0, \omega \neq \kappa} \hat{\eta}_{\kappa}(\omega),
\]

\[
\hat{\gamma}_{\psi_{\varphi}}(\kappa) = \hat{\gamma}_{\psi}(\kappa) \lor \bigwedge_{\psi(\omega) > 0, \omega \neq \kappa} \hat{\gamma}_{\kappa}(\omega),
\]

\[
\hat{\delta}_{\psi_{\varphi}}(\kappa) = \hat{\delta}_{\psi}(\kappa) \lor \bigwedge_{\psi(\omega) > 0, \omega \neq \kappa} \hat{\delta}_{\kappa}(\omega),
\]
\begin{align*}
\mathcal{G}_{\psi^\alpha}(\kappa) &= \mathcal{G}_{\psi}(\kappa) \lor \bigwedge_{\psi(\alpha) > 0, \alpha \neq \kappa} \mathcal{G}_{[\kappa]}(\omega), \\
\delta_{\psi^\alpha}(\kappa) &= \delta_{\psi}(\kappa) \lor \bigwedge_{\psi(\alpha) > 0, \alpha \neq \kappa} \delta_{[\kappa]}(\omega),
\end{align*}
(1.3)

\begin{align*}
\tilde{\eta}_{(\psi^\alpha)}(\kappa) &= \left\{ \begin{array}{ll}
0, & \text{if } \tilde{\eta}_{\psi^\alpha}(\kappa) \leq \tilde{\eta}_{\psi}(\kappa), \\
(\tilde{\eta}_{\psi^\alpha} \land \tilde{\eta}_{(\psi^\alpha)^c})(\kappa), & \text{otherwise}.
\end{array} \right.
\end{align*}
(1.4)

\begin{align*}
\tilde{\eta}_{(\psi^\alpha)}(\kappa) &= \left\{ \begin{array}{ll}
1, & \text{if } \tilde{\eta}_{\psi^\alpha}(\kappa) \geq \tilde{\eta}_{\psi}(\kappa), \\
(\tilde{\eta}_{\psi^\alpha} \lor \tilde{\eta}_{(\psi^\alpha)^c})(\kappa), & \text{otherwise}.
\end{array} \right.
\end{align*}

**Lemma 1.1.** For every SVNS $\psi \in \xi^\chi$ we obtain simply that:

1. $(\psi \lor \rho)^\psi \geq \psi^\alpha \lor \rho^\psi$,
2. $(\phi \land \rho)^\psi \leq \psi^\alpha \land \rho^\psi$,
3. if $\psi \leq \rho$ then $\psi^\psi \leq \rho^\psi$ and $\psi^\psi \leq \rho^\psi$,
4. $(\psi \lor \rho)^\psi = \psi^\psi \lor \rho^\psi$,
5. $(\psi \land \rho)^\psi = \psi^\psi \land \rho^\psi$,
6. $(\psi^\psi)^c = (\psi^c)^\psi$ and $\psi^\psi = \psi^\psi$,
7. $(\psi^\psi) = \psi^\psi$,
8. $(\psi^\psi)^\psi = \psi^\psi$.

Associated with a SVNS $\psi \in \xi^\chi$ in a SVNA-space $(\chi, \varphi)$, we can define a single-valued neutrosophic interior operator $\text{int}_{\psi}^\psi : \xi^\chi \to \xi^\chi$ as follows:

\begin{equation}
\text{int}_{\psi}^\psi(\rho) = \psi^\psi \land \rho^\psi, \forall \rho \neq (1, 0, 0) \text{ and } \text{int}_{\psi}^\psi((1, 0, 0)) = (1, 0, 0).
\end{equation}
(1.5)

Also, it was defined a $\text{cl}_{\psi}^\psi : \xi^\chi \to \xi^\chi$ as next:

\begin{equation}
\text{cl}_{\psi}^\psi(\rho) = (\psi^\psi)^c \lor \rho^\psi, \forall \rho \neq (0, 1, 1) \text{ and } \text{cl}_{\psi}^\psi((0, 1, 1)) = (0, 1, 1).
\end{equation}
(1.6)

Remembrance that:

\begin{align*}
\text{cl}_{\psi}^\psi(\rho^\psi) &= \text{cl}_{\psi}^\psi(\rho), \forall \rho \in \xi^\chi, \text{int}_{\psi}^\psi(\rho^\psi) = \text{int}_{\psi}^\psi(\rho), \forall \rho \in \xi^\chi, \\
\text{int}_{\psi}^\psi(\rho^c) &= [\text{cl}_{\psi}^\psi(\rho)]^c \text{ and } \text{cl}_{\psi}^\psi(\rho^c) = [\text{int}_{\psi}^\psi(\rho)]^c, \forall \rho \in \xi^\chi.
\end{align*}
(1.7)

**Definition 1.6.** Let $(\chi, \varphi)$ be a SVNA-space. Subsequently, for every $\psi \in \xi^\chi$. Therefore,

1. $\rho$ is single-valued neutrosophic preopen (SVN-preopen) [resp. preclosed (SVN-preclosed)] set iff $\rho \leq \text{int}_{\psi}^\psi(\text{cl}_{\psi}^\psi(\rho))$ [resp. $\rho \geq \text{cl}_{\psi}^\psi(\text{int}_{\psi}^\psi(\rho))$].

2. The single-valued neutrosophic preinterior of $\rho$ (abbreviated, $\text{pint}_{\psi}^\psi$) can be defined as follows:

\begin{equation}
\text{pint}_{\psi}^\psi(\rho) = \bigvee \{ \pi \in \xi^\chi : \rho \geq \pi, \pi \text{ is SVN-preopen} \}.
\end{equation}
(3) The single-valued neutrosophic preclosure of ρ (abbreviated, pclψ ρ) can be defined as follows:

\[ \text{pcl}_\psi (\rho) = \bigwedge \{ \pi \in \xi^\varphi : \rho \leq \pi, \pi \text{ is SVN-preclosed} \}. \]

2. Single-valued neutrosophic ideal approximation spaces

In this section, we first introduce and analyze the single-valued neutrosophic ideal approximation space (abbreviated, SVNIA-space) and single-valued neutrosophic operator clψ ρ, pclψ ρ, intψ ρ. Subsequently, we analyze the local single valued neutrosophic closed set (Φψ (ρ)(φ, h), for brevity) and local single valued neutrosophic preclosed set (Φpψ (ρ)(φ, h), for brevity).

**Definition 2.1.** A subset h ⊆ ξ^\varphi is known as the single-valued neutrosophic ideal (SVNI) on \( \chi \) if it meets the next criteria.

1. \((0, 1, 1) \in h, \]
2. If \( \tilde{\eta}_\psi (\kappa) \leq \tilde{\eta}_\rho (\kappa), \tilde{\gamma}_\psi (\kappa) \geq \gamma_\rho (\kappa), \delta_\psi (\kappa) \geq \delta_\rho (\kappa) \) and \( \psi, \rho \in h \), then, \( \rho, \rho \in h, \forall \kappa \in \chi \) and \( \psi, \rho \in \xi^\varphi, \)
3. If \( \psi \in h \) and \( \rho \in h \), then \( \langle (\tilde{\eta}_\psi \lor \tilde{\eta}_\rho) (\kappa), (\tilde{\gamma}_\psi \land \gamma_\rho) (\kappa), (\delta_\psi \land \delta_\rho) (\kappa) \rangle \in h, \forall \kappa \in \chi \) and \( \psi, \rho \in \xi^\varphi. \)

If \( h_1 \) and \( h_2 \) are SVNIs on \( \chi \), we have \( h_1 \) is finer than \( h_1 \) if \( h_1 \subseteq h_2 \). The triple \((\chi, \varphi, h)\) is known as an single valued neutrosophic ideal approximation space (abbreviated, SVNIA-space). Occasionally, \( h = (0, 1, 1) \) is written as \( h^\varphi \) herein to avoid ambiguity.

**Definition 2.2.** Let \((\chi, \varphi, h)\) be an SVNIA-space related with \( \psi \in \xi^\varphi \). Therefore

1. The local single valued neutrosophic closed set \( \Phi^p_\psi (\rho)(\varphi, h) \) of a set \( \rho \in \xi^\varphi \) is defined by:

\[ \Phi^p_\psi (\rho)(\varphi, h) = \bigwedge \{ \pi \in \xi^\varphi : \rho \overline{\land} \pi = (\tilde{\eta}_\rho \overline{\land} \tilde{\eta}_\psi (\kappa), \tilde{\gamma}_\rho \overline{\lor} \gamma_\psi (\kappa), \delta_\rho \overline{\land} \delta_\psi (\kappa)) \in h, \text{pcl}_\psi (\pi) = \pi \}. \]  

Occasionally, \( \Phi^p_\psi (\rho) \) or \( \Phi^p_\psi (\rho)(h) \) herein to avoid ambiguity.

2. The local single valued neutrosophic pre-closed set \( \Phi^p_\psi (\rho)(\varphi, h) \) of a set \( \rho \in \xi^\varphi \) is defined by:

\[ \Phi^p_\psi (\rho)(\varphi, h) = \bigwedge \{ \pi \in \xi^\varphi : \rho \overline{\land} \pi = (\tilde{\eta}_\rho \overline{\land} \tilde{\eta}_\psi (\kappa), \tilde{\gamma}_\rho \overline{\lor} \gamma_\psi (\kappa), \delta_\rho \overline{\land} \delta_\psi (\kappa)) \in h, \text{pcl}_\psi (\pi) = \pi \}. \]

We will write \( \Phi^p_\psi (\rho) \) or \( \Phi^p_\psi (\rho)(h) \) instead of \( \Phi^p_\psi (\rho)(\varphi, h) \).

