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Estimations with Step-Stress Partially Accelerated Life Tests for Ailamujia Distribution under Type-I Censored Data

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ABSTRACT. This paper addresses the problem of estimating parameters in partial accelerated life tests following the Ailamujia distribution under Type-I censoring, employing both the maximum likelihood approach and the least squares method. The assessment of estimation methods involves a comprehensive simulation study, complemented by the analysis of a real dataset for illustrative purposes. The findings reveal the least square estimation method outperforms maximum likelihood estimation, considering biases and mean square errors.

1. Introduction

Partial accelerated life tests (PALTs) are crucial methods employed to collect sufficient failure data for test items in a more time-efficient manner. In fact, PALT involves running some test items under normal usage conditions while subjecting others to accelerated conditions. This approach is a reasonable way to estimate the acceleration factor.

Stress can be applied using various methods, including constant-stress and step-stress approaches. In constant-stress PALT (CSPALT), a set of test items is subjected to testing either under normal usage conditions or at accelerated conditions, continuing until either a failure occurs or the test is stopped. In other words, each item experiences a consistent stress level until

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the test is halted. On the other hand, in step-stress PALT (SSPALT), a group of test items initially undergo testing under normal usage conditions. If no failures occur within a specified timeframe, they are then subjected to accelerated conditions until a predetermined number of failures transpire or a pre-set time limit is reached. See Nelson [1].

Attia et al. [2] examined the use of the maximum likelihood method to estimate the acceleration factor and Weibull distribution parameters in SSPALT when dealing with Type-I censoring. Abdel-Ghani [3] explored both maximum likelihood and Bayesian methods for estimating Weibull distribution parameters and the acceleration factor in both SSPALT and CSPALT, considering both Type-I and Type-II censored data. Abdel-Ghaly et al. [4] and Abdel-Ghaly et al. [5] conducted research on both estimation and optimal design issues related to the Pareto distribution in SSPALT, considering both Type-I and Type-II censoring. Additionally, Abdel-Ghani [6] focused on the estimation problem of log-logistic distribution parameters in the context of SSPALT. Ismail [7] calculated MLEs for the acceleration factor and Weibull distribution parameters, utilizing adaptive Type-II progressively hybrid censoring (AT-II PHC) data. Ismail [8] applied both likelihood and Bayesian methods to estimate the parameter of the Pareto distribution with Type-I censoring data in the context of a step-stress life test model. Recently, Fawzy and Athar [9] estimated the parameters of the Ailamujia distribution within a PALT framework, focusing on Progressive Type-II Censored Data.

1.1 Tampered Random Variable Model

Goel [10] introduced this model and an associated application. We shall denote the lifespan of an item tested under standard conditions using the random variable T. If the item doesn't fail by a certain time τ, it's subjected to higher stress until it fails. Degroot and Goel [11] have assumed this stress switch multiplies the remaining lifespan of the item by an unknown factor $\beta^{-1} > 0$. β is typically constant because we use only one higher stress level, and it usually shortens the life of the item.

In describing the model for this PALT, we use the variable X to represent the total lifespan of a test item, specified as:

$$X = \begin{cases} T, & \text{if} \quad T \leq \tau \\ \tau + \beta^{-1}(T - \tau), & \text{if} \quad T > \tau \end{cases}. \tag{1.1}$$

Switching to the higher stress level is a form of tampering with the ordinary life test, so X is termed a tampered random variable (TRV), τ is referred to as the tampering point, and β is known as the tampering coefficient.

1.2 Type-I Censored Samples

When terminating a life-testing experiment at a predetermined time τ and censoring the lifetimes of units exceeding τ, the resulting observed failure data constitute a Type-I censored sample. While having control over the experiment duration is advantageous, setting it in advance may lead to a drawback: the potential for very few or even no failures occurring before time τ. Additionally, another challenge of this censoring type is the randomness in the number of failures, which contributes to the complexity of maximum likelihood estimation (MLE) for parameters.

For additional information regarding Type-I censoring and the related inferential issues, one can consult references such as Lawless [12], Cohen and Whitten [13], and Cohen [14].

1.3 The Ailamujia Distribution

The Ailamujia distribution, introduced by Lv et al. [15], is a novel model designed for various engineering applications. It is a univariate lifetime distribution that relies on a single parameter, denoted as α, where α is greater than zero.

Multiple authors have worked on the Ailamujia distribution. Pan et al. [16] examined this distribution, focusing on interval estimation and hypothesis testing, particularly in cases with small sample sizes. Long [17] employed Bayesian methods to estimate the parameters of the Ailamujia Distribution under Type-II censoring, utilizing three different priors derived from missing data. Li [18] evaluated the minimax estimation of the Ailamujia model's parameter, using three distinct loss functions and a non-informative prior. Furthermore, Rather et al. [19] presented a size-biased version of the Ailamujia distribution and employed it to analyze data in engineering and medical science. Aijaz et al. [20] introduced the inverse analogue of the Ailamujia distribution, providing its statistical properties and applications.

