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Fuzzy (Almost, δ) Ideal Continuous Mappings

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Abstract. In this paper, we introduce the concept of fuzzy δ-ideal continuous, fuzzy θ-ideal continuous, fuzzy strongly δ-ideal continuous and fuzzy almost ideal continuous mappings in fuzzy ideal topological spaces given the definition of Šostak. In addition, we study some properties between them.

1. Introduction and Preliminaries

The concept of fuzzy topology was first defined in 1968 by Chang [1] and later redefined in a somewhat different way by Lowen [21] and by Hutton and Reilly [18]. According to Ŝostak's [27], in all these definitions, a fuzzy topology is a crisp subfamily of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore Ŝostak's introduced a new definition of fuzzy topology in 1985 [28]. Later on, he developed the theory of fuzzy topological spaces in [29]. After that several authors [2,3,5,19,20,23,25] have introduced the smooth definition and studied smooth fuzzy topological spaces being unaware of Ŝostak's works. In fuzzy topology, by introducing the notion of ideal, [27], and several other authors [17,22] carried out such analysis.

The notion of continuity is an important concept in fuzzy topology and fuzzy topology in Ŝostak sense as well as in all branches of mathematics and quantum physics (see [6,7,10–14]). We must state that this subject has been researched by physicists [7,10–13] as well as by others. El-Naschie has shown that the notion of fuzzy topology in Ŝostak sense has very important applications in quantum particle physics especially about both string theory and $\varepsilon^{(\infty)}$ theory [8,9,12,15,16] and also Saber et al. [30–39] who familiarized the concepts of single-valued neutrosophic ideal open local function and single-valued neutrosophic topological space. In this paper, we give a decomposition of fuzzy continuity, fuzzy ideal continuity and fuzzy ideal α-continuity, and we

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obtain several characterizations of fuzzy α-I-continuous functions. Moreover, we introduce the concept of fuzzy α-I-open functions in fuzzy ideal topological spaces and obtain their properties

Throughout this paper, let X be a nonempty set $I = [0,1]$ and $I_0 = (0,1]$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X denoted by I^X. For two fuzzy sets we write $\lambda q \mu$ to mean that λ is quasi-coincident (q-coincident, for short) with μ, i.e, there exists at least one point $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Negation of such a statement is denoted as $\lambda \overline{q} \mu$.

Definition 1.1. *[27]. A mapping $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:*

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$.
(O2) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, *for* $\{\mu_i\}_{i\in\Gamma} \in I^X$.
(O3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, *for* $\mu_1, \mu_2 \in I^X$.

Definition 1.2. *[27]. A mapping $\boldsymbol{I} : I^X \to I$ is called fuzzy ideal on X iff:*

(I_1) $\boldsymbol{I}(\underline{0}) = 1, \boldsymbol{I}(\underline{1}) = 0$.
(I_2) *If* $\lambda \leq \mu$, *then* $\boldsymbol{I}(\lambda) \geq \boldsymbol{I}(\mu)$, *for each* $\lambda, \mu \in I^X$.
(I_3) *For each* $\lambda, \mu \in I^X$, $\quad \boldsymbol{I}(\lambda \vee \mu) \geq \boldsymbol{I}(\lambda) \wedge \boldsymbol{I}(\mu)$.

The pair $(X, \tau, \boldsymbol{I})$ is called fuzzy ideal topological space (fits, for short)

Corollary 1.1. *[17]. Let $(X, \tau, \mathcal{I})$ be a fits. The simplest fuzzy ideal on X are $\mathcal{I}^0, \mathcal{I}^1 : I^X \to I$ where*

$$\boldsymbol{I}^0(\lambda) = \begin{cases} 1, \text{ if } \lambda = \underline{0}, \\ 0, \text{otherwise.} \end{cases} \quad \boldsymbol{I}^1(\lambda) = \begin{cases} 0, \text{if } \lambda = \underline{1}, \\ 1, \text{ otherwise.} \end{cases}$$

If $\boldsymbol{I} = \boldsymbol{I}^0$, for each $\mu \in I^X$ we have $\mu_r^* = C_\tau(\mu, r)$.
If $\boldsymbol{I} = \boldsymbol{I}^1$, for each $\mu \in \Theta'$ we have $\mu_r^* = \underline{0}$, where, $\underline{1} \notin \Theta'$ be a subset of I^X.

Definition 1.3. *[17]. Let $(X, \tau, \boldsymbol{I})$ be a fits. Let $\mu, \lambda \in I^X$, the r-fuzzy open local function μ_r^* of μ is the union of all fuzzy points x_t such that if $\rho \in Q(x_t, r)$ and $\boldsymbol{I}(\lambda) \geq r$ then there is at least one $y \in X$ for which $\rho(y) + \mu(y) - 1 > \lambda(y)$.*

Theorem 1.1. *[17]. Let (X, τ) be a fts. Then for each $r \in I_0$, $\lambda \in I^X$ we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows:*

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \ \lambda \leq \mu, \ \tau(\overline{1} - \mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\overline{0}, r) = \overline{0}$.
(2) $\lambda \leq C_\tau(\lambda, r)$.
(3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.
(4) $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$ *if* $r \leq s$.
(5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Theorem 1.2. *[17]. Let (X,τ) be a fts. Then for each $r \in I_0$, $\lambda \in I^X$ we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows:*

$$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X : \ \lambda \geq \mu, \ \tau(\mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following conditions:

(1) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$
(2) $I_\tau(\overline{1}, r) = \overline{1}$.
(3) $\lambda \geq I_\tau(\lambda, r)$.
(4) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.
(5) $I_\tau(\lambda, r) \leq I_\tau(\lambda, s)$ *if* $r \geq s$.
(6) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

Theorem 1.3. *[17]. Let (X,τ) be a fts and $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ be two fuzzy ideals of X. Then for each $r \in I_0$ and $\mu, \eta, \rho \in I^X$.*

(1) $\mu \leq \eta$, *then* $\mu^*_r \leq \eta^*_r$.
(2) $\boldsymbol{I}_1 \leq \boldsymbol{I}_2$, $\Rightarrow$ $\mu^*_r(\boldsymbol{I}_1, \tau) \leq \eta^*_r(\boldsymbol{I}_2, \tau)$.
(3) $\mu^*_r = C_\tau(\mu^*_r, r) \leq C_\tau(\mu, r)$.
(4) $(\mu^*_r)^* \leq \mu^*_r$.
(5) $(\mu^*_r \vee \eta^*_r) = (\mu \vee \eta)^*_r$.
(6) *If* $\boldsymbol{I}(\rho) \geq r$ *then* $(\mu \vee \rho)^*_r = \mu^*_r \vee \rho^*_r = \mu^*_r$.
(7) *If* $\tau(\rho) \geq r$, *then* $(\rho \wedge \mu^*_r) \leq (\rho \wedge \mu)^*_r$.
(8) $(\mu^*_r \wedge \eta^*_r) \geq (\mu \wedge \eta)^*_r$.

Theorem 1.4. *[17]. Let $(X,\tau,\boldsymbol{I})$ be a fits. Then for each $r \in I_0$, $\mu \in I^X$ we define $C^* : I^X \times I_0 \to I^X$ as follows:*

$$Cl^*(\mu, r) = \mu \vee \mu^*_r$$

For $\mu, \eta \in I^X$, the Cl^* satisfies the following conditions:

(1) *If* $\mu \leq \eta$, *then* $Cl^*(\mu, r) \leq Cl^*(\eta, r)$.
(2) $Cl^*(Cl^*(\mu, r), r) = Cl^*(\mu, r)$.
(3) $Cl^*(\mu \vee \eta, r) = Cl^*(\mu, r) \vee Cl^*(\eta, r)$.
(4) $Cl^*(\mu \wedge \eta, r) \leq Cl^*(\mu, r) \wedge Cl^*(\eta, r)$.

Definition 1.4. *[17] Let (X,τ) be a fts. For $\lambda \in I^X$ and $r \in I_0$.*

(1) *λ is called r-fuzzy semiopen (**r-FSO**, for short) iff* $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$.
(2) *λ is called r-fuzzy semiclosed (**r-FSC**, for short) iff $\overline{1} - \lambda$ is r-fuzzy semiopen set of X.*
(3) *λ is called r-fuzzy β-closed (**r-F**β**C**, for short) iff* $\lambda \leq C_\tau(I_\tau(C_\tau(\lambda, r), r), r)$.

Definition 1.5. *[17]. Let (X,τ,I) be a fuzzy ideal topological space. For each $\mu \in I^X$ and $r \in I_0$.*

(1) *μ is called r-fuzzy ideal open (r-**FIO**, for short) iff* $\mu \leq I_\tau(\mu^*_r, r)$.
(2) *μ is called r-fuzzy ideal closed (r-**FIC**, for short) iff $\overline{1} - \mu$ is r-**FIO**.*

Lemma 1.1. *[17]. Let $(X, \tau, \boldsymbol{I})$ be a fits.*

(1) *Any union of r-**FIO** sets is r-**FIO**.*
(2) *Any intersection of r-**FIC** sets is r-**FIC** .*

Definition 1.6. *[17]. Let (X, τ) and (X, η) be fts's. Let $f : X \to Y$ be a mapping.*

(1) *f is called fuzzy continuous iff $\eta(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^X$.*
(2) *f is called fuzzy open iff $\tau(\mu) \leq \eta(f(\mu))$ for each $\mu \in I^X$.*
(3) *f is called fuzzy closed iff $\tau(\overline{1} - \mu) \leq \eta(f(\overline{1} - \mu))$ for each $\mu \in I^X$.*

2. r-fuzzy $\theta\mathcal{I}$-Open and r-Fuzzy $\delta\mathcal{I}$-Open Sets

Definition 2.1. *Let $(X, \tau, \mathcal{I})$ be a fits. For $\mathcal{A} \in I^X$, $x_t \in P_t(X)$ and $r \in I_0$. Then,*

(1) *$\mathcal{A}$ is called r-fuzzy $\mathfrak{R}_{\tau_{\mathcal{I}}}$-neighborhood of x_t if $x_t q \mathcal{A}$ and $\mathcal{A}$ is r-**FRIO**.*
We denote

$$\mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r) = \{\mathcal{A} \in I^X | x_t q \mathcal{A}, \ \ \mathcal{A} \ \text{ is } r - \boldsymbol{FRIO}\}.$$

(2) *x_t is called r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$ if for every $\mathcal{B} \in Q_\tau(x_t, r)$, we have $\mathcal{A} q Cl^\star(\mathcal{B}, r)$.*
(3) *$\theta\mathcal{I}$-closure operator is mapping $C_{\theta\mathcal{I}\tau} : I^X \times I_0 \to I^X$ defined as:*

$$C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) = \bigvee\{x_t \in P_t(X) : \ \ x_t \text{ is } r - \theta\mathcal{I} - \text{cluster point of } \mathcal{A}\}.$$

Theorem 2.1. *Let $(X, \tau, \mathcal{I})$ be a fits, for each $\mathcal{A}, \mathcal{B} \in I^X$ and $r \in I_0$. Then the following properties hold:*

(1) *$\mathcal{A} \leq C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)$.*
(2) *If $\mathcal{A} \leq \mathcal{B}$, then $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \leq C_{\theta\mathcal{I}\tau}(\mathcal{B}, r)$.*
(3) *$C_\tau(\mathcal{A}, r) \leq \bigvee\{x_t \in P_t(X) | x_t$ is r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}\}$.*
(4) *$C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) = \bigwedge\{\mathcal{B} \in I^X | \mathcal{A} \leq int^\star(\mathcal{B}, r), \ \tau(\underline{1} - \mathcal{B}) \geq r\}$.*
(5) *$C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) = \bigwedge\{\mathcal{B} \in I^X | \mathcal{A} \leq \mathcal{B}, \ \mathcal{B}$ is r-fuzzy δI-closed$\}$.*
(6) *x_t is r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$ iff $x_t \in C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)$.*
(7) *x_t is r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$ iff $x_t \in C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)$.*
(8) *If $\mathcal{A} = C_\tau(int^\star(\mathcal{A}, r), r)$, then $C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) = \mathcal{A}$.*
(9) *$\mathcal{A} \leq C_\tau(\mathcal{A}, r) \leq C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) \leq C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \leq T_\tau(\mathcal{A}, r)$.*
(10) *$\mathcal{W}(\mathcal{A} \vee \mathcal{B}, r) = \mathcal{W}(\mathcal{A}, r) \vee \mathcal{W}(\mathcal{B}, r)$ for each $\mathcal{W} = \{C_{\delta\mathcal{I}\tau}, C_{\theta\mathcal{I}\tau}\}$.*
(11) *$C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\mathcal{A}, r), r) = C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)$.*

Proof. (1) and (2) are easily proved from Definition 2.1.