**Lemma 2.1.** Let \((\chi, \varphi, h^\varphi)\) be a SVNIA-space, \( \psi \in \xi^\varphi \). Then for any \( \rho \in \xi^\varphi \) we have \( \Phi^p_\psi (\rho) = \text{cl}_\psi (\rho) \) and \( \Phi^p_\psi (\rho) = \text{pcl}_\psi (\rho). \)

**Proposition 2.1.** Let \((\chi, \varphi, h)\) be a SVNIA-space related with \( \psi \in \xi^\varphi \). Then,

1. \( \rho \leq \pi \) implies \( \Phi^p_\psi (\rho) \leq \Phi^p_\psi (\pi) \) and \( \Phi^p_\psi (\rho) \leq \Phi^p_\psi (\pi). \)
2. If \( h_1 \) and \( h_2 \) are SVNIs on \( \chi \) and \( h_1 \subseteq h_2 \), then \( \Phi^p_\psi (\rho)(h_1) \geq \Phi^p_\psi (\rho)(h_2) \) and \( \Phi^p_\psi (\rho)(h_1) \geq \Phi^p_\psi (\rho)(h_2). \)
3. \( \Phi^p_\psi (\rho) \leq \Phi^p_\psi (\rho) \leq \text{cl}_\psi (\Phi^p_\psi (\rho)) \leq \text{cl}_\psi (\Phi^p_\psi (\rho)) \leq \text{pcl}_\psi (\Phi^p_\psi (\rho)) \leq \text{pcl}_\psi (\rho) \leq \text{cl}_\psi (\rho). \)
4. \( \Phi^p_\psi (\Phi^p_\psi (\rho)) \leq \text{cl}_\psi (\Phi^p_\psi (\rho)) = \Phi^p_\psi (\rho). \)
5. \( \Phi^p_\psi (\Phi^p_\psi (\rho)) \leq \text{pcl}_\psi (\Phi^p_\psi (\rho)) = \Phi^p_\psi (\rho). \)
6. \( \Phi^p_\psi (\rho) \lor \Phi^p_\psi (\pi) \leq \Phi^p_\psi (\rho \lor \pi) \) and \( \Phi^p_\psi (\rho) \land \Phi^p_\psi (\pi) \geq \Phi^p_\psi (\rho \land \pi). \)
Definition 2.3. Let \((\bar{\chi}, \bar{\varphi}, \bar{h})\) be a SVNIA-space associated with \(\psi \in \bar{\xi}^r\). Then for each \(\rho \in \bar{\xi}^r\) define the single valued neutrosophic operator \(\text{cl}^\psi_{\bar{\phi}}, \text{pcl}^\psi_{\bar{\phi}}, \text{int}^\psi_{\bar{\phi}}, \text{pint}^\psi_{\bar{\phi}}: \bar{\xi}^r \to \bar{\xi}^r\) as next:

\[
\text{cl}^\psi_{\bar{\phi}}(\rho) = \rho \lor \Phi^\psi(\rho) \quad \text{pcl}^\psi_{\bar{\phi}}(\rho) = \rho \lor \Phi^p_{\psi} \quad \forall \rho \in \bar{\xi}^r.
\]

\[
\text{int}^\psi_{\bar{\phi}}(\rho) = \rho \land (\Phi^\psi(\rho^c))^c \quad \text{pint}^\psi_{\bar{\phi}}(\rho) = \rho \land (\Phi^p_{\psi}(\rho^c))^c \quad \forall \rho \in \bar{\xi}^r.
\]

(2.3)

(2.4)

Proposition 2.2. Let \((\bar{\chi}, \bar{\varphi}, \bar{h})\) be a SVNIA-space related with \(\psi \in \bar{\xi}^r\). Then, for any \(\rho, \pi \in \bar{\xi}^r\), we have:

1. \(\text{int}^\psi_{\bar{\phi}}(\rho) \preceq \text{pint}^\psi_{\bar{\phi}}(\rho) \preceq \text{int}^\psi_{\bar{\phi}}(\rho) \preceq \text{pcl}^\psi_{\bar{\phi}}(\rho) \preceq \text{cl}^\psi_{\bar{\phi}}(\rho)\).
2. \(\text{cl}^\psi_{\bar{\phi}}(\rho^c) = (\text{int}^\psi_{\bar{\phi}}(\rho))^c\) and \(\text{int}^\psi_{\bar{\phi}}(\rho^c) = (\text{cl}^\psi_{\bar{\phi}}(\rho))^c\).
3. \(\text{cl}^\psi_{\bar{\phi}}(\rho \lor \pi) \preceq \text{cl}^\psi_{\bar{\phi}}(\rho) \land \text{cl}^\psi_{\bar{\phi}}(\pi)\) and \(\text{cl}^\psi_{\bar{\phi}}(\rho \land \pi) \preceq \text{cl}^\psi_{\bar{\phi}}(\rho) \land \text{cl}^\psi_{\bar{\phi}}(\pi)\).
4. \(\text{int}^\psi_{\bar{\phi}}(\rho \lor \pi) \preceq \text{int}^\psi_{\bar{\phi}}(\rho) \land \text{int}^\psi_{\bar{\phi}}(\pi)\) and \(\text{int}^\psi_{\bar{\phi}}(\rho \land \pi) \preceq \text{int}^\psi_{\bar{\phi}}(\rho) \land \text{int}^\psi_{\bar{\phi}}(\pi)\).
5. \(\text{cl}^\psi_{\bar{\phi}}(\text{cl}^\psi_{\bar{\phi}}(\rho)) \preceq \text{cl}^\psi_{\bar{\phi}}(\rho)\) and \(\text{int}^\psi_{\bar{\phi}}(\text{int}^\psi_{\bar{\phi}}(\rho)) \preceq \text{int}^\psi_{\bar{\phi}}(\rho)\).
6. If \(\rho \leq \pi\), then \(\text{cl}^\psi_{\bar{\phi}}(\rho) \leq \text{cl}^\psi_{\bar{\phi}}(\pi)\) and \(\text{int}^\psi_{\bar{\phi}}(\rho) \leq \text{int}^\psi_{\bar{\phi}}(\pi)\).
7. \(\text{pcl}^\psi_{\bar{\phi}}(\rho) \preceq \text{pcl}^\psi_{\bar{\phi}}(\rho)\).

Proof. From (7): Suppose that \(\text{pcl}^\psi_{\bar{\phi}}(\rho) \not\preceq \text{pcl}^\psi_{\bar{\phi}}(\rho)\), and if \(\text{pcl}^\psi_{\bar{\phi}}(\rho) = \pi\), then \(\rho \leq \pi\) and \(\pi\) is SVN-preclosed set with \(\text{pcl}^\psi_{\bar{\phi}}(\rho) \not\preceq \pi\). But \(\rho \leq \pi\) implies \(\rho \setminus \pi = \langle \tilde{\eta}, \tilde{\eta}, \tilde{\eta}, \tilde{\eta}, \tilde{\eta}, \tilde{\eta}, \tilde{\eta}, \tilde{\eta} \rangle \in \bar{h}\), and thus \(\Phi^p_{\psi}(\rho) \leq \pi\) which means that \(\text{pcl}^\psi_{\bar{\phi}}(\rho) = \rho \lor \Phi^p_{\psi}(\rho) \preceq \rho \land \pi \leq \pi\), which is a contradiction. Hence, \(\text{pcl}^\psi_{\bar{\phi}}(\rho) \preceq \text{pcl}^\psi_{\bar{\phi}}(\rho)\).

(1)-(6): Clear.

Definition 2.4. Let \((\bar{\chi}, \bar{\varphi}, \bar{h})\) be a SVNIA-space related with \(\psi \in \bar{\xi}^r\). Then,

1. \(\rho \in \bar{\xi}^r\) is termed a single valued neutrosophic ideal pre-open \((SVNI-preopen, \text{abbreviated})\) if \(\rho \preceq \text{int}^\psi_{\bar{\phi}}(\text{cl}^\psi_{\bar{\phi}}(\rho))\).
2. \(\rho \in \bar{\xi}^r\) is termed a single valued neutrosophic \(\Phi\)-open \((SVNI\Phi-open, \text{abbreviated})\) if \(\rho \preceq \text{int}^\psi_{\bar{\phi}}(\Phi^\psi(\rho))\).

The complement of \(SVNI\Phi\)-closed \((\text{resp., SVNI-preclosed})\) is termed \(SVNI\Phi\)-open \((\text{resp., SVNI-preopen})\).

3. \(\rho \in \bar{\xi}^r\) is termed a single valued neutrosophic dense in itself \((SVNI-dense\ in\ itself, \text{abbreviated})\) if \(\rho \preceq \Phi^\psi(\rho)\).

Theorem 2.1. Let \((\bar{\chi}, \bar{\varphi}, \bar{h})\) be a SVNIA-space related with \(\psi \in \bar{\xi}^r\). Then,

1. If \(\rho \in \bar{\xi}^r\) is \(SVNI\Phi\)-closed, then \(\rho \preceq \Phi^\psi(\text{int}^\psi_{\bar{\phi}}(\rho))\).
2. If \(\rho \in \bar{\xi}^r\) is \(SVNI\)-preclosed, then \(\rho \preceq \text{cl}^\psi_{\bar{\phi}}(\text{int}^\psi_{\bar{\phi}}(\rho))\).