This distribution is characterized by its probability density function (PDF), which is defined as follows:

$$g(t) = 4t\alpha^2 e^{-2\alpha t}, \qquad t \geq 0, \qquad \alpha > 0. \tag{1.2}$$

The corresponding cumulative distribution function (CDF) is formulated as

$$G(t) = 1 - (1 + 2\alpha t)e^{-2\alpha t}, \qquad t \geq 0, \qquad \alpha > 0. \quad (1.3)$$

The reliability function (RF) for a random variable T is represented as $R(t)$, and it can be derived as follows:

$$R(t) = 1 - G(t),$$

by employing Eq. (1.3), we obtain

$$R(t) = (1 + 2\alpha t)e^{-2\alpha t}, \quad t \geq 0, \qquad \alpha > 0. \qquad (1.4)$$

The hazard rate function (HRF), indicated as $h(t)$, for a random variable T can be determined as follows:

$$h(t) = \frac{g(t)}{\mathrm{R(t)}},$$

utilizing Eq.s (1.2) and (1.4), we get

$$h(t) = \frac{4t\alpha^2}{1 + 2\alpha t}, \qquad t \geq 0, \qquad \alpha > 0. \qquad (1.5)$$

A distribution RF indicates the probability of surviving beyond a given time point, which consistently decreases as time progresses. Additionally, the HRF in a distribution represents the instantaneous failure rate at a specific time point.

The Ailamujia distribution displays a right-skewed and bell-shaped PDF, accompanied by an increasing HRF. This distribution is a flexible probability distribution used for modeling repair time and ensuring the distribution delay time. See Jan et al. [21].

Utilizing the R programming language, The PDF, CDF, RF and HRF plots have been created using various values of the parameter α, as depicted in Fig. 1, Fig. 2, Fig. 3 and Fig. 4, respectively.

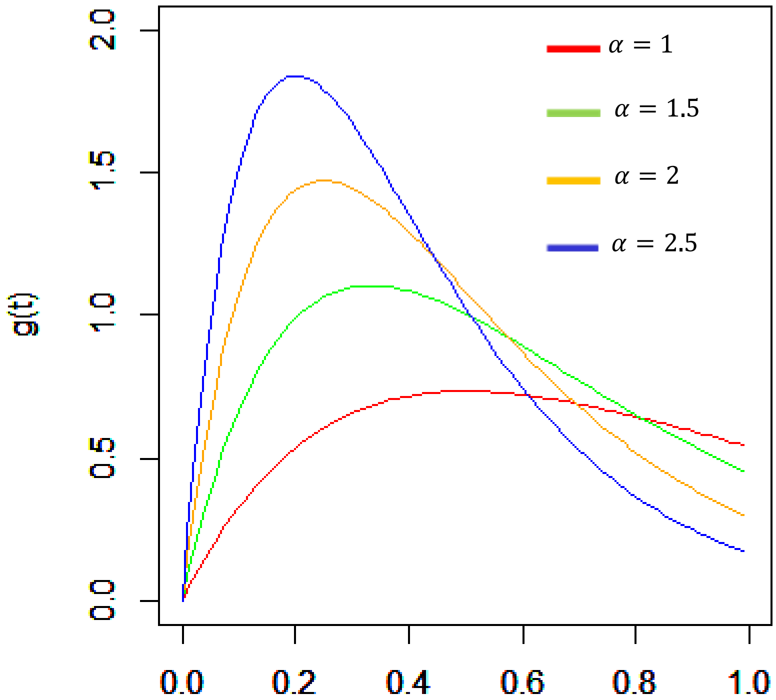

Fig. 1: PDF Plot of the Ailamujia distribution considering different values of α.

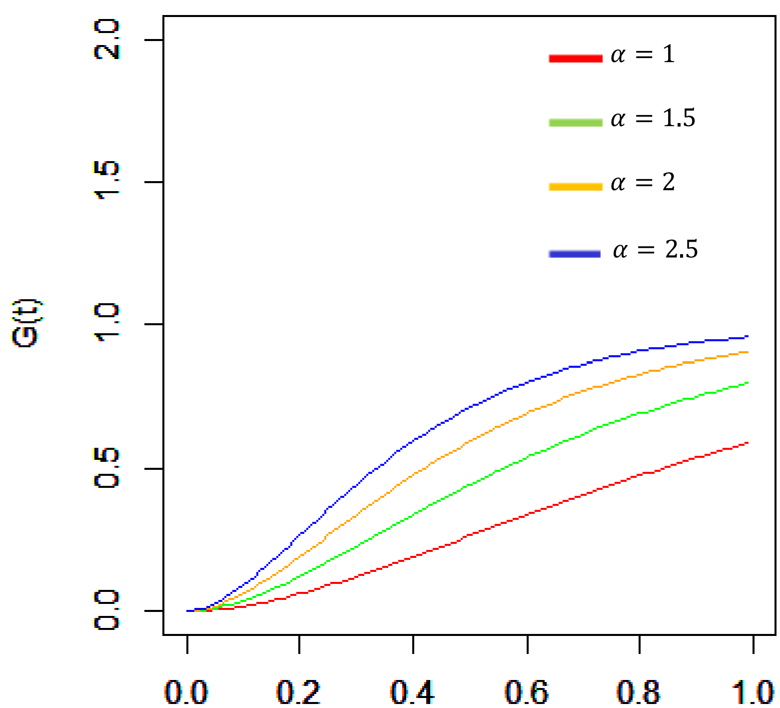

Fig. 2: CDF Plot of the Ailamujia distribution considering different values of α.