(3) Put $\mathcal{P} = \bigvee\{x_t \in P_t(X) | x_t$ is r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}\}$.

Suppose that $C_\tau(\mathcal{A}, r) \nleq \mathcal{P}$, there exists $x \in X$ and $t \in (0, 1)$ such that

$$C_\tau(\mathcal{A}, r)(x) > t > \mathcal{P}(x). \tag{2.1}$$

Then x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$. So, there exists $\mathcal{B} \in Q_\tau(x_t, r)$, $\mathcal{A} \leq \underline{1} - int_\tau(Cl^\star(\mathcal{B}, r), r) \leq \underline{1} - \mathcal{B}$. By definition of C_τ, $C_\tau(\mathcal{A}, r)(x) \leq (\underline{1} - \mathcal{B})(x) < t$. It is a contradiction for equation (2.1). Thus $C_\tau(\mathcal{A}, r) \leq \mathcal{P}$.

(4) $\bigcirc = \bigwedge\{\mathcal{B} \in I^X | \mathcal{A} \le int^\star(\mathcal{B}, r),\ \tau(\underline{1} - \mathcal{B}) \ge r\}$.

Suppose that $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \ngeq \bigcirc$, then there exists $x \in X$ and $t \in (0,1)$ such that

$$C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x) < t \le \bigcirc(x). \tag{2.2}$$

Then x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$. So, there exists $\mathcal{B} \in Q_\tau(x_t, r)$, and $\mathcal{A} \le \underline{1} - Cl^\star(\mathcal{B}, r)$. Thus, $\mathcal{A} \le \underline{1} - Cl^\star(\mathcal{B}, r) = int^\star(\underline{1} - \mathcal{B}, r),\ \tau(\mathcal{B}) \ge r$. Hence,

$$\bigcirc(x) \le (\underline{1} - \mathcal{B})(x) < t.$$

It is a contradiction for equation (2.2). Thus $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \ge \bigcirc$.

Suppose that $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \nleq \bigcirc$, then there exists r-fuzzy $\theta\mathcal{I}$-cluster point $y_s \in P_t(X)$ of $\mathcal{A}$, such that

$$C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(y) > s > \bigcirc(y). \tag{2.3}$$

By definition of $\bigcirc$, there exists $\mathcal{B} \in I^X$ with $\mathcal{A} \le int^\star(\mathcal{B}, r),\ \tau(\underline{1} - \mathcal{B}) \ge r$ such that $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(y) > s > \mathcal{B}(y) \ge \bigcirc(y)$. Then $\underline{1} - \mathcal{B} \in Q_\tau(y_s, r)$. Furthermore, $\mathcal{A} \le int^\star(\mathcal{B}, r) = \underline{1} - Cl^\star(\underline{1} - \mathcal{B}, r)$ implies $\mathcal{A}\overline{q}Cl^\star(\underline{1} - \mathcal{B}, r)$. Hence y_s is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$. It is a contradiction for equation (2.3). Thus $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \le \bigcirc$.

(5) It is similarly proved as in (3) and (4).

(6) ($\Rightarrow$) It is trivial.

($\Leftarrow$) Suppose that x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$. Then there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $Cl^\star(\mathcal{B}, r) \le \underline{1} - \mathcal{A}$. Thus

$$\mathcal{A} \le \underline{1} - Cl^\star(\mathcal{B}, r) = int^\star(\underline{1} - \mathcal{B}, r).$$

By (4), $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x) \le (\underline{1} - \mathcal{B})(x) < t$. Hence $x_t \notin C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)$.

(7) is similarly proved as in (6).

(8) Obvious from Theorem 1.1(4).

(9) Form Theorem 1.1(5), we show that only $C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) \le C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)$. Suppose that $C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) \nleq C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)$, then there exists $x \in X$ and $t \in I_0$ such that

$$C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)(x) > t > C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x). \tag{2.4}$$

Since $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x) < t$, x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$. So, there exists $\mathcal{B} \in Q_\tau(x_t, r)$, $A \le \underline{1} - Cl^\star(\mathcal{B}, r)$ implies $A\overline{q}int_\tau(Cl^\star(\mathcal{B}, r), r)$. Hence, x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$, by (7), we have

$$C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)(x) < t.$$

It is a contradiction for equation (2.4). Thus $C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) \le C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)$.

On the other hand, suppose that $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \nleq T_\tau(\mathcal{A}, r)$, then there exists $x \in X$ and $t \in I_0$ such that

$$C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x) > t > T_\tau(\mathcal{A}, r)(x). \tag{2.5}$$

Since $T_\tau(\mathcal{A},r)(x) < t$, x_t is not r-fuzzy θ-cluster point of $\mathcal{A}$. So, there exists $\mathcal{B} \in Q_\tau(x_t,r)$, $A \leq \underline{1} - C_\tau(\mathcal{B},r)$ implies $A\bar{q}Cl^\star(\mathcal{B},r)$. Hence, x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$, by (6), we have

$$C_{\theta\mathcal{I}\tau}(\mathcal{A},r)(x) < t.$$

It is a contradiction for equation (2.5). Thus $C_{\theta\mathcal{I}\tau}(\mathcal{A},r) \leq T_\tau(\mathcal{A},r)$.

(10) Let $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \vee C_{\delta\mathcal{I}\tau}(\mathcal{B},r) \not\geq C_{\delta\mathcal{I}\tau}(\mathcal{A} \vee \mathcal{B},r)$. Then there exists $x \in X$ and $x = (0,1)$ such that

$$C_{\delta\mathcal{I}\tau}(\mathcal{A},r)(x) \vee C_{\delta\mathcal{I}\tau}(\mathcal{B},r)(x) < t < C_{\delta\mathcal{I}\tau}(\mathcal{A} \vee \mathcal{B},r)(x). \tag{2.6}$$

Since $C_{\delta\mathcal{I}\tau}(\mathcal{A},r)(x) < t$ and $C_{\delta\mathcal{I}\tau}(\mathcal{B},r)(x) < t$, x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$ and $\mathcal{B}$ So, there exists $\mathcal{A}_1, \mathcal{B}_1 \in Q_\tau(x_t,r)$, and $\mathcal{A} \leq \underline{1} - int_\tau(Cl^\star(\mathcal{A}_1,r),r)$, $\mathcal{B} \leq \underline{1} - int_\tau(Cl^\star(\mathcal{B}_1,r),r)$. Thus, $(\mathcal{A}_1 \wedge \mathcal{B}_1) \in Q_\tau(x_t,r)$ and

$$\begin{aligned} \mathcal{A} \vee \mathcal{B} &\leq \underline{1} - (int_\tau(Cl^\star(\mathcal{A}_1,r),r) \wedge int_\tau(Cl^\star(\mathcal{B}_1,r),r)) \\ &= \underline{1} - (int_\tau(Cl^\star(\mathcal{A}_1,r) \wedge Cl^\star(\mathcal{B}_1,r),r),r) \\ &\leq \underline{1} - (int_\tau(Cl^\star(\mathcal{A}_1 \wedge \mathcal{B}_1,r),r)). \end{aligned}$$

Thus, $\mathcal{A} \vee \mathcal{B}\bar{q}int_\tau(Cl^\star(\mathcal{A}_1 \wedge \mathcal{B}_1,r),r)$. Hence, x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A} \vee \mathcal{B}$, by (7), we have

$$C_{\delta\mathcal{I}\tau}(\mathcal{A} \vee \mathcal{B},r)(x) < t.$$

It is a contradiction of equation (2.6) and $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \vee C_{\delta\mathcal{I}\tau}(\mathcal{B},r) \geq C_{\delta\mathcal{I}\tau}(\mathcal{A} \vee \mathcal{B},r)$.

On the other hand, $\mathcal{A}, \mathcal{B} \geq \mathcal{A} \vee \mathcal{B}$. Hence $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \vee C_{\delta\mathcal{I}\tau}(\mathcal{B},r) \leq C_{\delta\mathcal{I}\tau}(\mathcal{A} \vee \mathcal{B},r)$. Thus,

$$C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \vee C_{\delta\mathcal{I}\tau}(\mathcal{B},r) = C_{\delta\mathcal{I}\tau}(\mathcal{A} \vee \mathcal{B},r).$$

(11) Since $\mathcal{A} \leq C_{\delta\mathcal{I}\tau}(\mathcal{A},r)$, $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \leq C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\mathcal{A},r),r)$. On the other hand, suppose that $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \not\geq C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\mathcal{A},r),r)$, there exists $x \in X$ and $t \in I_0$ such that

$$C_{\delta\mathcal{I}\tau}(\mathcal{A},r)(x) < t < C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\mathcal{A},r),r)(x). \tag{2.7}$$

Since $C_{\delta\mathcal{I}\tau}(\mathcal{A},r)(x) < t$, x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$. So, there exists $\mathcal{B} \in Q_\tau(x_t,r)$ such that $A \leq \underline{1} - int_\tau(Cl^\star(\mathcal{B},r),r) = C_\tau(int^\star(\mathcal{B},r),r)$. Since, $C_\tau(int^\star(\mathcal{B},r),r)$ is **r-FRIC** and $A \leq C_\tau(int^\star(\mathcal{B},r),r)$. Then by Theorem 1.1(4), $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \leq C_\tau(int^\star(\mathcal{B},r),r)$. Again,

$$C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\mathcal{A},r),r) \leq C_{\delta\mathcal{I}\tau}(C_\tau(int^\star(\mathcal{B},r),r),r) = C_\tau(int^\star(\mathcal{B},r),r).$$

Hence, $C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\mathcal{A},r),r)(x) \leq C_\tau(int^\star(\mathcal{B},r),r)(x) < t$. It is a contradiction for equation (2.7). □

Theorem 2.2. *Let $(X,\tau,\mathcal{I})$ be a fits, $\mathcal{A} \in I^X$ and $r \in I_0$. Then the following properties are holds:*