Proof. For (1): Let \(\rho \in \bar{\xi}^r\) is \(SVNI\Phi\)-closed. Then,

\[
\rho^c \preceq \text{int}^\psi_{\bar{\phi}}(\Phi^\psi(\rho^c)) \leq \text{int}^\psi_{\bar{\phi}}(\text{cl}^\psi_{\bar{\phi}}(\rho^c)) = \text{int}^\psi_{\bar{\phi}}((\text{int}^\psi_{\bar{\phi}}(\rho))^c) = (\text{cl}^\psi_{\bar{\phi}}(\text{int}^\psi_{\bar{\phi}}(\rho))^c) \preceq (\Phi^\psi(\text{int}^\psi_{\bar{\phi}}(\rho))^c).}
\]
Hence, $\Phi_{\psi}(\text{int}_{\psi}(\rho)) \leq \rho$.

(2), it is easy. □

It is clear that:

$$
\begin{array}{c}
\text{SVN}\Phi\text{-open}(\text{SVN}\Phi\text{-open}) 
\quad \longrightarrow 
\text{SVNI}\text{-preopen}(\text{SVNI}\text{-preclosed})
\end{array}
$$

\[ \downarrow \quad \downarrow \]

\[ \text{SVNI}\text{-preopen}(\text{SVNI}\text{-preclosed}) 
\quad \text{SVNI}\text{-preopen}(\text{SVNI}\text{-preclosed}) \]

**Example 2.1.** Let $\chi$ be SVNR on a set $\tilde{\chi} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ defined as next.

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\kappa_3$</th>
<th>$\kappa_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>(1, 0, 0)</td>
<td>(1, 0, 0)</td>
<td>(0, 1, 1)</td>
<td>(0, 1, 1)</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>(1, 0, 0)</td>
<td>(1, 0, 0)</td>
<td>(0, 1, 1)</td>
<td>(0, 1, 1)</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>(0, 1, 1)</td>
<td>(0, 1, 1)</td>
<td>(1, 0, 0)</td>
<td>(0.6, 0.4, 0.6)</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>(0, 1, 1)</td>
<td>(0, 1, 1)</td>
<td>(0.6, 0.4, 0.6)</td>
<td>(1, 0, 0)</td>
</tr>
</tbody>
</table>

**Table 1. Caption**

Assume that $\psi = \langle (1, 0, 0), (1, 0, 0), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5) \rangle$. Then,

$$
\tilde{\eta}_{\psi}(\kappa_1) = \tilde{\eta}_\psi(\kappa_1) \wedge \bigvee_{\psi(\omega) > 0, \omega \neq \kappa_1} \tilde{\eta}_{\psi}[\omega] = 1,
$$

$$
\tilde{\gamma}_{\psi}(\kappa_1) = \tilde{\gamma}_\psi(\kappa_1) \vee \bigwedge_{\psi(\omega) > 0, \omega \neq \kappa_1} (\tilde{\gamma}_{\psi}[\omega]) = 0,
$$

$$
\tilde{\delta}_{\psi}(\kappa_1) = \tilde{\delta}_\psi(\kappa_1) \vee \bigwedge_{\psi(\omega) > 0, \omega \neq \kappa_1} (\tilde{\delta}_{\psi}[\omega]) = 0,
$$

Hence, $(\psi)^\varphi(\kappa_1) = (0, 1, 1)$ and similarly, we can obtain $(\psi)^\varphi(\kappa_2) = (0, 1, 1)$ and

$$
\tilde{\eta}_{\psi}(\kappa_3) = \tilde{\eta}_\psi(\kappa_3) \wedge \bigvee_{\psi(\omega) > 0, \omega \neq \kappa_3} \tilde{\eta}_{\psi}[\omega] = 0.5,
$$

$$
\tilde{\gamma}_{\psi}(\kappa_3) = \tilde{\gamma}_\psi(\kappa_3) \vee \bigwedge_{\psi(\omega) > 0, \omega \neq \kappa_3} (\tilde{\gamma}_{\psi}[\omega]) = 0.5,
$$

$$
\tilde{\delta}_{\psi}(\kappa_3) = \tilde{\delta}_\psi(\kappa_3) \vee \bigwedge_{\psi(\omega) > 0, \omega \neq \kappa_3} (\tilde{\delta}_{\psi}[\omega]) = 0.6.
$$

Hence, $\psi^\varphi(\kappa_3) = (0.5, 0.5, 0.6)$ and similarly, we can obtain $\psi^\varphi(\kappa_4) = (0.5, 0.5, 0.6)$. Thus and by equation (1.3), we have

$$
\psi^\varphi = \langle (1, 0, 0), (1, 0, 0), (0.5, 0.5, 0.6), (0.5, 0.5, 0.6) \rangle,
$$

$$
\psi_5 = \langle (1, 0, 0), (1, 0, 0), (0.6, 0.5, 0.5), (0.6, 0.5, 0.5) \rangle.$$
(ψ_φ)^c = \langle 0, 1, 1, 0, 1, 1, 0.5, 0.5, 0.6 \rangle, (0.5, 0.5, 0.6),

Let, \( \rho = \langle 0.3, 0.3, 0.3 \rangle, (0.3, 0.3, 0.3), (1, 1), (1, 1) \rangle, then by equations (3) and (4), we obtain \( \rho^\phi = \langle (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.6, 0.6, 0.6) \rangle \) and \( \rho_\chi = \langle (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.6, 0.6, 0.6) \rangle \). Hence,

\[ \text{cl}_\psi^\phi (\rho) = (\psi_\phi)^c \cup \rho^\phi = \langle (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.5, 0.5, 0.6), (0.5, 0.5, 0.6) \rangle \]

implies that

\[ \rho \leq \text{int}_\psi^\phi (\text{cl}_\psi^\phi (\rho)) = \langle (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.5, 0.6, 0.6), (0.5, 0.6, 0.6) \rangle \).

Thus, \( \rho \) is SVN-preopen.

Assume that a SVN1 is defined on \( \check{x} \) as next

\[ h = \{ \pi_n \in \xi^\check{x} : \pi_n \leq \langle (0.5, 0.3, 0.3), (0.5, 0.3, 0.3), (1, 0, 0), (1, 0, 0) \rangle \} . \]

By equations (1.1), (1.2) and (1.3) we get that, \( \rho \) it is neither SVN-preopen nor SVNΦ-open.

**Theorem 2.2.** Let \( (\check{x}, \varphi, h) \) be a SVNIA-space related with \( \psi \in \xi^\check{x} \). Then, the next are equivalent.

1. \( \rho \in \xi^\check{x} \) is SVN-open.
2. \( \rho \in \xi^\check{x} \) is SVN-preopen and SVN-dense in itself

**Proof.** (1) \( \Rightarrow \) (2): It is very easy to see that all SVNΦ-open set is SVN-preopen. On the other hand \( \rho \text{int}_\psi^\phi (\Phi_\varphi (\rho)) \), which means \( \rho \) is SVN-dense in itself.

(2) \( \Rightarrow \) (1): By assumption, \( \rho \leq \text{int}_\psi^\phi (\text{cl}_\psi^\phi (\rho)) = \text{int}_\psi^\phi (\rho \lor \Phi_\varphi (\rho)) = \text{int}_\psi^\phi (\Phi_\varphi (\rho)) \), and hence \( \rho \) is SVNΦ-open. \( \square \)

3. Separation axioms in single-valued neutrosophic ideal approximation spaces

The goal of this unit is to familiarize the concepts of SVNIT_0^{(t,s)}, SVNIT_1^{(t,s)}, SVNIT_2^{(t,s)} and SVNIT_0^{(t,s)}, SVNIT_1^{(t,s)}, SVNIT_2^{(t,s)}.

**Definition 3.1.** Let \( (\check{x}, \varphi, h) \) be a SVNIA-space related with \( \psi \in \xi^\check{x} \). Then,

1. A SVNIA-space \( (\check{x}, \varphi, h) \) (resp. a SVNIA-space \( (\check{x}, \varphi) \)) is termed a SVNIT_0^{(t,s)} (resp. SVNIT_0^{(t,s)}) if for every \( x \neq y \in \check{x} \), there exists \( \rho \in \xi^\check{x} \), \( t \in \xi_0 \) with \( \check{\eta}_{\text{int}_\psi^\phi (\rho)} (x) \geq t, \check{\gamma}_{\text{int}_\psi^\phi (\rho)} (x) \leq t \) and \( \check{\delta}_{\text{int}_\psi^\phi (\rho)} (x) \leq t \) [resp. \( \check{\eta}_{\text{int}_\psi^\phi (\rho)} (y) \geq t, \check{\gamma}_{\text{int}_\psi^\phi (\rho)} (y) \leq t \) such that \( \check{\eta}_{\rho} (y) < t, \check{\gamma}_{\rho} (y) > t \delta_{\rho} (y) > t \)].

2. A SVNIA-space \( (\check{x}, \varphi, h) \) (resp. a SVNIA-space \( (\check{x}, \varphi) \)) is termed a SVNIT_1^{(t,s)} (resp. SVNIT_1^{(t,s)}) if for every \( x \neq y \in \check{x} \), there exists \( \rho, \tau \in \xi^\check{x} \), \( t, \tau \in \xi_0 \) with \( \check{\eta}_{\text{int}_\psi^\phi (\rho)} (x) \geq t, \check{\gamma}_{\text{int}_\psi^\phi (\rho)} (x) \leq t, \check{\delta}_{\text{int}_\psi^\phi (\rho)} (x) \leq t \) and \( \check{\eta}_{\text{int}_\psi^\phi (\tau)} (y) \geq \tau, \check{\gamma}_{\text{int}_\psi^\phi (\tau)} (y) \leq \tau, \check{\delta}_{\text{int}_\psi^\phi (\tau)} (y) \leq \tau \) such that \( \check{\eta}_{\rho} (y) < t, \check{\gamma}_{\rho} (y) > t \delta_{\rho} (y) > t \).
Example 3.1. Let $\varphi$ be a $SVNR$ on a set $\tilde{\chi} = \{x, y, z\}$ as shown down.