(1) *$\mathcal{A}$ is r-**FPIC** iff $C_\tau(\mathcal{A},r) = C_{\delta\mathcal{I}\tau}(\mathcal{A},r)$.*
(2) *$\mathcal{A}$ is r-**FSIC** iff $C_\tau(\mathcal{A},r) = C_{\theta\mathcal{I}\tau}(\mathcal{A},r)$.*
(3) *$\mathcal{A}$ is r-**FαIO** iff $C_\tau(\mathcal{A},r) = C_{\delta\mathcal{I}\tau}(\mathcal{A},r) = C_{\theta\mathcal{I}\tau}(\mathcal{A},r)$.*

Proof. (1) Let $\mathcal{A}$ be r-**FPIC**. Then $\mathcal{A} \leq C_\tau(int^\star(\mathcal{A}, r), r)$ and by Theorem 1.1(3) and (4), we have

$$\begin{aligned} C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) &\leq C_{\delta\mathcal{I}\tau}(C_\tau(int^\star(\mathcal{A}, r), r), r) \\ &= C_\tau(int^\star(\mathcal{A}, r), r) \\ &\leq C_\tau(\mathcal{A}, r) \leq C_{\delta\mathcal{I}\tau}(\mathcal{A}, r). \end{aligned}$$

Conversely, suppose that there exist $\mathcal{A} \in I^X$, $r \in I_0$ $x \in X$ and $t \in (0, 1)$ such that

$$C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)(x) > t > C_\tau(\mathcal{A}, r)(x).$$

Then x_t is not r-fuzzy δ-cluster point of $\mathcal{A}$. So, there exists $\mathcal{B} \in Q_\tau(x_t, r)$, with $\mathcal{A} \leq \underline{1} - \mathcal{B}$. Since x_t is r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$, for $\mathcal{B} \in Q_\tau(x_t, r)$, we have $int_\tau(Cl^\star(\mathcal{B}, r), r)q\mathcal{A}$. Since

$$int_\tau(Cl^\star(\mathcal{B}, r), r) \leq int_\tau(Cl^\star(\underline{1} - \mathcal{A}, r), r),$$

and

$$\mathcal{A} \geq \underline{1} - int_\tau(Cl^\star(\mathcal{B}, r), r) \geq \underline{1} - int_\tau(Cl^\star(\underline{1} - \mathcal{A}, r), r) = C_\tau(int^\star(\mathcal{A}, r), r).$$

Hence, $\mathcal{A}$ is not r-**FPIC**.

(2) Let $\mathcal{A}$ be r-**FSIC**. Then $\mathcal{A} \leq int^\star(C_\tau(\mathcal{A}, r), r)$, $\tau(\underline{1} - C_\tau(\mathcal{A}, r)) \geq r$, by Theorem 4.3.2(4), we have $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \leq C_\tau(\mathcal{A}, r)$.

Conversely, suppose that there exist $\mathcal{A} \in I^X$, $r \in I_0$ $x \in X$ and $t \in (0, 1)$ such that

$$C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x) > t > C_\tau(\mathcal{A}, r)(x).$$

Then $\underline{1} - C_\tau(\mathcal{A}, r) = int_\tau(\underline{1} - \mathcal{A}, r) \in Q_\tau(x_t, r)$. Since x_t is r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$, $Cl^\star(int_\tau(\underline{1} - \mathcal{A}, r), r)q\mathcal{A}$. It implies $\mathcal{A} \nleq \underline{1} - Cl^\star(int_\tau(\underline{1} - \mathcal{A}, r), r) = int^\star(C_\tau(\mathcal{A}, r), r)$. Thus $\mathcal{A}$ is not r-**FSIC**.

(3) It is similarly proved as in (1) and (2). □

Definition 2.2. *Let $(X, \tau, \mathcal{I})$ be a fits, for $\mathcal{A}, \mathcal{B} \in I^X$ and $r \in I_0$. Then,*

1. *$\mathcal{A}$ is called is r-fuzzy $\delta\mathcal{I}$-closed (resp. r-fuzzy $\theta\mathcal{I}$-closed) iff $C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) = \mathcal{A}$ (resp. $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) = \mathcal{A}$). We define*
$$\Delta_{\tau_\mathcal{I}}(\mathcal{A}, r) = \bigwedge\{\mathcal{B} | \ \mathcal{A} \leq \mathcal{B}, \ \mathcal{B} = C_{\delta\mathcal{I}\tau}(\mathcal{B}, r)\}.$$
$$\Theta_{\tau_\mathcal{I}}(\mathcal{A}, r) = \bigwedge\{\mathcal{B} | \ \mathcal{A} \leq \mathcal{B}, \ \mathcal{B} = C_{\theta\mathcal{I}\tau}(\mathcal{B}, r)\}.$$
2. *The complement of r-fuzzy $\delta\mathcal{I}$-closed (resp. r-fuzzy $\theta\mathcal{I}$-closed) set is called r-fuzzy $\delta\mathcal{I}$-open (resp. r-fuzzy $\theta\mathcal{I}$-open).*

Theorem 2.3. *Let $(X, \tau, \mathcal{I})$ be a fits, for $\mathcal{A} \in I^X$ and $r \in I_0$. Then the following properties are holds:*

1. $\Delta_{\tau_\mathcal{I}}(\mathcal{A}, r) = C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)$
2. *$\Delta_{\tau_\mathcal{I}}(\mathcal{A}, r)$ is r-fuzzy $\delta\mathcal{I}$-closed.*
3. $\Theta_{\tau_\mathcal{I}}(\mathcal{A}, r) = C_{\theta\mathcal{I}\tau}(\Theta_{\tau_\mathcal{I}}(\mathcal{A}, r), r)$.
4. *$\Theta_{\tau_\mathcal{I}}(\mathcal{A}, r)$ is r-fuzzy $\theta\mathcal{I}$-closed.*
5. $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \leq \Theta_{\tau_\mathcal{I}}(\mathcal{A}, r)$.

Proof. From Theorem 2.1 (9,11), $\mathcal{A} \leq C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) = C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\mathcal{A}, r), r)$ implies $\Delta_{\tau_{\mathcal{I}}}(\mathcal{A}, r) \leq C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)$.

Suppose that $\Delta_{\tau_{\mathcal{I}}}(\mathcal{A}, r) \not\geq C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)$, there exist $x \in X$ and $t \in I_0$ such that

$$\Delta_{\tau_{\mathcal{I}}}(\mathcal{A}, r)(x) < t < C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)(x).$$

Form the definition of $\Delta_{\tau_{\mathcal{I}}}(\mathcal{A}, r)$. There exist $\mathcal{B} \in I^X$ and $\mathcal{A} \leq \mathcal{B} = C_{\delta\mathcal{I}\tau}(\mathcal{B}, r)$ such that

$$\Delta_{\tau_{\mathcal{I}}}(\mathcal{A}, r)(x) \leq \mathcal{B}(x) < t < C_{\delta\mathcal{I}\tau}(\mathcal{A}, r)(x).$$

On the other hand, $C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) \leq C_{\delta\mathcal{I}\tau}(\mathcal{B}, r) = \mathcal{B}$. It is a contradiction. Hence, $\Delta_{\tau_{\mathcal{I}}}(\mathcal{A}, r) \geq C_{\delta\mathcal{I}\tau}(\mathcal{B}, r)$.

(2) Form Theorem 2.1(11), it is trivial.

(3) Let $\mathcal{A} \leq \mathcal{B}_i = C_{\theta\mathcal{I}\tau}(\mathcal{B}_i, r)$ for each $i \in \Gamma$. Then

$$\bigwedge_{i\in\Gamma} \mathcal{B}_i \leq C_{\theta\mathcal{I}\tau}(\bigwedge_{i\in\Gamma} \mathcal{B}_i, r) \leq C_{\theta\mathcal{I}\tau}(\mathcal{B}_i, r) = \mathcal{B}_i.$$

So, $\bigwedge_{i\in\Gamma} \mathcal{B}_i \leq C_{\theta\mathcal{I}\tau}(\bigwedge_{i\in\Gamma} \mathcal{B}_i, r)$. Hence, $\Theta_{\tau_{\mathcal{I}}}(\mathcal{A}, r) = C_{\theta\mathcal{I}\tau}(\Theta_{\tau_{\mathcal{I}}}(\mathcal{A}, r), r)$.

(4) Form (3), it is trivial.

(5) Since $\mathcal{A} \leq \Theta_{\tau_{\mathcal{I}}}(\mathcal{A}, r)$, by (3), $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \leq C_{\theta\mathcal{I}\tau}(\Theta_{\tau_{\mathcal{I}}}(\mathcal{A}, r), r) = \Theta_{\tau_{\mathcal{I}}}(\mathcal{A}, r)$. □

Definition 2.3. *Let $(X, \tau, \mathcal{I})$ be a fits, $\mathcal{A}, \mathcal{B} \in I^X$ and $r \in I_0$. Then X is called:*

1. *Fuzzy $\mathcal{I}$-regular if for each $\mathcal{A} \in Q_\tau(x_t, r)$, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $Cl^\star(\mathcal{B}, r) \leq \mathcal{A}$.*
2. *Fuzzy almost $\mathcal{I}$-regular if for each $\mathcal{A} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$, there exists $\mathcal{B} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$ such that $Cl^\star(\mathcal{B}, r) \leq \mathcal{A}$.*

Theorem 2.4. *Let $(X, \tau, \mathcal{I})$ be a fits, for $\mathcal{A}, \mathcal{B} \in I^X$ and $r \in I_0$. Then the following statements are equivalent:*

1. *$(X, \tau, \mathcal{I})$ is called fuzzy almost $\mathcal{I}$-regular.*
2. *For each $x_t \in P_t(X)$ and each $\mathcal{A} \in Q_\tau(x_t, r)$, there exists $\mathcal{B} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$ such that $Cl^\star(\mathcal{B}, r) \leq int_\tau(Cl^\star(\mathcal{A}, r), r)$.*
3. *For each $x_t \in P_t(X)$ and each $\mathcal{A} \in Q_\tau(x_t, r)$, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $Cl^\star(\mathcal{B}, r) \leq int_\tau(Cl^\star(\mathcal{A}, r), r)$.*
4. *For each $x_t \in P_t(X)$ and r-**FRIC** set $\mathcal{D} \in I^X$ with $x_t \notin \mathcal{D}$, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ and $\mathcal{A}$ is r-fuzzy $\star$-open set such that $\mathcal{D} \leq \mathcal{A}$ and $Cl^\star(\mathcal{A}, r)\bar{q}Cl^\star(\mathcal{B}, r)$.*
5. *For each $x_t \in P_t(X)$ and r-**FRIC** set $\mathcal{D} \in I^X$ with $x_t \notin \mathcal{D}$, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ and $\mathcal{A}$ is r-fuzzy $\star$-open set such that $\mathcal{D} \leq \mathcal{A}$ and $Cl^\star(\mathcal{B}, r)\bar{q}\mathcal{A}$.*
6. *For each r-**FRIO** set $\mathcal{A} \in I^X$ with $\mathcal{D}q\mathcal{A}$, there exists r-**FRIO** set $\mathcal{B} \in I^X$ such that $\mathcal{D}q\mathcal{B} \leq Cl^\star(\mathcal{B}, r) \leq \mathcal{A}$.*
7. *For each r-**FRIC** set $\mathcal{A} \in I^X$ with $\mathcal{D} \not\leq \mathcal{A}$, there exist r-**FRIO** set $\mathcal{B} \in I^X$ and is r-fuzzy $\star$-open set $C \in I^X$ such that $\mathcal{D}q\mathcal{B}$, $\mathcal{A} \leq C$ and $\mathcal{B}\bar{q}C$.*

Proof. The proof of (1)⇒(2) and (2)⇒(3) are obvious.

(3)⇒(1): Let $x_t \in P_t(X)$ and $\mathcal{A} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$. Then, by (3), there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $Cl^\star(\mathcal{B}, r) \leq int_\tau(Cl^\star(\mathcal{A}, r), r) = \mathcal{A}$. Since $\mathcal{B} \in Q_\tau(x_t, r)$, $int_\tau(Cl^\star(\mathcal{B}, r), r) \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$. Also, since $\mathcal{D} = int_\tau(Cl^\star(\mathcal{B}, r), r) \leq Cl^\star(\mathcal{B}, r)$, $Cl^\star(\mathcal{D}, r) \leq Cl^\star(\mathcal{B}, r)$ and hence $x_t q\mathcal{D} \leq Cl^\star(\mathcal{D}, r) \leq Cl^\star(\mathcal{B}, r) \leq \mathcal{A}$ where $\mathcal{D} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$.

(3)⇒(4): Let $\mathcal{D}$ be r-**FRIC** set in X and $x_t \in P_t(X)$ with $x_t \notin \mathcal{D}$. Then $x_t q\underline{1} - \mathcal{D}$ and $\underline{1} - \mathcal{D} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r) \subset Q_\tau(x_t, r)$. By (3), there exists $C \in Q_\tau(x_t, r)$ such that $Cl^\star(C, r) \leq int_\tau(Cl^\star(\underline{1} - \mathcal{D}, r), r) = \underline{1} - \mathcal{D}$.

Now, $x_t q int_\tau(Cl^\star(C, r), r)$, then, $int_\tau(Cl^\star(C, r), r) \in Q_{\tau_{\mathcal{I}}}(x_t, r)$, and hence by hypothesis, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $Cl^\star(\mathcal{B}, r) \leq int_\tau(Cl^\star(C, r), r)$. Then, $\mathcal{D} \leq \underline{1} - Cl^\star(C, r)$. Put $\mathcal{A} = \underline{1} - Cl^\star(C, r)$, then $\mathcal{A}$ is r-fuzzy $\star$-open set. Hence

$$Cl^\star(\mathcal{A}, r) \leq \underline{1} - int_\tau(Cl^\star(C, r), r) \leq \underline{1} - Cl^\star(\mathcal{B}, r).$$

Hence, $Cl^\star(\mathcal{B}, r)\overline{q}Cl^\star(\mathcal{A}, r)$

(4)⇒(5): It is trivial.

(5)⇒(6): Suppose that $\mathcal{A}$ is r-**FRIO** set with $\mathcal{D}q\mathcal{A}$, then $\mathcal{D} \nleq \underline{1} - \mathcal{A}$. Hence there exists $x_t \in P_t(X)$ such that $x_t \in \mathcal{D}$ and $\mathcal{D}_t \nleq \underline{1} - \mathcal{A}$ where $\underline{1} - \mathcal{A}$ is r-**FRIC** set. By (5), there exists $\mathcal{B} \in Q_\tau(x_t, r)$ and $C \in I^X$ is r-fuzzy $\star$-open set such that $\underline{1} - \mathcal{A} \leq C$ and $Cl^\star(\mathcal{B}, r)\overline{q}C$. From $\mathcal{B} \in Q_\tau(x_t, r)$ we have $x_t q\mathcal{B} \leq int_\tau(Cl^\star(\mathcal{B}, r), r)$. Put $\mathcal{B}_1 = int_\tau(Cl^\star(\mathcal{B}, r), r)$, we have $\mathcal{D}q\mathcal{B}_1$ and $\mathcal{B}_1$ is r-**FRIO** set such that

$$\mathcal{D}q\mathcal{B}_1 \leq Cl^\star(\mathcal{B}_1, r) \leq Cl^\star(\mathcal{B}, r) \leq \underline{1} - C \leq \mathcal{A}.$$

(6)⇒(7): Let $\mathcal{A}$ be r-**FRIC** set $\mathcal{A} \in I^X$ with $\mathcal{D} \nleq \mathcal{A}$. Then, $\mathcal{D}q\underline{1} - \mathcal{A}$ and hence by (6), there exists r-**FRIO** set $\mathcal{B} \in I^X$ such that $\mathcal{D}q\mathcal{B} \leq Cl^\star(\mathcal{B}, r) \leq \underline{1} - \mathcal{A}$. Then, $\mathcal{B}$ is r-**FRIO** set and $\underline{1} - Cl^\star(\mathcal{B}, r)$ is r-fuzzy $\star$-open set such that $\mathcal{D}q\mathcal{B}$, $\mathcal{A} \leq \underline{1} - Cl^\star(\mathcal{B}, r)$ and $\mathcal{B}\overline{q}\underline{1} - Cl^\star(\mathcal{B}, r)$.

(7)⇒(1): Let $\mathcal{A} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$. Then $x_t \notin \underline{1} - \mathcal{A}$ and $\underline{1} - \mathcal{A}$ is r-**FRIC** set. By (7), there exist r-**FRIO** set $\mathcal{B} \in I^X$ and is r-fuzzy $\star$-open set $C \in I^X$ such that $x_t q\mathcal{B}$, $\underline{1} - \mathcal{A} \leq C$ and $\mathcal{B}\overline{q}C$. Then, $\mathcal{B} \in \mathfrak{R}_{\tau_{\mathcal{I}}}(x_t, r)$. Since C is r-fuzzy $\star$-open set, $Cl^\star(\mathcal{B}, r)\overline{q}C$. Thus $x_t q\mathcal{B} \leq Cl^\star(\mathcal{B}, r) \leq \underline{1} - C \leq \mathcal{A}$. Hence $(X, \tau, \mathcal{I})$ is called fuzzy almost $\mathcal{I}$-regular. □

The following theorem is similarly proved in Theorem 2.4.

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ be a fits, for $\mathcal{A} \in I^X$ and $r \in I_0$. Then the following statements are equivalent:*

1. *$(X, \tau, \mathcal{I})$ is called fuzzy $\mathcal{I}$-regular.*
2. *For each $x_t \in P_t(X)$ and each $\mathcal{A} \in I^X$ with $\tau(\underline{1} - \mathcal{A}) \geq r$ and $x_t \notin \mathcal{A}$, there exists $\mathcal{B} \in I^X$ with $\mathcal{B}$ is r-fuzzy $\star$-open such that $x_t \notin C_\tau(\mathcal{B}, r)$ and $\mathcal{A} \leq \mathcal{B}$.*
3. *For each $x_t \in P_t(X)$ and each $\mathcal{A} \in I^X$ with $\tau(\underline{1} - \mathcal{A}) \geq r$ and $x_t \notin \mathcal{A}$, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ and $C \in I^X$ with C is r-fuzzy $\star$-open such that $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B}\overline{q}C$.*
4. *For each $\mathcal{D} \in I^X$ and $\mathcal{A} \in I^X$ with $\tau(\underline{1} - \mathcal{A}) \geq r$ and $\mathcal{D} \nleq \mathcal{A}$, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ and $\mathcal{B}, C \in I^X$ with $\tau(\mathcal{B}) \geq r$ and C is r-fuzzy $\star$-open sets such that $\mathcal{D}q\mathcal{B}$, $\mathcal{A} \leq C$ and $\mathcal{B}\overline{q}C$.*

Theorem 2.6. *An fits $(X, \tau, \mathcal{I})$ is fuzzy almost $\mathcal{I}$-regular iff for each $\mathcal{A} \in I^X$ and $r \in I_0$, $C_{\delta\mathcal{I}\tau}(\mathcal{A}, r) = C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)$.*

Proof. From Theorem 4.3.2(9), we only show that $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \geq C_{\theta\mathcal{I}\tau}(\mathcal{A},r)$. Let $C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \ngeq C_{\theta\mathcal{I}\tau}(\mathcal{A},r)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$C_{\delta\mathcal{I}\tau}(\mathcal{A},r)(x) < t < C_{\theta\mathcal{I}\tau}(\mathcal{A},r)(x). \tag{2.8}$$

Since $C_{\delta\mathcal{I}\tau}(\mathcal{A},r)(x) < t$, x_t is not a r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$. So, there exists $\mathcal{B} \in Q_\tau(x_t,r)$, with $\mathcal{A} \leq \underline{1} - int_\tau(Cl^\star(\mathcal{B},r),r)$. Since $\mathcal{B} \in Q_\tau(x_t,r)$, $int_\tau(Cl^\star(\mathcal{B},r),r) \in \Re_{\tau_\mathcal{I}}(x_t,r)$. By fuzzy almost $\mathcal{I}$-regularity of X, there exists $\mathcal{D} \in \Re_{\tau_\mathcal{I}}(x_t,r)$ such that $Cl^\star(\mathcal{D},r) \leq int_\tau(Cl^\star(\mathcal{B},r),r)$. Thus,

$$\mathcal{A} \leq \underline{1} - int_\tau(Cl^\star(\mathcal{B},r),r) \leq \underline{1} - Cl^\star(\mathcal{D},r) = int^\star(\underline{1} - \mathcal{D}), \tau(\mathcal{D}) \geq r.$$

By Theorem 2.1(4), $C_{\theta\mathcal{I}\tau}(\mathcal{A},r)(x) \leq (\underline{1} - \mathcal{D})(x) < t$. It is a contradiction for equation (4.9).

Conversely, Let $\mathcal{A} \in \Re_{\tau_\mathcal{I}}(x_t,r) \subset Q_\tau(x_t,r)$. Then by Theorem 2.1(8), $t > (\underline{1} - \mathcal{A})(x) = C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{A},r)(x)$. Since, $C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{A},r) = C_{\theta\mathcal{I}\tau}(\underline{1} - \mathcal{A},r)$, x_t x_t is not a r-fuzzy $\theta\mathcal{I}$-cluster point of $\underline{1} - \mathcal{A}$. Then there exists $\mathcal{B} \in Q_\tau(x_t,r)$ such that $\underline{1} - \mathcal{A}\bar{q}Cl^\star(\mathcal{B},r)$ implies $Cl^\star(B,r) \leq \mathcal{A} = int_\tau(Cl^\star(\mathcal{A},r),r)$ and by Theorem 2.4(3), $(X,\tau,\mathcal{I})$ is fuzzy almost $\mathcal{I}$-regular. □

Theorem 2.7. *An fits $(X,\tau,\mathcal{I})$ is fuzzy almost $\mathcal{I}$-regular iff for each r-**FRIC** set $\mathcal{A} \in I^X$ and $r \in I_0$, $C_{\theta\mathcal{I}\tau}(\mathcal{A},r) = \mathcal{A}$.*

Proof. The necessary part follows from Theorem 4.3.9 and the fact that r-**FRIC** set is r-fuzzy $\delta\mathcal{I}$-closed.