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>(1, 0.0)</td>
<td>(0.3, 0.7, 0.3)</td>
<td>(0, 1, 1)</td>
</tr>
<tr>
<td>$y$</td>
<td>(0.3, 0.7, 0.3)</td>
<td>(1, 0, 0)</td>
<td>(0.1, 0.9, 0.1)</td>
</tr>
<tr>
<td>$z$</td>
<td>(0, 1, 1)</td>
<td>(0.1, 0.9, 0.1)</td>
<td>(1, 0, 0)</td>
</tr>
</tbody>
</table>

Assume that $\psi = \langle (1, 0, 0), (0.8, 0.8, 0.2), (0, 1, 1) \rangle$ and $t = s = 0.5$. Then,

\[
\bar{\eta}_{\psi^x} (x) = \bar{\eta}_{\psi} (x) \lor \bigwedge_{\psi^x (\omega) > 0, \omega \neq x} \bar{\delta}_{x} (\omega) = 1,
\]

\[
\bar{\gamma}_{\psi^x} (x) = \bar{\gamma}_{\psi} (x) \land \bigvee_{\psi^x (\omega) > 0, \omega \neq x} (1 - \bar{\gamma}_{x} (\omega)) = 0,
\]

\[
\bar{\delta}_{\psi^x} (x) = \bar{\delta}_{\psi} (x) \land \bigvee_{\psi^x (\omega) > 0, \omega \neq x} \bar{\eta}_{x} (\omega) = 0,
\]

\[
\bar{\eta}_{\psi^y} (y) = \bar{\eta}_{\psi} (y) \lor \bigwedge_{\psi^y (\omega) > 0, \omega \neq y} \bar{\delta}_{y} (\omega) = 0.8,
\]

\[
\bar{\gamma}_{\psi^y} (y) = \bar{\gamma}_{\psi} (y) \land \bigvee_{\psi^y (\omega) > 0, \omega \neq y} (1 - \bar{\gamma}_{y} (\omega)) = 0.3,
\]

\[
\bar{\delta}_{\psi^y} (y) = \bar{\delta}_{\psi} (y) \land \bigvee_{\psi^y (\omega) > 0, \omega \neq y} \bar{\eta}_{y} (\omega) = 0.2,
\]

and

\[
\bar{\eta}_{\psi^z} (z) = \bar{\eta}_{\psi} (z) \lor \bigwedge_{\psi^z (\omega) > 0, \omega \neq z} \bar{\delta}_{z} (\omega) = 0.1,
\]
\[ \tilde{\gamma}_\varphi(z) = \tilde{\gamma}_\varphi(z) \land \bigvee_{\varphi'(\omega) > 0, \omega \neq z} (1 - \tilde{\gamma}_\varphi(z))(\omega) = 0.1, \]

\[ \tilde{\delta}_\varphi(z) = \tilde{\delta}_\varphi(z) \land \bigvee_{\varphi'(\omega) > 0, \omega \neq z} \tilde{\eta}_\varphi(z)(\omega) = 0.1. \]

Hence, \( \psi_\varphi = \langle (1, 0, 0), (0.8, 0.3, 0.2), (0.1, 0.1, 0.1) \rangle \) and by equation (3), we have

\[ \psi^{\varphi} = \langle (0.3, 0.7, 0.3), (0.3, 0.8, 0.2), (0, 1, 1) \rangle, \]

and hence, we get

\[ (\psi_\varphi)^c = \langle (0, 1, 1), (0.2, 0.7, 0.8), (0.1, 0.9, 0.1) \rangle. \]

Now, for the case \( x \neq y \), there exists \( \rho = \langle (0.8, 0.2, 0.8), (0, 1, 1), (0.6, 0.4, 0.6) \rangle \), and then

\[ \rho_\varphi = \langle (0.8, 0.2, 0.3), (0.1, 0.3, 0.3), (0.6, 0.1, 0.1) \rangle, \]

which means \( \text{int}\psi_\varphi(\rho) = \psi_\varphi \land \rho_\varphi = \langle (0.8, 0.2, 0.3), (0.1, 0.3, 0.3), (0.1, 0.1, 0.1) \rangle \), and thus

\[ \tilde{\eta}_{\text{int}\psi_\varphi(\rho)}(x) \geq 0.5, \quad \tilde{\gamma}_{\text{int}\psi_\varphi(\rho)}(x) \leq 0, \quad \tilde{\delta}_{\text{int}\psi_\varphi(\rho)}(x) \leq 0.5, \]

with \( \tilde{\eta}_\rho(y) < 0.5, \tilde{\gamma}_\rho(y) > 0.5 \tilde{\delta}_\rho(y) > 0.5. \)

Also, we can find \( \pi = \langle (0, 1, 1), (0.6, 0.4, 0.4), (0.1, 0.1, 0.1) \rangle \), and then

\[ \pi_\varphi = \langle (0, 1, 1), (0.6, 0.3, 0.3), (0.1, 0.1, 0.1) \rangle \]

which means \( \text{int}\psi_\varphi(\pi) = \psi_\varphi \land \pi_\varphi = \langle (0, 1, 1), (0.6, 0.3, 0.3), (0.1, 0.1, 0.1) \rangle \), and thus

\[ \tilde{\eta}_{\text{int}\psi_\varphi(\pi)}(y) \geq 0.5, \quad \tilde{\gamma}_{\text{int}\psi_\varphi(\pi)}(y) \leq 0.5, \quad \tilde{\delta}_{\text{int}\psi_\varphi(\pi)}(y) \leq 0.5, \]

with \( \tilde{\eta}_\pi(x) < 0.5, \tilde{\gamma}_\pi(x) > 0.5 \tilde{\delta}_\pi(x) > 0.5. \)

For the cases \( x \neq z \) and \( y \neq z \) we can find \( \sigma \in \xi^k \) with

\[ \tilde{\eta}_{\text{int}\psi_\varphi(\sigma)}(x) \geq 0.5, \quad \tilde{\gamma}_{\text{int}\psi_\varphi(\sigma)}(x) \leq 0.5, \quad \tilde{\delta}_{\text{int}\psi_\varphi(\sigma)}(x) \leq 0.5, \]

or

\[ \tilde{\eta}_{\text{int}\psi_\varphi(\sigma)}(y) \geq 0.5, \quad \tilde{\gamma}_{\text{int}\psi_\varphi(\sigma)}(y) \leq 0.5, \quad \tilde{\delta}_{\text{int}\psi_\varphi(\sigma)}(y) \leq 0.5, \]

such that \( \tilde{\eta}_\sigma(z) < 0.5, \tilde{\gamma}_\sigma(z) > 0.5 \tilde{\delta}_\sigma(z) > 0.5, \) while we can not find \( \sigma \in \xi^k \) with

\[ \tilde{\eta}_{\text{int}\psi_\varphi(\sigma)}(z) \geq 0.5, \quad \tilde{\gamma}_{\text{int}\psi_\varphi(\sigma)}(z) \leq 0.5, \quad \tilde{\delta}_{\text{int}\psi_\varphi(\sigma)}(z) \leq 0.5. \]

Hence, \( (\tilde{\chi}, \varphi) \) is a single-valued neutrosophic approximation \( T^{(0.5,0.5)}_0 \) – space related with \( \psi \). \( (\tilde{\chi}, \varphi) \) but its not be a single-valued neutrosophic approximation \( T^{(0.5,0.5)}_1 \) – space or \( T^{(0.5,0.5)}_2 \) – space.

In this example given to show that \( (\tilde{\chi}, \varphi, h) \) is a single-valued neutrosophic ideal approximation \( T^{(0.5,0.5)}_i \) – space, \( i = 1, 2, 3 \) while \( (\tilde{\chi}, \varphi) \) is even not single-valued neutrosophic approximation \( T^{(0.5,0.5)}_1 \) – space.
If \((\tilde{\chi}, \varphi)\) and \((\tilde{\mathcal{F}}, \varphi^*)\) are SVNAs-spaces related with \(\psi \in \Xi^\mathcal{L}\) and \(\rho \in \Xi^{\mathcal{L}^\mathcal{F}}\), respectively, then a mapping \(f : (\tilde{\chi}, \varphi) \rightarrow (\tilde{\mathcal{F}}, \varphi^*)\) is said to be single-valued neutrosophic approximation continuous (SVNAC-map), if
\[
\text{int}_\psi^\mathcal{L}(f^{-1}(\sigma)) \geq f^{-1}(\text{int}_{\psi^*}(\sigma)), \forall, \sigma \in \Xi^{\mathcal{L}^\mathcal{F}}.
\]

Obviously, it corresponds to
\[
\text{cl}_\psi^\mathcal{L}(f^{-1}(\sigma)) \leq f^{-1}(\text{cl}_{\psi^*}(\sigma)), \forall, \sigma \in \Xi^{\mathcal{L}^\mathcal{F}}.
\]

Now, with respect to \(\psi \in \Xi^\mathcal{L}\) and \(\rho \in \Xi^{\mathcal{L}^\mathcal{F}}\), provided that \(\mathcal{h}, \mathcal{h}^*\) are SVNIs on \(\tilde{\chi}, \tilde{\mathcal{F}}\), respectively, then a map \(f : (\tilde{\chi}, \varphi, \mathcal{h}) \rightarrow (\tilde{\mathcal{F}}, \varphi^*, \mathcal{h}^*)\) is termed single-valued neutrosophic ideal approximation continuous (birfy, SVNIA-map), provided that \(\text{int}_{\psi}^\mathcal{L}(f^{-1}(\sigma)) \geq f^{-1}(\text{int}_{\psi^*}(\sigma))\) for every \(\sigma \in \Xi^{\mathcal{L}^\mathcal{F}}\). Obviously, it corresponds to \(\text{cl}_{\psi}^\mathcal{L}(f^{-1}(\sigma)) \leq f^{-1}(\text{cl}_{\psi^*}(\sigma))\) for every \(\sigma \in \Xi^{\mathcal{L}^\mathcal{F}}\).