Conversely, let $\mathcal{A}$ be any r-**FRIC** set with $x_t \notin \mathcal{A}$. Then, $x_t \notin C_{\theta\mathcal{I}\tau}(\mathcal{A},r)$ and hence, x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$ so, there there exists $\mathcal{B} \in Q_\tau(x_t,r)$ such that $\mathcal{A}\bar{q}Cl^\star(\mathcal{B},r)$. Thus, $\mathcal{A} \leq \underline{1} - Cl^\star(\mathcal{B},r) = \mathcal{D}$ and $\mathcal{D}$ is r-fuzzy $\star$-open implies $\mathcal{D}\bar{q}Cl^\star(\mathcal{B},r)$. Hence, by Theorem 4.3.7(5), $(X,\tau,\mathcal{I})$ is fuzzy almost $\mathcal{I}$-regular. □

Lemma 2.1. *If $\mathcal{A},\mathcal{B} \in I^X$, $r \in I_0$ such that $\mathcal{A}\bar{q}\mathcal{B}$ where $\mathcal{B}$ is r-fuzzy $\delta\mathcal{I}$-open, then $C_{\delta\mathcal{I}\tau}(\mathcal{A},r)\bar{q}\mathcal{B}$.*

Proof. Let $\mathcal{A}\bar{q}\mathcal{B}$ where $\mathcal{B}$ is r-fuzzy $\delta\mathcal{I}$-open. Then, $\mathcal{A} \leq \underline{1} - \mathcal{B} = C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{B},r)$, by Theorem 2.1(11),

$$C_{\delta\mathcal{I}\tau}(\mathcal{A},r) \leq C_{\delta\mathcal{I}\tau}(C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{B},r),r) = C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{B},r) = \underline{1} - \mathcal{B}.$$

Hence, $C_{\delta\mathcal{I}\tau}(\mathcal{A},r)\bar{q}\mathcal{B}$. □

Lemma 2.2. *Let $(X,\tau,\mathcal{I})$ be a fits and $\mathcal{A} \in I^X$ is r-fuzzy $\delta\mathcal{I}$-open set iff for every $x_t \in P_t(X)$ with $x_tq\mathcal{A}$, there exists r-**FRIO** set $\mathcal{B} \in I^X$ such that $x_tq\mathcal{B} \leq \mathcal{A}$.*

Proof. Let $x_t \in P_t(X)$ with $x_tq\mathcal{A}$. Then $x_t \notin \underline{1} - \mathcal{A}$. Since $\mathcal{A}$ is r-fuzzy $\delta\mathcal{I}$-open set, $x_t \notin \underline{1} - \mathcal{A} = C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{A},r)$. Thus, x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\underline{1} - \mathcal{A}$. So, there exists $\mathcal{D} \in Q_\tau(x_t,r)$ such that $\underline{1} - \mathcal{A}\bar{q}int_\tau(Cl^\star(\mathcal{D},r),r)$. Put $\mathcal{B} = int_\tau(Cl^\star(\mathcal{D},r),r)$, so, $\mathcal{B}$ is r-**FRIO** set with $x_tq\mathcal{B} \leq \mathcal{A}$.

Conversely, suppose $\underline{1} - \mathcal{A} \neq C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{A},r)$, then there exist $x \in X$ and $t \in I_0$ such that

$$(\underline{1} - \mathcal{A})(x) < t < C_{\delta\mathcal{I}\tau}(\underline{1} - \mathcal{A},r)(x).$$

Since $x_tq\mathcal{A}$, there exists a r-**FRIO** set $\mathcal{B}$ such that $x_tq\mathcal{B} \leq \mathcal{A}$. It implies

$$\underline{1} - \mathcal{A} \leq \underline{1} - \mathcal{B} = C_\tau(int^\star(\underline{1} - \mathcal{B},r),r).$$

By Theorem 1.1(4), $C_{\delta\mathcal{I}\tau}(\underline{1}-\mathcal{A},r)(x) \le (\underline{1}-\mathcal{B})(x) < t$. It is a contradiction. Hence, $\underline{1}-\mathcal{A} = C_{\delta\mathcal{I}\tau}(\underline{1}-\mathcal{A},r)$, i.e., $\mathcal{A}$ is r-fuzzy $\delta\mathcal{I}$-open set. □

Lemma 2.3. *If $\tau(\mathcal{A}) \ge r$, then $C_\tau(\mathcal{A},r) = C_{\delta\mathcal{I}\tau}(\mathcal{A},r)$.*

Proof. Let $\tau(\mathcal{A}) \ge r$. Then, $\mathcal{A}$ is r-fuzzy $\star$-open set and so, $\mathcal{A} = int^\star(\mathcal{A},r)$. Then, by Theorem 2.1(8),

$$C_\tau(\mathcal{A},r) = C_\tau(int^\star(\mathcal{A},r),r) = C_{\delta\mathcal{I}\tau}(\mathcal{A},r).$$

□

Theorem 2.8. *Let $(X,\tau,\mathcal{I})$ be a fits. Then the following statements are equivalent:*

(1) *$(X,\tau,\mathcal{I})$ is fuzzy almost $\mathcal{I}$-regular.*
(2) *For each r-fuzzy $\delta\mathcal{I}$-open set $\mathcal{A} \in^X$ and each $x_t \in P_t(X)$ with $x_t q\mathcal{A}$, there exists r-fuzzy $\delta\mathcal{I}$-open set $\mathcal{B} \in I^X$ such that $x_t q\mathcal{B} \le Cl^\star(\mathcal{B},r) \le \mathcal{A}$.*

Proof. (1)⇒(2): Let $\mathcal{A}$ be r-fuzzy $\delta\mathcal{I}$-open set such each $x_t q\mathcal{A}$. Then by Lemma 2.3., there exists r-**FRIO** set $C \in I^X$ such that $x_t qC \le \mathcal{A}$. By fuzzy almost $\mathcal{I}$-regularity of X, there exists r-**FRIO** set $\mathcal{B}$ (which is also r-fuzzy $\delta\mathcal{I}$-open) such that $x_t q\mathcal{B} \le Cl^\star(\mathcal{B},r) \le C \le \mathcal{A}$.

(2)⇒(1): It is obvious. □

3. Fuzzy $\theta\mathcal{I}$-Continuous

Definition 3.1. *Let $(X,\tau,\mathcal{I}_1) \to (Y,\eta,\mathcal{I}_2)$ be a mapping. Then,*

(1) *f is called fuzzy δ-ideal continuous (**F**$\delta\mathcal{I}$-continuous, for short) iff for each $\mathcal{A} \in Q_\eta(f(x_t),r)$, there exists $\mathcal{B} \in Q_\tau(x_t,r)$ such that*

$$f(int_\tau(Cl^\star(\mathcal{B},r),r)) \le int_\eta(Cl^\star(\mathcal{A},r),r).$$

(2) *f is called fuzzy θ-ideal continuous (**F**$\theta\mathcal{I}$-continuous, for short) iff for each $\mathcal{A} \in Q_\eta(f(x_t),r)$, there exists $\mathcal{B} \in Q_\tau(x_t,r)$ such that*

$$f(Cl^\star(\mathcal{B},r)) \le Cl^\star(\mathcal{A},r).$$

(3) *f is called fuzzy strongly θ-ideal continuous (**FS**$\theta\mathcal{I}$-continuous, for short) iff for each $\mathcal{A} \in Q_\eta(f(x_t),r)$, there exists $\mathcal{B} \in Q_\tau(x_t,r)$ such that*

$$f(Cl^\star(\mathcal{B},r)) \le \mathcal{A}.$$

(4) *f is called fuzzy almost ideal continuous (**FA**$\mathcal{I}$-continuous, for short) iff for each $\mathcal{A} \in Q_\eta(f(x_t),r)$, there exists $\mathcal{B} \in Q_\tau(x_t,r)$ such that*

$$f(\mathcal{B}) \le int_\eta(Cl^\star(\mathcal{A},r),r).$$

From the above definition, we obtain the following diagram:

F strongly continuous $\Rightarrow$ **F super continuous**

$\Downarrow$ $\Downarrow$

FSθI-continuous $\Rightarrow$ **F$\delta\mathcal{I}$-continuous** $\Rightarrow$ **FA$\mathcal{I}$-continuous**.

Theorem 3.1. *Let $f : (X, \tau, \mathcal{I}_1) \to (Y, \eta, \mathcal{I}_2)$ be a mapping. Then the following statements are equivalent:*

(1) *f is **F**$\delta\mathcal{I}$-continuous.*
(2) *For each $\mathcal{A} \in \mathfrak{R}_{\eta_{\mathcal{I}_2}}(f(x_t), r)$ there exists $\mathcal{B} \in \mathfrak{R}_{\tau_{\mathcal{I}_1}}(x_t, r)$ such that $f(\mathcal{B}) \leq \mathcal{A}$.*
(3) *$f(C_{\delta\mathcal{I}_1\tau}(\mathcal{A}, r)) \leq C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)$ for each $\mathcal{A} \in I^X$ and $r \in I_0$.*
(4) *$C_{\delta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r)) \leq f^{-1}(C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r))$ for each $\mathcal{B} \in I^Y$ and $r \in I_0$.*
(5) *For each r-fuzzy $\delta\mathcal{I}$-closed (resp. r-fuzzy $\delta\mathcal{I}$-open) set $\mathcal{B} \in I^Y$, $f^{-1}(\mathcal{B})$ is r-fuzzy $\delta\mathcal{I}$-closed (resp. r-fuzzy $\delta\mathcal{I}$-open) set in X.*
(6) *For each r-**FRIO** (resp. r-**FRIC**) set $\mathcal{D} \in I^Y$, $f^{-1}(\mathcal{D})$ is r-fuzzy δI-open (resp. r-fuzzy δI-closed) set in X.*

Proof. (1)$\Rightarrow$(2): This follows immediately from Definition 3.1.

(2)$\Rightarrow$(3): Suppose there exists $\mathcal{A} \in I^X$ and $r \in I_0$ such that

$$f(C_{\delta\mathcal{I}_1\tau}(\mathcal{A}, r)) \nleq C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r).$$

Then there exists $y \in Y$ and $t \in I_0$ such that

$$f(C_{\delta\mathcal{I}_1\tau}(\mathcal{A}, r))(y) > t > C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, provides a contradiction that $f(C_{\delta\mathcal{I}_1\tau}(\mathcal{A}, r))(y) = 0$.

If $f^{-1}(\{y\}) \neq \emptyset$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_{\delta\mathcal{I}_1\tau}(\mathcal{A}, r))(y) \geq C_{\delta\mathcal{I}_1\tau}(\mathcal{A}, r)(x) > t > C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)(f(x)). \tag{3.1}$$

Since $C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)(f(x)) < t$, by Theorem 2.1(7), $f(x)_t$ is not r-fuzzy $\delta\mathcal{I}$-cluster point of $f(\mathcal{A})$. So, there exists $\mathcal{D} \in Q_\eta(f(x)_t, r)$ such that $f(\mathcal{A}) \leq \underline{1} - int_\eta(Cl^\star(\mathcal{D}, r), r)$. Since $\mathcal{D} \in Q_\eta(f(x)_t, r)$, $int_\eta(Cl^\star(\mathcal{D}, r), r) \in \mathfrak{R}_{\eta_{\mathcal{I}_2}}(f(x)_t, r)$. By (2), there exists $\mathcal{B} \in \mathfrak{R}_{\tau_{\mathcal{I}_1}}(x_t, r)$ such that $f(\mathcal{B}) \leq int_\eta(Cl^\star(\mathcal{D}, r), r)$. Hence, $\mathcal{B} \in Q_\tau(x_t, r)$ and $f(\mathcal{A}) \leq \underline{1} - f(\mathcal{B}) = \underline{1} - f(int_\tau(Cl^\star(\mathcal{B}, r), r))$ implies that $\mathcal{A} \leq \underline{1} - int_\tau(Cl^\star(\mathcal{B}, r), r)$. Thus x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\mathcal{A}$, by Theorem 2.1(7), $C_{\delta\mathcal{I}_1\tau}(\mathcal{A}, r)(x) < t$. It is a contradiction for equation (3.1).

(3)$\Rightarrow$(4): For all $\mathcal{B} \in I^Y$ and $r \in I_0$. Put $\mathcal{A} = f^{-1}(\mathcal{B})$ form (3). Then

$$f(C_{\delta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r)) \leq C_{\delta\mathcal{I}_2\eta}(f(f^{-1}(\mathcal{B})), r) \leq C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r).$$

It implies

$$C_{\delta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r) \leq f^{-1}(f(C_{\delta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r))) \leq f^{-1}(C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r)).$$

(4)$\Rightarrow$(5): Let $\mathcal{B} \in I^Y$ be r-fuzzy δI-closed. By (4), we have

$$C_{\delta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r) \leq f^{-1}(C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r)) = f^{-1}(\mathcal{B}),$$

and always $f^{-1}(\mathcal{B}) \leq C_{\delta\mathcal{I}\tau}(f^{-1}(\mathcal{B}), r)$, implies $f^{-1}(\mathcal{B}) = C_{\delta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r)$. Hence, $f^{-1}(\mathcal{B})$ is r-fuzzy $\delta\mathcal{I}$-closed set. Other case is similarly proved

(5)⇒(6): Let $\mathcal{D}$ be r-**FRIO** set in Y. Then, by Theorem 2.1(8), $\mathcal{D}$ is r-fuzzy $\delta\mathcal{I}$-open set. By (5), we have $f^{-1}(\mathcal{D})$ is r-fuzzy $\delta\mathcal{I}$-open set. Other cases are similarly proved

(6)⇒(1): Let $\mathcal{A} \in Q_\eta(f(x_t), r)$. Then, $int_\eta(Cl^\star(\mathcal{A}, r), r) \in \mathfrak{R}_{\eta\mathcal{I}_2}(f(x)_t, r)$. By (6), we have

$$\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)) = C_{\delta\mathcal{I}_1\tau}(\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)), r).$$

Since $f(x_t)q\mathcal{A} \leq int_\eta(Cl^\star(\mathcal{A}, r), r)$, $x_t q f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r))$, that is

$$t > (\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)))(x) = C_{\delta\mathcal{I}_1\tau}(\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)), r).$$

Thus, x_t is not r-fuzzy $\delta\mathcal{I}$-cluster point of $\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r))$. Then, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)) \leq \underline{1} - int_\tau(Cl^\star(\mathcal{B}, r), r)$. Hence,

$$f(int_\tau(Cl^\star(\mathcal{B}, r), r)) \leq int_\eta(Cl^\star(\mathcal{A}, r), r).$$

Therefore, f is **F**$\delta\mathcal{I}$-continuous. □

Theorem 3.2. *$f : (X, \tau, \mathcal{I}_1) \to (Y, \eta, \mathcal{I}_2)$ be a mapping. Then the following statements are equivalent:*

(1) *f is **F**$\theta\mathcal{I}$-continuous.*
(2) *for each $\mathcal{A} \in \mathfrak{R}_{\eta_{\mathcal{I}_2}}(f(x_t), r)$ there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $f(Cl^\star(\mathcal{B}, r)) \leq \mathcal{A}$.*
(3) *$f(C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r)) \leq C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)$ for each $\mathcal{A} \in I^X$ and $r \in I_0$.*
(4) *$C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r)) \leq f^{-1}(C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r))$ for each $\mathcal{B} \in I^Y$ and $r \in I_0$.*
(5) *For each r-fuzzy $\delta\mathcal{I}$-closed (resp. r-fuzzy $\delta\mathcal{I}$-open) set $\mathcal{B} \in I^Y$, $f^{-1}(\mathcal{B})$ is r-fuzzy $\theta\mathcal{I}$-closed (resp. r-fuzzy $\theta\mathcal{I}$-open) set.*

Proof. (1)⇒(2): This follows immediately from Definition 3.1.

(2)⇒(3): Suppose there exists $\mathcal{A} \in I^X$ and $r \in I_0$ such that

$$f(C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r)) \nleq C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r).$$

Then there exists $y \in Y$ and $t \in I_0$ such that

$$f(C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r))(y) > t > C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, provides a contradiction that $f(C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r))(y) = 0$.
If $f^{-1}(\{y\}) \neq \emptyset$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r))(y) \geq C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r)(x) > t > C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)(f(x)). \tag{3.2}$$

Since $C_{\delta\mathcal{I}_2\eta}(f(\mathcal{A}), r)(f(x)) < t$, $f(x)_t$ is not r-fuzzy $\delta\mathcal{I}$-cluster point of $f(\mathcal{A})$. So, there exists $\mathcal{D} \in Q_\eta(f(x)_t, r)$ such that $f(\mathcal{A}) \leq \underline{1} - int_\eta(Cl^\star(\mathcal{D}, r), r)$. Since $\mathcal{D} \in Q_\eta(f(x)_t, r)$, $int_\eta(Cl^\star(\mathcal{D}, r), r) \in \mathfrak{R}_{\eta_{\mathcal{I}_2}}(f(x_t), r)$. By (2), there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $f(Cl^\star(\mathcal{B}, r)) \leq int_\eta(Cl^\star(\mathcal{D}, r), r)$. Hence, $f(\mathcal{A}) \leq \underline{1} - f(Cl^\star(\mathcal{B}, r))$ implies that $\mathcal{A} \leq \underline{1} - Cl^\star(\mathcal{B}, r)$. Thus, x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\mathcal{A}$, by Theorem 2.1(6), $C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r)(x) < t$. it is a contradiction for equation (3.2).

(3)⇒(4): For all $\mathcal{B} \in I^Y$ and $r \in I_0$. Put $\mathcal{A} = f^{-1}(\mathcal{B})$ form (2). Then

$$f(C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}, r)) \leq C_{\delta\mathcal{I}_2\eta}(f(f^{-1}(\mathcal{B}), r) \leq C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r).$$

It implies that

$$C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}, r) \leq f^{-1}(f(C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r))) \leq f^{-1}(C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r)).$$

(4)⇒(5): Let $\mathcal{B} \in I^Y$ be r-fuzzy $\delta\mathcal{I}$-closed by (4), we have

$$C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r) \leq f^{-1}(C_{\delta\mathcal{I}_2\eta}(\mathcal{B}, r)) = f^{-1}(\mathcal{B}),$$

and since $f^{-1}(\mathcal{B}) \leq C_{\theta\mathcal{I}1\tau}(f^{-1}(\mathcal{B}), r)$, $f^{-1}(\mathcal{B}) = C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r)$. Another case is similarly proved.

(5)⇒(1): Let $\mathcal{A} \in Q_\eta(f(x_t), r)$. Then, $int_\eta(Cl^\star(\mathcal{A}, r), r) \in \mathfrak{R}_{\eta_{\mathcal{I}_2}}(f(x)_t, r)$, by Theorem 2.1(8), we have

$$\underline{1} - int_\eta(Cl^\star(\mathcal{A}, r), r) = C_{\delta\mathcal{I}_2\eta}(\underline{1} - int_\eta(Cl^\star(\mathcal{A}, r), r), r).$$

Hence, $int_\eta(Cl^\star(\mathcal{A}, r), r)$ is r-fuzzy $\delta\mathcal{I}$-open set. By (5),

$$\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)) = C_{\theta\mathcal{I}_1\tau}(\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)), r).$$

Since $f(x_t)q\mathcal{A} \leq int_\eta(Cl^\star(\mathcal{A}, r), r)$, $x_t q f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r))$, that is

$$t > (\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)))(x) = C_{\theta\mathcal{I}_1\tau}(\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)), r).$$

Thus, x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r))$. Then, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $\underline{1} - f^{-1}(int_\eta(Cl^\star(\mathcal{A}, r), r)) \leq \underline{1} - Cl^\star(\mathcal{B}, r)$. Hence,

$$f(Cl^\star(\mathcal{B}, r)) \leq int_\eta(Cl^\star(\mathcal{A}, r), r) \leq Cl^\star(\mathcal{A}, r).$$

Therefore, f is $\mathbf{F}\theta\mathcal{I}$-continuous. □

The following theorem is similarly proved as in Theorem 3.2.

Theorem 3.3. *Let $f : (X, \tau, \mathcal{I}_1) \to (Y, \eta, \mathcal{I}_2)$ be a mapping. Then the following statements are equivalent:*

(1) *f is $\mathcal{F}\theta\mathcal{I}$-continuous.*
(2) *$f(C_{\theta\mathcal{I}_1\tau}(\mathcal{A}, r)) \leq C_{\theta\mathcal{I}_2\eta}(f(\mathcal{A}), r)$ for each $\mathcal{A} \in I^X$ and $r \in I_0$.*
(3) *$C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r)) \leq f^{-1}(C_{\theta\mathcal{I}_2\eta}(\mathcal{B}, r))$ for each $\mathcal{B} \in I^Y$ and $r \in I_0$.*

Theorem 3.4. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \eta)$ be a mapping. Then the following statements are equivalent:*

(1) *f is **FS**$\theta\mathcal{I}$-continuous.*
(2) *$f(C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)) \leq C_\eta(f(\mathcal{A}), r)$ for each $\mathcal{A} \in I^X$ and $r \in I_0$.*
(3) *$C_{\theta\mathcal{I}\tau}(f^{-1}(\mathcal{B}), r)) \leq f^{-1}(C_\eta(\mathcal{B}, r))$ for each $\mathcal{B} \in I^Y$ and $r \in I_0$.*
(4) *For each $\eta(\underline{1} - \mathcal{B}) \geq r$ (resp. $\eta(\mathcal{B}) \geq r$), $f^{-1}(\mathcal{B})$ is r-fuzzy $\theta\mathcal{I}$-closed (resp. r-fuzzy $\theta\mathcal{I}$-open) set in X.*

Proof. (1)⇒(2): Suppose there exists $\mathcal{A} \in I^X$ and $r \in I_0$ such that

$$f(C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)) \nleq C_\eta(f(\mathcal{A}), r).$$

Then there exists $y \in Y$ and $t \in I_0$ such that

$$f(C_{\theta\mathcal{I}\tau}(\mathcal{A}, r))(y) > t > C_\eta(f(\mathcal{A}), r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, provides a contradiction that $f(C_{\theta\mathcal{I}\tau}(\mathcal{A}, r))(y) = 0$.
If $f^{-1}(\{y\}) \neq \emptyset$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_{\theta\mathcal{I}\tau}(\mathcal{A}, r))(y) \geq C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x) > t > C_\eta(f(\mathcal{A}), r)(f(x)). \quad (3.3)$$

Since $C_\eta(f(\mathcal{A}), r)(f(x)) < t$, we have, $f(x)_t$ is not r-fuzzy δ-cluster point of $f(\mathcal{A})$. So, there exists $\mathcal{D} \in Q_\eta(f(x)_t, r)$ such that $f(\mathcal{A}) \leq \underline{1} - \mathcal{D}$. Since f is **FS**$\theta\mathcal{I}$-continuous, for $\mathcal{D} \in Q_\eta(f(x)_t, r)$, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $f(Cl^\star(\mathcal{B}, r)) \leq \mathcal{D}$. Hence, $f(\mathcal{A}) \leq \underline{1} - f(C^\star(\mathcal{B}, r))$ implies $\mathcal{A} \leq \underline{1} - Cl^\star(\mathcal{B}, r) = int^\star(\underline{1} - \mathcal{B}, r)$. Since $\tau(\mathcal{B}) \geq r$, by Theorem 2.1(4), we have $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r) \leq \underline{1} - \mathcal{B}$. Since $x_t q \mathcal{B}$, we have $C_{\theta\mathcal{I}\tau}(\mathcal{A}, r)(x) \leq (\underline{1} - \mathcal{B})(x) < t$. It is a contradiction of equation (3.3).