Also, let us call \(f : (\tilde{\chi}, \varphi) \rightarrow (\tilde{\mathcal{F}}, \varphi^*)\) is termed single-valued neutrosophic approximation open (birfy, SVNIAO-map), provided that \(\text{int}_{\psi}^\mathcal{L}(f(\pi)) \geq f(\text{int}_{\psi^*}(\pi))\) for all \(\pi \in \Xi^\mathcal{L}\), \(f : (\tilde{\chi}, \varphi) \rightarrow (\tilde{\mathcal{F}}, \varphi^*, \mathcal{h}^*)\) is termed single-valued neutrosophic ideal approximation open (birfy, SVNIAO-map), provided that \(\text{int}_{\psi}^\mathcal{L}(f(\pi)) \geq f(\text{int}_{\psi^*}(\pi))\) for all \(\pi \in \Xi^\mathcal{L}\).

From (1) in Proposition 2, we can prove that (SVNAC-map) (resp. (SVNIAO-map)) will be (SVNAC-map) (resp. (SVNIAO-map)).

**Theorem 3.1.** Let \((\tilde{\chi}, \varphi), (\tilde{\mathcal{F}}, \varphi^*)\) be SVNA-spaces related with \(\psi \in \Xi^\mathcal{L}\), \(\rho \in \Xi^{\mathcal{L}^\mathcal{F}}\), respectively. Subsequently, let \(\mathcal{h}\) be a SVNNI on \(\tilde{\chi}\), \(f : (\tilde{\chi}, \varphi) \rightarrow (\tilde{\mathcal{F}}, \varphi^*)\) is an injective (SVNAC-map) with \(f(\psi) = \rho\). Therefore, \((\tilde{\chi}, \varphi, \mathcal{h})\) is a single-valued neutrosophic ideal approximation \(T_i^{(\psi, \mathcal{h})}\)–space if \((\tilde{\mathcal{F}}, \varphi^*)\) is a single-valued neutrosophic approximation \(T_i^{(\psi, \mathcal{h})}\)–space, \(i = 1, 2, 3\).

**Proof.** Because \(\ell \neq j\) in \(\tilde{\chi}\), we get than \(f(\ell) \neq f(j)\) in \(\tilde{\mathcal{F}}\), wing to \(\tilde{\mathcal{F}}\) is a single-valued neutrosophic approximation \(T_2^{(\psi, \mathcal{h})}\)–space, then there are \(\sigma, \rho \in \Xi^{\mathcal{L}^\mathcal{F}}\) with \(\ell \leq \tilde{\text{int}}_{\psi^*}(\sigma)\) \((f(\ell)), \ell \geq \tilde{\text{int}}_{\psi^*}(\rho)\) \((f(\ell)), \ell \geq \tilde{\delta}_{\text{int}}(\psi^*)\) \((f(\ell))\) and \(s \leq \tilde{\text{sup}}_{\psi}(\sigma)\) \((f(\ell)), s \geq \tilde{\sup}_{\psi^*}(\rho)\) \((f(\ell)), s \geq \tilde{\delta}_{\text{int}}(\psi^*)\) \((f(\ell))\) such that \(\tilde{\text{sup}}(\sigma) < (t \cap s), \tilde{\sup}(\rho) > (t \cap s), \tilde{\sup}(\rho) > (t \cap s)\), this implies, \(t \leq f^{-1}(\tilde{\text{int}}_{\psi}(\sigma))\) \((\ell)), t \geq f^{-1}(\tilde{\text{sup}}_{\psi^*}(\rho))\) \((\ell)), t \geq f^{-1}(\tilde{\delta}_{\text{int}}(\psi^*))\) \((\ell))\) and hence
\[
t \leq f^{-1}(\tilde{\text{int}}_{\psi^*}(\sigma))\) \((\ell)), t \geq f^{-1}(\tilde{\sup}_{\psi^*}(\rho))\) \((\ell)), t \geq f^{-1}(\tilde{\delta}_{\text{int}}(\psi^*))\) \((\ell))\).

Since \(\ell\) is SVNAC-map, we have \(t \leq \tilde{\text{int}}_{\psi^*}(f^{-1}(\sigma))\) \((\ell)), t \geq \tilde{\sup}_{\psi^*}(f^{-1}(\sigma))\) \((\ell)), t \geq \tilde{\delta}_{\text{int}}(f^{-1}(\sigma))\) \((\ell))\) and hence
\[
t \leq \tilde{\text{int}}_{\psi^*}(f^{-1}(\sigma))\) \((\ell)), t \geq \tilde{\sup}_{\psi^*}(f^{-1}(\sigma))\) \((\ell)), t \geq \tilde{\delta}_{\text{int}}(f^{-1}(\sigma))\) \((\ell))\).

Therefore, \((\tilde{\chi}, \varphi)\) and \((\tilde{\mathcal{F}}, \varphi^*)\) are SVNAs-spaces related with \(\psi \in \Xi^\mathcal{L}\), \(\rho \in \Xi^{\mathcal{L}^\mathcal{F}}\), respectively, then a mapping \(f : (\tilde{\chi}, \varphi) \rightarrow (\tilde{\mathcal{F}}, \varphi^*)\) is said to be single-valued neutrosophic approximation continuous (SVNAC-map), if
\[
\text{int}_\psi^\mathcal{L}(f^{-1}(\sigma)) \geq f^{-1}(\text{int}_{\psi^*}(\sigma)), \forall, \sigma \in \Xi^{\mathcal{L}^\mathcal{F}}.
\]
That is, there is $E = f^{-1}(a)$, $\omega = f^{-1}(q)$ with $t \leq \tilde{h}_{\text{int}}(\omega)(t)$, $\tilde{f}(a) \leq \tilde{\delta}_{\text{int}}(\omega)(t)$ and $s \leq \tilde{g}_{\text{int}}(a)(t)$, $s \leq \tilde{\delta}_{\text{int}}(a)(t)$. Hence, $(\tilde{h}, \tilde{f}, \tilde{g})$ is a single-valued neutrosophic ideal approximation $T_2^{(t,s)}$-space. Likewise, we can establish that other cases have a similar line of reasoning.

\[ \square \]

**Theorem 3.2.** Let $(\tilde{x}, \psi), (\tilde{F}, \psi^*)$ be a SVNA-spaces related with $\psi \in \xi^\tilde{F}, \rho \in \xi^{\tilde{F}}$, respectively. Subsequently, let $\tilde{h}$ be a SVNI on $\tilde{F}$ and $f: (\tilde{x}, \psi) \longrightarrow (\tilde{F}, \psi^*)$ is an surjective (SVNAO-map) with $f^{-1}(\rho) = \psi$. Therefore, $(\tilde{x}, \psi^*, \tilde{h}^*)$ is a single-valued neutrosophic ideal approximation $T_i^{(t,s)}$-space if $(\tilde{F}, \psi)$ is a single-valued neutrosophic approximation $T_i^{(t,s)}$-space, $i = 1, 2, 3$.

**Proof.** Because $f$ is surjective, we obtain $a \neq b$ in $\tilde{F}$ which implies $f^{-1}(a) \neq f^{-1}(b)$ in $\tilde{x}$. Owing to $(\tilde{x}, \psi)$ is single-valued neutrosophic approximation $T_2^{(t,s)}$-space, $\alpha, \beta \in \xi^{\tilde{x}}$ with $t \leq \tilde{h}_{\text{int}}(a)(f^{-1}(a))$, $t \geq \tilde{g}_{\text{int}}(a)(f^{-1}(a))$ and $s \leq \tilde{g}_{\text{int}}(s)(f^{-1}(b))$, $s \geq \tilde{\delta}_{\text{int}}(s)(f^{-1}(b))$, exist such that $\tilde{h}_{\text{sup}}(\alpha) \leq (t \cap s), \tilde{g}_{\text{sup}}(\alpha) \geq (t \cap s), \tilde{\delta}_{\text{sup}}(\alpha) \geq (t \cap s)$, and since $f$ is surjective, we obtain $t \leq \tilde{h}_{\text{int}}(a)(f^{-1}(a))$, $s \leq \tilde{g}_{\text{int}}(s)(f^{-1}(b))$, $s \geq \tilde{\delta}_{\text{int}}(s)(f^{-1}(b))$. Subsequently, based on $f$ is SVNAO-map,

\[
\begin{align*}
& t \leq \tilde{h}_{\text{int}}(f(a))(a), \quad t \geq \tilde{g}_{\text{int}}(f(a))(a), \quad t \geq \tilde{\delta}_{\text{int}}(f(a))(a), \\
& s \leq \tilde{h}_{\text{int}}(f(b))(b), \quad s \geq \tilde{g}_{\text{int}}(f(b))(b), \quad s \geq \tilde{\delta}_{\text{int}}(f(b))(b).
\end{align*}
\]