(3)⇒(4): For all $\mathcal{B} \in I^Y$ and $r \in I_0$. Put $\mathcal{A} = f^{-1}(\mathcal{B})$ form (3). Then

$$f(C_{\theta\mathcal{I}\tau}(f^{-1}(\mathcal{B}, r)) \leq C_\eta(f(f^{-1}(\mathcal{B}), r) \leq C_\eta(\mathcal{B}, r).$$

Implies $C_{\theta\mathcal{I}\tau}(f^{-1}(\mathcal{B}, r) \leq f^{-1}(f(C_{\theta\mathcal{I}\tau}(f^{-1}(\mathcal{B}), r))) \leq f^{-1}(C_\eta(\mathcal{B}, r))$.

(4)⇒(5): Let $\eta(\underline{1} - \mathcal{B}) \geq r$. Then $\mathcal{B} = C_\eta(\mathcal{B}, r)$. By (4), we have

$$C_{\theta\mathcal{I}\tau}(f^{-1}(\mathcal{B}), r)) \leq f^{-1}(C_\eta(\mathcal{B}, r)) = f^{-1}(\mathcal{B}).$$

And always $f^{-1}(\mathcal{B}) \leq C_{\theta\mathcal{I}\tau}(f^{-1}(\mathcal{B}), r)$. Hence $f^{-1}(\mathcal{B}) = C_{\theta\mathcal{I}_1\tau}(f^{-1}(\mathcal{B}), r)$. Another case is similarly proved.

(5)⇒(1): Let $\mathcal{A} \in Q_\eta(f(x_t), r)$. Then, $\tau(\mathcal{A}) \geq r$. By (5), $\underline{1} - f^{-1}(\mathcal{A}) = C_{\theta\mathcal{I}_1\tau}(\underline{1} - f^{-1}(\mathcal{A}), r)$. Since $f(x_t) q \mathcal{A}$, $x_t q f^{-1}(\mathcal{A})$, that is

$$t > (\underline{1} - f^{-1}(\mathcal{A}))(x) = C_{\theta\mathcal{I}\tau}(\underline{1} - f^{-1}(\mathcal{A}), r).$$

Thus, x_t is not r-fuzzy $\theta\mathcal{I}$-cluster point of $\underline{1} - f^{-1}(\mathcal{A})$. Then, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $\underline{1} - f^{-1}(\mathcal{A}) \leq \underline{1} - (Cl^\star(\mathcal{B}, r)$. Hence, $f(Cl^\star(\mathcal{B}, r)) \leq \mathcal{A}$. Therefore, f is **FS**$\theta\mathcal{I}$-continuous. □

The following theorem is similarly proved as in Theorem 3.4.

Theorem 3.5. *$f : (X, \tau) \to (Y, \eta, \mathcal{I})$ be a mapping. Then the following statements are equivalent:*

(1) *f is **FA**$\mathcal{I}$-continuous.*
(2) *$f(C_\tau(\mathcal{A}, r)) \leq C_{\delta\mathcal{I}\eta}(f(\mathcal{A}), r)$ for each $\mathcal{A} \in I^X$ and $r \in I_0$.*
(3)[*$C_\tau(f^{-1}(\mathcal{B}), r)) \leq f^{-1}(C_{\delta\mathcal{I}\eta}(\mathcal{B}, r))$ for each $\mathcal{B} \in I^Y$ and $r \in I_0$.*
(4) *For each r-fuzzy $\delta\mathcal{I}$-closed (resp. r-fuzzy $\delta\mathcal{I}$-open) set $\mathcal{B} \in I^Y$, $\tau(\underline{1} - f^{-1}(\mathcal{B})) \geq r$ (resp. $\tau(f^{-1}(\mathcal{B})) \geq r$).*
(5) *For each r-**FRIO** (resp. r-**FRIC**) set $\mathcal{B} \in I^Y$, $\tau(f^{-1}(\mathcal{B})) \geq r$ (resp. $\tau(\underline{1} - f^{-1}(\mathcal{B})) \geq r$).*

Example 3.1. *Define* $\tau_1, \tau_2, \mathcal{I}_1, \mathcal{I}_2 : I^X \to I$ *as follows:*

$$\tau_1(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.4}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_2(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.4}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{I}_1(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} = \underline{0}, \\ \frac{2}{3}, & \text{if } \mathcal{B} = \underline{0.5}, \\ \frac{1}{2}, & \text{if } \underline{0} < \mathcal{B} < \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}_2(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} = \underline{0}, \\ \frac{2}{3}, & \text{if } \mathcal{B} = \underline{0.7}, \\ \frac{1}{2}, & \text{if } \underline{0} < \mathcal{B} < \underline{0.7}, \\ 0, & \text{otherwise}. \end{cases}$$

From Theorem 2.1(4), and 2.3, we obtain $C_\tau, D_\tau, C_{\delta\mathcal{I}\tau} : I^X \times I_0 \to I^X$ *as follows:*

$$(D_{\tau_1} = C_{\delta\mathcal{I}_1\tau_1})(\mathcal{B}, r) = \begin{cases} \underline{0}, & \text{if } \mathcal{B} = \underline{0}, \\ \underline{0.6}, & \text{if } \underline{0} \neq \mathcal{B} \leq \underline{0.6}, \ 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

$$C_{\tau_2}(\mathcal{B}, r) = \begin{cases} \underline{0}, & \text{if } \mathcal{B} = \underline{0}, \\ \underline{0.6}, & \text{if } \underline{0} \neq \mathcal{B} \leq \underline{0.6}, \ 0 < r \leq \frac{1}{2}, \\ \underline{0.7}, & \text{if } \underline{0.6} < \mathcal{B} \leq \underline{0.7}, \ 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

$$C_{\delta\mathcal{I}_2\tau_2}(\mathcal{B}, r) = \begin{cases} \underline{0}, & \text{if } \mathcal{B} = \underline{0}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

By Theorem 3.1(3), the identity mapping $id_X : (X, \tau_1, \mathcal{I}_1) \to (Y, \tau_2, \mathcal{I}_2)$ *is* ***F***$\delta\mathcal{I}$*-continuous but it is not* ***F****-super continuous because, by Theorem 1.4.6,* $\underline{1} = D_{\tau_1}(\underline{0.7}, \frac{1}{2}) \geq C_{\tau_2}(\underline{0.7}, \frac{1}{2}) = \underline{0.7}$.

Example 3.2. *Define* $\tau_1, \tau_2, \mathcal{I}_1, \mathcal{I}_2 : I^X \to I$ *as follows:*

$$\tau_1(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.6}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_2(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{B} \in \{\underline{0.6}, \underline{0.3}\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{I}_1(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} = \underline{0}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.3}, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{B} < \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}_2(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} = \underline{0}, \\ \frac{1}{2}, & \text{if } \mathcal{B} = \underline{0.4}, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{B} < \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases}$$

From Theorem 2.1(4), and 2.3, we obtain $C_{\theta\mathcal{I}\tau}, T_\tau, C_\tau : I^X \times I_0 \to I^X$ *as follows:*

$$C_{\theta\mathcal{I}_1\tau_1}(\mathcal{B}, r) = \begin{cases} \underline{0}, & \text{if } \mathcal{B} = \underline{0}, \\ \underline{0.4}, & \text{if } \underline{0} \neq \mathcal{B} \leq \underline{0.4}, \ 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

$$C_{\tau_2}(\mathcal{B},r)=\begin{cases}\underline{0}, & \text{if } \mathcal{B}=\underline{0},\\ \underline{0.4}, & \text{if } \underline{0}\neq\mathcal{B}\leq\underline{0.4},\ \ 0<r\leq\frac{1}{2},\\ \underline{0.7}, & \text{if } \underline{0.4}<\mathcal{B}\leq\underline{0.7},\ \ 0<r\leq\frac{1}{2},\\ \underline{1}, & \text{otherwise},\end{cases}$$

$$C_{\theta\mathcal{I}_2\tau_2}(\mathcal{B},r)=\begin{cases}\underline{0}, & \text{if } \mathcal{B}=\underline{0},\\ \underline{0.7}, & \text{if } \underline{0}\neq\mathcal{B}\leq\underline{0.4},\ \ 0<r\leq\frac{1}{2},\\ \underline{1}, & \text{otherwise},\end{cases}$$

$$T_{\tau_1}(\mathcal{B},r)=\begin{cases}\underline{0}, & \text{if } \mathcal{B}=\underline{0},\\ \underline{1}, & \text{otherwise},\end{cases}$$

By Theorem 3.1(3), the identity mapping $id_X:(X,\tau_1,\mathcal{I}_1)\to(Y,\tau_2,\mathcal{I}_2)$ *is* ***FS***$\theta\mathcal{I}$*-continuous but* f *it is not* ***F****-strongly continuous because,* $T_{\tau_1}(\mathcal{B},r)\nleq C_{\tau_2}(\mathcal{B},r)$.

Example 3.3. *Define* $\tau_1,\tau_2,\mathcal{I}_1,\mathcal{I}_2:I^X\to I$ *as follows:*

$$\tau_1(\mathcal{B})=\begin{cases}1, & \text{if } \mathcal{B}\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if } \mathcal{B}=\underline{0.4},\\ 0, & \text{otherwise},\end{cases}\qquad \tau_2(\mathcal{B})=\begin{cases}1, & \text{if } \mathcal{B}\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if } \mathcal{B}\in\{\underline{0.7},\underline{0.4}\},\\ 0, & \text{otherwise},\end{cases}$$

$$\mathcal{I}_1(\mathcal{B})=\begin{cases}1, & \text{if } \mathcal{B}=\underline{0},\\ \frac{2}{3}, & \text{if } \underline{0}<\mathcal{B}\leq\underline{0.2},\\ 0, & \text{otherwise},\end{cases}\qquad \mathcal{I}_2(\mathcal{B})=\begin{cases}1, & \text{if } \mathcal{B}=\underline{0},\\ \frac{2}{3}, & \text{if } \underline{0}<\mathcal{B}\leq\underline{0.4},\\ 0, & \text{otherwise}.\end{cases}$$

From Theorems 2.1 (4), we obtain $C_{\delta\mathcal{I}\tau},C_{\theta\mathcal{I}\tau}:I^X\times I_0\to I^X$ *as follows:*

$$C_{\tau_2}(\mathcal{B},r)=\begin{cases}\underline{0}, & \text{if } \mathcal{B}=\underline{0},\\ \underline{0.3}, & \text{if } \underline{0}\neq\mathcal{B}\leq\underline{0.3},\ \ 0<r\leq\frac{1}{2},\\ \underline{0.6}, & \text{if } \underline{0.3}<\mathcal{B}\leq\underline{0.6},\ \ 0<r\leq\frac{1}{2},\\ \underline{1}, & \text{otherwise},\end{cases}$$

$$(C_{\theta\mathcal{I}_1\tau_1}=C_{\delta\mathcal{I}_2\tau_2}=C_{\delta\mathcal{I}_1\tau_1})(\mathcal{B},r)=\begin{cases}\underline{0}, & \text{if } \mathcal{B}=\underline{0},\\ \underline{0.6}, & \text{if } \underline{0}\neq\mathcal{B}\leq\underline{0.6},\ \ 0<r\leq\frac{1}{2},\\ \underline{1}, & \text{otherwise},\end{cases}$$

By Theorem 3.2(3), the identity mapping $id_X:(X,\tau_1,\mathcal{I}_1)\to(Y,\tau_2,\mathcal{I}_2)$ *is* ***F***$\delta\mathcal{I}$*-continuous but it is not* ***FS***$\theta\mathcal{I}$*-continuous because,* $C_{\theta\mathcal{I}_1\tau_1}(\mathcal{B},r)\nleq C_{\tau_2}(\mathcal{B},r)$.

Definition 3.2. *Let* $(X,\tau,\mathcal{I})$ *be a fits,* $\mathcal{A},\mathcal{B}\in I^X$ *and* $r\in I_0$. *Then* X *is called fuzzy* $\mathcal{I}$*-semiregular (for short, F*$\mathcal{I}$*S-regular) if for each* $\mathcal{A}\in Q_\tau(x_t,r)$, *there exists* $\mathcal{B}\in Q_\tau(x_t,r)$ *such that* $int_\tau(Cl^\star(\mathcal{B},r),r)\leq\mathcal{A}$.

Theorem 3.6. *Let* $f:(X,\tau,\mathcal{I}_1)\to(Y,\eta,\mathcal{I}_2)$ *be a mapping. Then the following statements are hold:*

(1) *If* Y *isF*$\mathcal{I}$*S-regular and* f *is* ***F***$\delta\mathcal{I}$*-continuous, then* f *is F-continuous.*

(2) *If X is $F\mathcal{I}S$-regular and f is $\mathbf{FA}\mathcal{I}$-continuous, then f is $\mathbf{F}\delta\mathcal{I}$-continuous.*
(3) *If Y is fuzzy almost $\mathcal{I}$-regular and f is $\mathbf{F}\theta\mathcal{I}$-continuous, then f is $\mathbf{F}\delta\mathcal{I}$-continuous.*
(4) *If X is fuzzy almost $\mathcal{I}$-regular and f is $\mathbf{F}\delta\mathcal{I}$-continuous, then f is $\mathbf{FS}\theta\mathcal{I}$-continuous.*

Proof. (1) Let $\eta(\mathcal{A}) \geq r$ for each $f(x)_t \in \mathcal{A}$. Then, $\mathcal{A} \in Q_\eta(f(x_t), r)$. Since $(Y, \eta, \mathcal{I}_2)$ is fuzzy $\mathcal{I}$-semiregular, there exists $\mathcal{D} \in Q_\eta(f(x_t), r)$ with $int_\eta(Cl^\star(\mathcal{D}, r), r) \leq \mathcal{A}$. By $\mathbf{F}\delta\mathcal{I}$-continuity of f, there exists $\mathcal{B} \in Q_\tau(x_t, r)$ such that $f(int_\tau(Cl^\star(\mathcal{B}, r), r)) \leq int_\eta(Cl^\star(\mathcal{D}, r), r)$. Since $\tau(\mathcal{B}) \geq r$,

$$f(\mathcal{B}) \leq f(int_\tau(Cl^\star(\mathcal{B}, r), r)) \leq int_\eta(Cl^\star(\mathcal{D}, r), r) \leq \mathcal{A}.$$

Thus, $f(\mathcal{B}) \leq \mathcal{A}$ and hence f is F-continuous.

(2-4) are similar. □

Lemma 3.1. *Let If $(X, \tau_1, \mathcal{I}_1)$, $(Y, \tau_2, \mathcal{I}_2)$ and $(Z, \tau_3, \mathcal{I}_3)$ be fits's. Let $f : X \to Y$ and $g : Y \to Z$ be a mappings. If f is $\mathbf{FS}\theta\mathcal{I}$-continuous and g is $\mathbf{FA}\mathcal{I}$-continuous, then $g \circ f$ is $\mathbf{F}\delta\mathcal{I}$-continuity.*

Proof. Obvious. □

Definition 3.3. *Let $(X, \tau, \mathcal{I})$ be a fits. Then,*

(1) *the pair $(\mathcal{A}, \mathcal{B})$ is said to be fuzzy ideal r-θ-separation relative to X iff $\mathcal{A}\bar{q}\mathcal{B}$, $\mathcal{A}\bar{q}\Theta_{\tau_\mathcal{I}}(\mathcal{B}, r)$ and $\Theta_{\tau_\mathcal{I}}(\mathcal{A}, r)\bar{q}\mathcal{B}$.*

   A fuzzy set $\mathcal{D} \in I^X$ is said to be fuzzy ideal r-θ-connected iff there do not exist two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ in X such that $(\mathcal{A}, \mathcal{B})$ is fuzzy ideal r-θ-separation relative to X and $\mathcal{D} = \mathcal{A} \vee \mathcal{B}$.

(2) *The pair $(\mathcal{A}, \mathcal{B})$ is said to be fuzzy ideal r-δ-separation relative to X iff $\mathcal{A}\bar{q}\mathcal{B}$, $\mathcal{A}\bar{q}\Delta_{\tau_\mathcal{I}}(\mathcal{B}, r)$ and $\Delta_{\tau_\mathcal{I}}(\mathcal{A}, r)\bar{q}\mathcal{B}$.*

   A fuzzy set $\mathcal{D} \in I^X$ is said to be fuzzy ideal r-δ-connected iff there do not exist two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ in X such that $(\mathcal{A}, \mathcal{B})$ is fuzzy ideal r-δ-separation relative to X and $\mathcal{D} = \mathcal{A} \vee \mathcal{B}$.

Lemma 3.2. *It is clear that every fuzzy ideal r-δ-connected is fuzzy ideal r-θ-connected.*

Example 3.4. *Define $\tau, \mathcal{I} : I^X \to I$ as follows:*

$$\tau_1(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if } \mathcal{B} \in \{\underline{0.7}, \underline{0.4}\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}_1(\mathcal{B}) = \begin{cases} 1, & \text{if } \mathcal{B} = \underline{0}, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{B} \leq \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

From Theorem 3.2 (2) $C_{\theta\mathcal{I}\tau} : I^X \times I_0 \to I^X$ as follows:

$$\Delta_{\tau_\mathcal{I}}(\mathcal{B}, r) = \begin{cases} \underline{0}, & \text{if } \mathcal{B} = \underline{0}, \\ \underline{0.6}, & \text{if } \underline{0} \neq \mathcal{B} \leq \underline{0.6}, \quad 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

$$C_{\theta\mathcal{I}\tau}(\mathcal{B}, r) = \begin{cases} \underline{0}, & \text{if } \mathcal{B} = \underline{0}, \\ \underline{0.6}, & \text{if } \underline{0} \neq \mathcal{B} \leq \underline{0.3}, \quad 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

Then by Definition 3.4, we have

$$\Theta_{\tau_{\mathcal{I}}}(\mathcal{B},r)=\begin{cases}\underline{0}, & \text{if } \mathcal{B}=\underline{0},\\ \underline{1}, & \text{otherwise,}\end{cases}$$

For $\underline{0.4}=\underline{0.3}\vee\underline{0.4}$ *we have* $\underline{0.3}\underline{q}\underline{0.4}$, $\underline{0.6}=\Delta_{\tau_{\mathcal{I}}}(\underline{0.3},\frac{1}{2})\underline{q}\underline{0.4}$ *and* $\underline{0.3}\underline{q}\Delta_{\tau_{\mathcal{I}}}(\underline{0.4},\frac{1}{2})=\underline{0.6}$. *Hence,* $(\underline{0.3},\underline{0.4})$ *is fuzzy ideal* $\frac{1}{2}-\delta$*-separation and* $\underline{0.4}$ *is not fuzzy ideal* $\frac{1}{2}-\delta$*-connected.*

For any representation $\underline{0.4}=\mathcal{A}\vee C$*, where* $\mathcal{A}$ *and* C *are non-empty,* $\Theta_{\tau_{\mathcal{I}}}(\mathcal{B},r)=\underline{1}$ *for* $\mathcal{B}\in\{\mathcal{A},C\}$*. Thus,* $\underline{0.4}$ *is fuzzy ideal* $\frac{1}{2}-\theta$*-connected.*

Lemma 3.3. *If* $(X,\tau,\mathcal{I})$ *is fuzzy almost* $\mathcal{I}$*-regular, then the concepts fuzzy ideal r-*δ*-connectedness and fuzzy ideal r-*θ*-connectedness are equivalent.*

Proof. The proof is easily from Theorem 4.3.9. □

Theorem 3.7. *Let Let* $f:(X,\tau,\mathcal{I}_1)\to(Y,\eta,\mathcal{I}_2)$ *be a mapping. Then the following statements are hold:*

(1) *If* $\mathcal{A}$ *is fuzzy ideal r-*θ*-connected and* f *is* $\boldsymbol{F}\theta\mathcal{I}$*-continuous, then* $f(\mathcal{A})$ *is fuzzy ideal r-*θ*-connected.*
(2) *If* $\mathcal{A}$ *is fuzzy ideal r-*δ*-connected and* f *is* $\boldsymbol{F}\delta\mathcal{I}$*-continuous, then* $f(\mathcal{A})$ *is fuzzy ideal r-*δ*-connected.*
(3) *If* $\mathcal{A}$ *is fuzzy ideal r-*δ*-connected and* f *is* $\boldsymbol{FS}\theta\mathcal{I}$*-continuous, then* $f(\mathcal{A})$ *is fuzzy ideal r-*δ*-connected.*

Proof. Let $(\mathcal{A},\mathcal{B})$ be fuzzy ideal r-θ-separation relative to Y such that $f(\mathcal{D})=\mathcal{A}\vee\mathcal{B}$. Suppose that $C_1=\mathcal{D}\wedge f^{-1}(\mathcal{A})$ and $C_2=\mathcal{D}\wedge f^{-1}(\mathcal{B})$. Then, $\mathcal{D}=C_1\vee C_2$. To arrive at a contradiction it suffices to show that (C_1,C_2) is fuzzy ideal r-θ-separation relative to X. Now, since $f(\mathcal{D})\vee\mathcal{A}\neq\underline{0}$ (otherwise $\mathcal{A}=\underline{0}$), there exists $y\in Y$ such that $f(\mathcal{D})(y)>0$. Then, for some $x_t\in P_t(X)$, $\mathcal{D}(x)>0$. Also, $f^{-1}(\mathcal{A})(x)=\mathcal{A}(f(x))>0$. Thus, $C_1=\mathcal{D}\vee f^{-1}(\mathcal{A})\neq\underline{0}$. Similarly $C_2=\mathcal{D}\vee f^{-1}(\mathcal{B})\neq\underline{0}$. Now $C_1\leq f^{-1}(\mathcal{A})$, by Theorem 3.5(1),

$$C_{\delta\mathcal{I}_1\tau}(C_1,r)\leq f^{-1}(C_{\delta\mathcal{I}_2\eta}(\mathcal{A}),r)=f^{-1}(\Delta_{\eta\mathcal{I}_2}(\mathcal{A},r).$$

Again, $\Delta_{\eta\mathcal{I}_2}(\mathcal{A},r)\bar{q}f^{-1}(\mathcal{B})$ implies that $C_{\delta\mathcal{I}_1\tau}(C_1,r)\bar{q}f^{-1}(\mathcal{B})$. But $C_2\leq f^{-1}(C_1)$. So, $C_2\bar{q}C_{\delta\mathcal{I}_1\tau}(C_1,r)$, thus (C_1,C_2) is fuzzy ideal r-θ-separation relative to X.

The proof (2) and (3) it is clear. □

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