Therefore,

\[
\begin{align*}
& t \leq \tilde{h}_{\text{int}}(f(a))(a), \quad t \geq \tilde{g}_{\text{int}}(f(a))(a), \quad t \geq \tilde{\delta}_{\text{int}}(f(a))(a), \\
& s \leq \tilde{h}_{\text{int}}(f(b))(b), \quad s \geq \tilde{g}_{\text{int}}(f(b))(b), \quad s \geq \tilde{\delta}_{\text{int}}(f(b))(b).
\end{align*}
\]

That is, there exist $a = f(a)$ and $b = f(b)$ with

\[
\begin{align*}
& t \leq \tilde{h}_{\text{int}}(a)(a), \quad t \geq \tilde{g}_{\text{int}}(a)(a), \quad t \geq \tilde{\delta}_{\text{int}}(a)(a), \\
& s \leq \tilde{h}_{\text{int}}(b)(b), \quad s \geq \tilde{g}_{\text{int}}(b)(b), \quad s \geq \tilde{\delta}_{\text{int}}(b)(b),
\end{align*}
\]

and

\[
\tilde{h}_{\text{sup}}(a \cap b) < (t \cap s), \quad \tilde{g}_{\text{sup}}(a \cap b) > (t \cap s), \quad \tilde{\delta}_{\text{sup}}(a \cap b) > (t \cap s).
\]

Thus, $(\tilde{x}, \psi^*, \tilde{h}^*)$ is a single-valued neutrosophic ideal approximation $T_i^{(t,s)}$-space. Likewise, we can establish that other cases have a similar line of reasoning. \[ \square \]

### 4. Connected single-valued neutrosophic ideal approximation spaces

**Definition 4.1.** Let $(\tilde{x}, \psi)$ be an SVNA-spaces related with $\psi \in \xi^{\tilde{x}}$. Therefore,

1. two non-null SVNSs $\rho, \pi \in \xi^{\tilde{x}}$ are single-valued neutrosophic approximation preseparated (abbreviated, SVNA-preseparated) [resp. separated, (abbreviated, SVNA-separated)] sets if

\[
pcl_{\psi}^{\psi}(\rho) \land \pi = \rho \land pcl_{\psi}^{\psi}(\pi) = \langle 0, 1, 1 \rangle, \quad [\text{resp. } cl_{\psi}^{\psi}(\rho) \land \pi = \rho \land cl_{\psi}^{\psi}(\pi) = \langle 0, 1, 1 \rangle].
\]
(2) a non-null SVNS $\rho \in \xi^\tilde{\chi}$ is termed single-valued neutrosophic approximation predisconnected (abbreviated, SVNA-predisconnected) [resp. disconnected, (abbreviated, SVNA-disconnected)] set if there exist SVNA-preseparated [resp. SVNA-separated] sets $\rho, \pi \in \xi^\tilde{\chi}$ such that

$$\rho \land \pi = \sigma.$$ 

A SVNS $\sigma \in \xi^\tilde{\chi}$ is termed single-valued neutrosophic approximation preconnected (abbreviated, SVNA-preconnected) [resp. connected, (abbreviated, SVNA-preconnected)] if it is not SVNA-predisconnected [resp. SVNA-disconnected].

(3) $(\tilde{\chi}, \varphi)$ is termed SVNA-predisconnected [resp. SVNA-disconnected] space if there exist SVNA-preseparated [resp. SVNA-separated] sets $\rho, \pi \in \xi^\tilde{\chi}$, such that

$$\rho \land \pi = (1, 0, 0).$$ 

A SVNA-space $(\tilde{\chi}, \varphi)$ is termed SVNA-preconnected [resp. SVNA-connected] space if it is not SVNA-predisconnected [resp. SVNA-disconnected].

**Definition 4.2.** Let $(\tilde{\chi}, \varphi)$ be an SVNA-spaces related with $\psi \in \xi^\tilde{\chi}$. Therefore,

(1) two non-null SVNSs $\rho, \pi \in \xi^\tilde{\chi}$ are single-valued neutrosophic ideal approximation preseparated (abbreviated, SVNIA-preseparated) [resp. separated, (abbreviated, SVNIA-separated)] sets if

$$\text{pcl}^\psi_\tilde{\chi}(\rho) \land \pi = \rho \land \text{pcl}^\psi_\tilde{\chi}(\pi) = (0, 1, 1),$$ 

$$\text{resp. } \text{cl}^\psi_\tilde{\chi}(\rho) \land \pi = \rho \land \text{cl}^\psi_\tilde{\chi}(\pi) = (0, 1, 1),$$

(2) a non-null SVNS $\rho \in \xi^\tilde{\chi}$ is termed single-valued neutrosophic ideal approximation predisconnected (abbreviated, SVNIA-predisconnected) [resp. disconnected, (abbreviated, SVNIA-disconnected)] set if there exist SVNIA-preseparated [resp. SVNIA-separated] sets $\rho, \pi \in \xi^\tilde{\chi}$ such that

$$\rho \land \pi = \sigma.$$ 

A SVNS $\sigma \in \xi^\tilde{\chi}$ is termed single-valued neutrosophic ideal approximation preconnected (abbreviated, SVNIA-preconnected) [resp. connected, (abbreviated, SVNIA-preconnected)] if it is not SVNIA-predisconnected [resp. SVNIA-disconnected].

(3) $(\tilde{\chi}, \varphi)$ is termed SVNIA-predisconnected [resp. SVNIA-disconnected] space if there exist SVNIA-preseparated [resp. SVNIA-separated] sets $\rho, \pi \in \xi^\tilde{\chi}$, such that

$$\rho \land \pi = (1, 0, 0).$$ 

A SVNIA-space $(\tilde{\chi}, \varphi)$ is termed SVNIA-preconnected [resp. SVNIA-connected] space if it is not SVNIA-predisconnected [resp. SVNIA-disconnected].
Remark 4.1. We have the following implications.

\[
\begin{array}{ccc}
SVNA\text{-preseparated} & \longrightarrow & SVNIA\text{-separated} \\
\downarrow & & \downarrow \\
SVNA\text{-preseparated} & \longrightarrow & SVNIA\text{-preseparated}
\end{array}
\]

and hence

\[
\begin{array}{ccc}
SVNIA\text{-preconnected} & \longrightarrow & SVNA\text{-preconnected} \\
\downarrow & & \downarrow \\
SVNIA\text{-connected} & \longrightarrow & SVNA\text{-connected}
\end{array}
\]

Example 4.1. Let \( \varphi \) be a SVNR on \( \bar{\chi} = \{a, b, c, d, e\} \) be a set defined by

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
<th>( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( (1, 0, 0) )</td>
<td>( (1, 0, 0) )</td>
<td>( (0.4, 0.2, 0.2) )</td>
<td>( (0, 1, 1) )</td>
<td>( (0, 1, 1) )</td>
</tr>
<tr>
<td>( b )</td>
<td>( (1, 0, 0) )</td>
<td>( (1, 0, 0) )</td>
<td>( (0.4, 0.2, 0.2) )</td>
<td>( (0, 1, 1) )</td>
<td>( (0, 1, 1) )</td>
</tr>
<tr>
<td>( c )</td>
<td>( (0.4, 0.2, 0.2) )</td>
<td>( (0.4, 0.2, 0.2) )</td>
<td>( (1, 0, 0) )</td>
<td>( (0, 1, 1) )</td>
<td>( (0, 1, 1) )</td>
</tr>
<tr>
<td>( d )</td>
<td>( (0, 1, 1) )</td>
<td>( (0, 1, 1) )</td>
<td>( (0, 1, 1) )</td>
<td>( (1, 0, 0) )</td>
<td>( (1, 0, 0) )</td>
</tr>
<tr>
<td>( e )</td>
<td>( (0, 1, 1) )</td>
<td>( (0, 1, 1) )</td>
<td>( (0, 1, 1) )</td>
<td>( (1, 0, 0) )</td>
<td>( (1, 0, 0) )</td>
</tr>
</tbody>
</table>

spouse that \( \psi = \langle (0, 1, 1), (0, 1, 1), (0.4, 0.4, 0.4), (0.8, 0.8, 0.8), (0, 1, 1) \rangle \). Then,

\[
\psi_{\varphi} = \langle (0, 1, 1), (0, 1, 1), (0.4, 0.4, 0.4), (0.8, 0.8, 0.8), (0, 1, 1) \rangle,
\]

\[
(\psi_{\varphi})^c = \langle (1, 0, 0), (1, 0, 0), (0.4, 0.6, 0.4), (0.8, 0.2, 0.8), (1, 0, 0) \rangle.
\]

Now, for

\[
\rho = \langle (0.6, 0.6, 0.6), (0, 1, 1), (0, 1, 1), (0, 1, 1), (0, 1, 1) \rangle,
\]

\[
\pi = \langle (0, 1, 1), (0.6, 0.6, 0.6), (0, 1, 1), (0, 1, 1), (0, 1, 1) \rangle
\]

Then,

\[
\rho^{\varphi} = \langle (0.6, 0.6, 0.6), (0, 1, 1), (0, 1, 1), (0, 1, 1), (0, 1, 1) \rangle,
\]

\[
\pi^{\varphi} = \langle (0, 1, 1), (0.6, 0.6, 0.6), (0, 1, 1), (0, 1, 1), (0, 1, 1) \rangle.
\]

Consequently,

\[
\text{cl}_{\varphi}^{\psi}(\rho) = (\psi_{\varphi})^c \lor \rho^{\varphi} = \langle (1, 0, 0), (1, 0, 0), (0.4, 0.6, 0.4), (0.8, 0.2, 0.8), (1, 0, 0) \rangle,
\]

and

\[
\text{cl}_{\varphi}^{\psi}(\pi) = (\psi_{\varphi})^c \lor \pi^{\varphi} = \langle (1, 0, 0), (1, 0, 0), (0.4, 0.6, 0.4), (0.8, 0.2, 0.8), (1, 0, 0) \rangle.
\]
Furthermore,
\[ \rho = \langle (0.6, 0.6, 0.6), (0.1, 1), (0.1, 1), (0.2, 0.8, 0.4), (0, 1, 1) \rangle, \]
\[ \pi = \langle (0, 1, 1), (0.6, 0.6, 0.6), (0, 1, 1), (0.2, 0.8, 0.4), (0, 1, 1) \rangle. \]

Consequently, \( \text{int}^{\psi}(\rho) = \psi \land \rho = \langle (0, 1, 1), (0, 1, 1), (0, 1, 1), (0.2, 0.8, 0.8), (0, 1, 1) \rangle \) and \( \text{int}^{\psi}(\pi) = \psi \land \pi = \langle (0, 1, 1), (0, 1, 1), (0.2, 0.8, 0.8), (0, 1, 1) \rangle \). Therefore,

1. \( \rho, \pi \) are SVNA-preconnected sets but not SVNA-separated sets.

2. We define an SVNI \( \xi \) as follows: \( \sigma \in h \) for all \( \sigma = \langle (0.6, 0.6, 0.6), (0.6, 0.6, 0.6), (0.6, 0.6, 0.6), (0.6, 0.6, 0.6) \rangle \). Then \( \rho \in h \) and \( \pi \in h \) which means that \( \Phi^{\psi}(\rho) = \langle 0, 1, 1 \rangle \) and \( \Phi^{\psi}(\pi) = \langle 0, 1, 1 \rangle \) and \( \text{cl}^{\psi}(\rho) = \rho \) and \( \text{cl}^{\psi}(\pi) = \pi \). Therefore, \( \text{cl}^{\psi}(\rho) \land \pi = \langle 0, 1, 1 \rangle \) and \( \text{cl}^{\psi}(\pi) \land \rho = \langle 0, 1, 1 \rangle \). Hence, \( \rho, \pi \) are SVNA-separated sets but not SVNA-separated sets.

**Theorem 4.1.** Let \( (\tilde{\chi}, \varphi, h) \) be an SVNA-space related with \( \psi \in \xi^{\tilde{\chi}} \). Therefore, the following statements are equivalent:

1. \( (\tilde{\chi}, \varphi, h) \) is SVNA-preconnected space.
2. \( \rho \land \pi = \langle 0, 1, 1 \rangle \), \( \text{pint}^{\psi}(\rho) = \rho \), \( \text{pint}^{\psi}(\pi) = \pi \) and \( \rho \lor \pi = \langle 1, 0, 0 \rangle \) imply \( \rho = \langle 0, 1, 1 \rangle \) or \( \pi = (0, 1, 1) \).
3. \( \rho \land \pi = \langle 0, 1, 1 \rangle \), \( \text{pcl}^{\psi}(\rho) = \rho \), \( \text{pcl}^{\psi}(\pi) = \pi \) and \( \rho \lor \pi = \langle 1, 0, 0 \rangle \) imply \( \rho = \langle 0, 1, 1 \rangle \) or \( \pi = (0, 1, 1) \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( \rho, \pi \in \xi^{\tilde{\chi}} \) with \( \text{pint}^{\psi}(\rho) = \rho \), \( \text{pint}^{\psi}(\pi) = \pi \) such that \( \rho \land \pi = \langle 0, 1, 1 \rangle \) and \( \rho \lor \pi = \langle 1, 0, 0 \rangle \). Therefore, \( \text{pcl}^{\psi}(\rho) = \text{pcl}^{\psi}(\pi) = \pi \) and \( \rho \lor \pi = \langle 1, 0, 0 \rangle \). Consequently, \( \rho, \pi \) are SVNA-separated sets so that \( \rho \lor \pi = \langle 1, 0, 0 \rangle \). Because \( (\tilde{\chi}, \varphi, h) \) is SVNA-preconnected space we obtain \( \rho = \langle 0, 1, 1 \rangle \) or \( \pi = (0, 1, 1) \).

(2) \( \Rightarrow \) (3); (3) \( \Rightarrow \) (1): Clear.

**Theorem 4.2.** Let \( (\tilde{\chi}, \varphi, h) \) be an SVNA-space related with \( \psi \in \xi^{\tilde{\chi}} \) for each \( \rho \in \xi^{\tilde{\chi}} \). Therefore, the following properties are equivalent:

1. \( \rho \) is SVNA-preconnected set.
2. If \( \pi, \sigma \) are SVNA-preseparated sets with \( \rho \leq \pi \lor \sigma \), then \( \rho \land \pi = \langle 0, 1, 1 \rangle \) or \( \rho \land \sigma = \langle 0, 1, 1 \rangle \).
3. If \( \pi, \sigma \) are SVNA-preseparated sets with \( \rho \leq \pi \lor \sigma \), then \( \rho \leq \pi \lor \sigma \leq \rho \).
Hence, \((\rho \land \sigma)\) and \((\rho \land \pi)\) are SVNIA-preseparated sets with \(\rho = (\rho \land \sigma) \lor (\rho \land \pi)\). But \(\rho\) is SVNIA-preconnected set means that \(\rho \land \sigma = \langle 0, 1, 1 \rangle\) or \(\rho \land \pi = \langle 0, 1, 1 \rangle\).

\[\text{(2) } \Rightarrow \text{(3): If } \rho \land \pi = \langle 0, 1, 1 \rangle \text{ and } \rho \leq \pi \lor \sigma \text{ means that } \rho = \rho \land (\pi \lor \sigma) = (\rho \land \pi) \lor (\rho \land \sigma) = \rho \land \sigma \text{ and hence } \rho \leq \sigma. \text{ Also, if } \rho \land \sigma = \langle 0, 1, 1 \rangle, \text{ then } \rho \leq \pi.\]

\[\text{(3) } \Rightarrow \text{(1): Clear. } \square\]

**Lemma 4.1.** Let \((\tilde{\chi}, \varphi)\) be an SVNIA-space related with \(\psi \in \Xi^\tilde{\chi}\) for each \(\rho \in \Xi^\tilde{\chi}\). Therefore, the following properties are equivalent:

1. \(\rho\) is SVNIA-preconnected set.
2. If \(\pi, \sigma\) are SVNIA-preseparated sets with \(\rho \leq \pi \lor \sigma\), then \(\rho \land \pi = \langle 0, 1, 1 \rangle\) or \(\rho \land \sigma = \langle 0, 1, 1 \rangle\).
3. If \(\pi, \sigma\) are SVNIA-preseparated sets with \(\rho \leq \pi \lor \sigma\), then \(\rho \leq \pi\) or \(\rho \leq \sigma\).

**Theorem 4.3.** Let \((\tilde{\chi}, \varphi)\) and \((\tilde{\mathcal{F}}, \varphi^*)\) are SVNIA-spaces related with \(\psi \in \Xi^\tilde{\chi}\) and \(\rho \in \Xi^\tilde{\mathcal{F}}\), respectively, \(\mathcal{H}\) a SVN on \(\tilde{\chi}\), and \(f : (\tilde{\chi}, \varphi, \mathcal{H}) \longrightarrow (\tilde{\mathcal{F}}, \varphi^*)\) is single-valued neutrosophic mapping such that \(\text{pcl}_{\psi}(f^{-1}(\pi)) \leq f^{-1}(\text{pcl}_{\varphi^*}(\pi))\) for all \(\pi \in \Xi^\tilde{\mathcal{F}}\). Then, \(f(\rho) \in \Xi^\tilde{\mathcal{F}}\) is a SVNIA-preconnected set if \(\rho\) is a SVNIA-preconnected set in \(\tilde{\chi}\).

**Proof.** Let \(\pi, \sigma \in \Xi^\tilde{\mathcal{F}}\) be SVNIA-preseparated sets with \(f(\rho) = \pi \lor \sigma\). Therefore, \(\text{pcl}_{\varphi^*}(\pi) \land \sigma = \pi \land \text{pcl}_{\varphi^*}(\sigma) = \langle 0, 1, 1 \rangle\). Then, \(\rho \leq (f^1(\pi) \lor f^{-1}(\sigma))\), and by hypothesis of \(f\), we get that

\[
\text{pcl}_{\psi}(f^{-1}(\pi)) \land f^{-1}(\sigma) \leq f^{-1}(\text{pcl}_{\varphi^*}(\pi)) \land f^{-1}(\sigma) = f^{-1}(\text{pcl}_{\varphi^*}(\pi) \land \sigma)
\]

\[
= f^{-1}(\langle 0, 1, 1 \rangle) = \langle 0, 1, 1 \rangle,
\]

and in similar way,

\[
\text{pcl}_{\psi}(f^{-1}(\sigma)) \land f^{-1}(\pi) \leq f^{-1}(\text{pcl}_{\varphi^*}(\sigma)) \land f^{-1}(\pi) = f^{-1}(\text{pcl}_{\varphi^*}(\sigma) \land \pi)
\]

\[
= f^{-1}(\langle 0, 1, 1 \rangle) = \langle 0, 1, 1 \rangle.
\]

Therefore, \(f^{-1}(\pi)\) and \(f^{-1}(\sigma)\) are SVNIA-preseparated sets in \(\tilde{\chi}\). Consequently, \(\rho \leq (f^1(\pi) \lor f^{-1}(\sigma))\). Because \(\rho\) is a SVNIA-preconnected set in \(\tilde{\chi}\), then by Theorem 6 (3), we get that \(\rho \leq f^1(\pi)\) or \(\rho \leq f^{-1}(\sigma)\) implies that \(f(\rho) \leq \pi\) or \(f(\rho) \leq \sigma\). Hence, from Corollary 2, \(f(\rho) \in \Xi^\tilde{\mathcal{F}}\) is a SVNIA-preconnected set in \(\tilde{\mathcal{F}}\). \(\square\)

5. Compactness in single-valued neutrosophic ideal approximation spaces

**Definition 5.1.** Let \((\tilde{\chi}, \varphi, \mathcal{H})\) be an SVNIA-space related with \(\psi \in \Xi^\tilde{\chi}\). Therefore, \(\tilde{\chi}\) is termed single-valued neutrosophic regular (abbreviated, SVN-regular) [resp. ideal regular (abbreviated, SVN-I-regular)] space if for every \(\rho \in \Xi^\tilde{\chi}\) with \(\text{int}_{\psi}(\rho) = \emptyset\),

\[
\rho = \bigvee_{i \in \Gamma} \{ q_i : \text{int}_{\psi}(q_i) = q_i, \text{cl}_{\psi}(q_i) \leq \emptyset \},
\]

\[
\text{resp. } \rho = \bigvee_{i \in \Gamma} \{ q_i : \text{int}_{\psi}(q_i) = q_i, \text{cl}_{\psi}(q_i) \leq \emptyset \}.
\]
For each SVN-regular space is a SVNI-regular space the validity of this relation is clear based on Definition 4.1. If \( h = (0, 1, 1) \) then the notions of SVN-regular and SVNI-regular are equivalent.

**Definition 5.2.** Let \( (\bar{\chi}, \varphi, h) \) be an SVNIA-space related with \( \psi \in \xi^{k} \). Therefore,

1. \( \rho \) is an single-valued neutrosophic approximation compact (abbreviated, SVNIA-compact) [resp. ideal approximation compact (abbreviated, SVNA-compact)] if for every family \( \rho_{i} \in \xi^{k} : \text{int}_{\psi}(\rho_{i}) = \rho_{i}, i \in \Gamma \) with \( \rho \leq \bigvee_{i \in \Gamma} \rho_{i} \) there exists a finite \( \Gamma_{0} \subseteq \Gamma \) such that \( \rho \leq \bigvee_{i \in \Gamma_{0}} \rho_{i} \)
2. \( \rho \) is an single-valued neutrosophic nearly approximation compact (abbreviated, SVNAA-compact) [resp. nearly ideal approximation compact (abbreviated, SVNAIA-compact)] if for every family \( \rho_{i} \in \xi^{k} : \text{int}_{\psi}(\rho_{i}) = \rho_{i}, i \in \Gamma \) with \( \rho \leq \bigvee_{i \in \Gamma} \rho_{i} \) there exists a finite \( \Gamma_{0} \subseteq \Gamma \) such that \( \rho \leq \bigvee_{i \in \Gamma_{0}} \text{int}_{\psi}(\rho_{i}) \)
3. \( \rho \) is an single-valued neutrosophic nearly approximation compact (abbreviated, SVNAA-compact) [resp. nearly ideal approximation compact (abbreviated, SVNAIA-compact)] if for every family \( \rho_{i} \in \xi^{k} : \text{int}_{\psi}(\rho_{i}) = \rho_{i}, i \in \Gamma \) with \( \rho \leq \bigvee_{i \in \Gamma} \rho_{i} \) there exists a finite \( \Gamma_{0} \subseteq \Gamma \) such that \( \rho \leq \bigvee_{i \in \Gamma_{0}} \text{cl}_{\psi}(\rho_{i}) \)

The SVNIA-space \( (\bar{\chi}, \varphi) \) [resp. SVNI-space \( (\bar{\chi}, \varphi, h) \)] will be termed SVNA-compact, SVNAA-compact, SVNNI-compact if we replaced \( \rho \) with \( (1, 0, 0) \).

It is clear that:

\[
\text{SVNA-compact} \quad \longrightarrow \quad \text{SVNIA-compact} \quad \longrightarrow \quad \text{SVNAA-compact}
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
\text{SVNI-compact} \quad \longrightarrow \quad \text{SVNAIA-compact} \quad \longrightarrow \quad \text{SVNNIA-compact}
\]

**Theorem 5.1.** Let \( (\bar{\chi}, \varphi, h) \) be SVNAIA-compact and SVNI-regular. Therefore, \( \bar{\chi} \) is a SVNI-compact space.

**Proof.** Assume a family \( \rho_{i} \in \xi^{k} : \text{int}_{\psi}(\rho_{i}) = \rho_{i}, i \in \Gamma \) with \( \bigvee_{i \in \Gamma} \rho_{i} = (1, 0, 0) \). By SVNI-regularity of \( \bar{\chi} \) then for every \( \text{int}_{\psi}(\rho_{i}) = \rho_{i} \), we have

\[
\rho_{i} = \bigvee_{i \in \Gamma} \{ \rho_{i} : \text{int}_{\psi}(\rho_{i}), \text{cl}_{\psi}(\rho_{i}) \leq \rho_{i} \}.
\]

Thus, \( \bigvee_{i \in \Gamma} (\bigvee_{i \in \Gamma} \rho_{i}) = (1, 0, 0) \), because, \( \bar{\chi} \) is SVNAIA-compact, therefore there exists a finite index subset \( \Gamma_{0} \times \Gamma_{K} \) of \( \Gamma \times \Gamma \) such that

\[
(1, 0, 0) \bar{\lambda}(\bigvee_{i \in \Gamma_{0}} (\bigvee_{i \in \Gamma_{K}} \rho_{i})) \in h.
\]

Since for any \( i \in \gamma_{0} \) we have \( \bigvee_{i \in \Gamma_{K}} \text{cl}_{\psi}(\rho_{i}) \leq \rho_{i} \) and hence we get that

\[
(1, 0, 0) \bar{\lambda}(\bigvee_{i \in \Gamma_{0}} (\bigvee_{i \in \Gamma_{K}} \rho_{i})) \geq (1, 0, 0) \bar{\lambda}(\bigvee_{i \in \Gamma_{0}} \rho_{i}).
\]
Thus, \((1,0,0)\bar{\lambda}(\bigvee_{i\in \Gamma_0} \rho_i) \in h\), and hence \(\check{\chi}\) is a SVNIA-compact space. \(\square\)

**Theorem 5.2.** Let \((\check{\chi},\varphi,h)\) be SVNIA-compact and SVNI-regular. Therefore, \(\check{\chi}\) is a SVNIA-compact space.

**Proof.** Similar to Theorem 8. \(\square\)

**Theorem 5.3.** Let \(f : (\check{\chi},\varphi,h_1) \longrightarrow (\check{\tilde{F}},\varphi',h_2)\) be injective (SVNAC-map) between two SVNAs associated with \(\psi \in \xi^{\check{\chi}}\) and \(\rho \in \xi^{\tilde{F}},\) respectively, and \(\pi \in h_1 \Rightarrow f(\pi) \in h_2,\) for all \(\sigma \in \xi^{\check{\chi}}\) is SVNIA-compact. Then, \(f(\sigma)\) is SVNIA-compact as well.

**Proof.** Let \(\{q_i \in \xi^{\tilde{F}} : \text{int}^{\psi}_{\varphi}(q_i) = q_i, i \in \Gamma\}\) be a family with \(f(\sigma) \leq \bigvee_{i \in \Gamma} q_i.\) Because of \(f\) is a SVNAC-map, then \(\text{int}^{\psi}_{\varphi}(f^{-1}(q_i)) = f^{-1}(q_i)\) with \(\sigma \leq \bigvee_{i \in \Gamma} f^{-1}(q_i).\) By SVNIA-compactness of \(\sigma\) there exists a finite \(\Gamma_0 \subseteq \Gamma\) such that \(\sigma \bar{\lambda}(\bigvee_{i \in \Gamma_0}(f^{-1}(q_i))) \in h_1.\) Since \(\pi \in h_1 \Rightarrow f(\pi) \in h_2,\) for all \(\sigma \in \xi^{\check{\chi}}\) then \(f(\sigma \bar{\lambda}(\bigvee_{i \in \Gamma_0}(f^{-1}(q_i)))) = f(\sigma \bar{\lambda}(\bigvee_{i \in \Gamma_0}(q_i))) \in h_2.\) Because of \(f\) is a injective, therefore, \(f(\sigma \bar{\lambda}(\bigvee_{i \in \Gamma_0}(f^{-1}(q_i)))) = f(\sigma \bar{\lambda}(\bigvee_{i \in \Gamma_0}(q_i))) \in h_2.\) Thus, \(f(\sigma)\) is SVNIA-compact. \(\square\)

**Conflicts of Interest:** The author declares that there are no conflicts of interest regarding the publication of this paper.

**References**