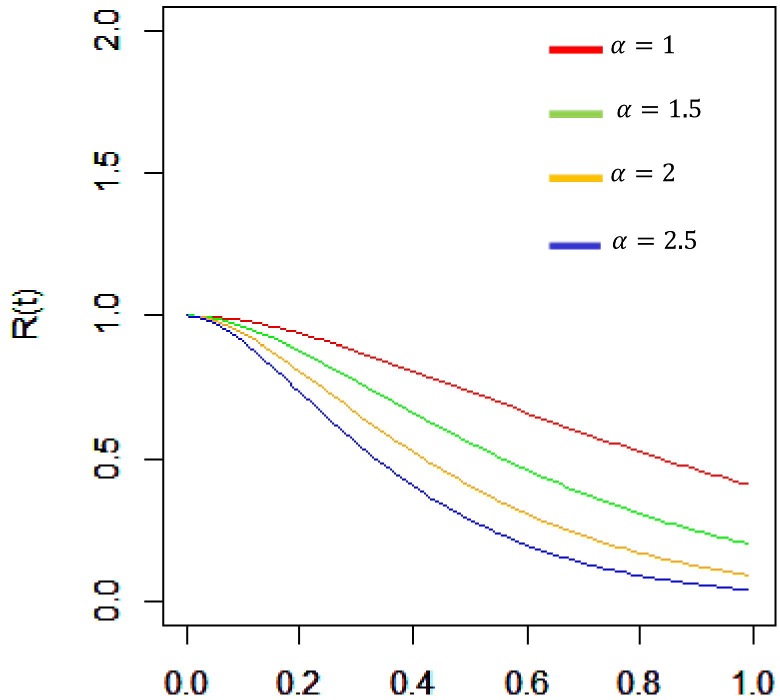

Fig. 3: RF Plot of the Ailamujia distribution considering different values of α.

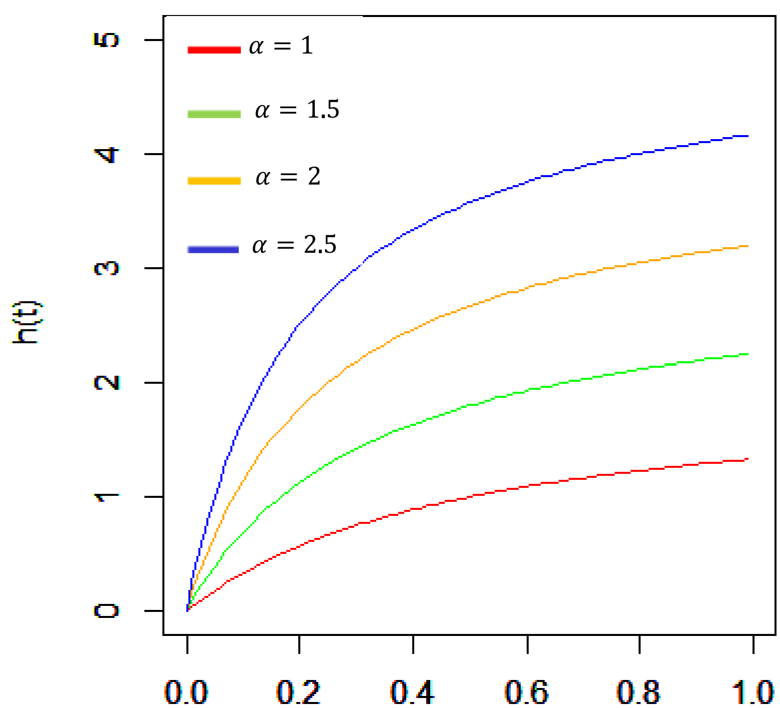

Fig. 4: HRF Plot of the Ailamujia distribution considering different values of α.

In this paper, we explore a TRV model within the framework of Type-I censoring, with a specific emphasis on Ailamujia's lifetimes. Section 2 provides a comprehensive overview of the model we are investigating. Moving on to Section 3, we derive the MLEs for the unknown parameters and assess confidence intervals (CIs) using the Fisher information matrix. In Section 4, we employ the least squares (LS) technique to obtain estimates for the model parameters. Section 5 presents a simulation study that compares various estimation methods and a real data set is analyzed for illustration. Lastly, the paper concludes in Section 6.

2. Model Description

We assume the failure time data is derived from a TRV model. We are examining a SSPALT model with stress levels S_1 and S_2, and incorporating Type-I censoring. Moreover, we presume that the lifetime distribution at stress levels S_1 and S_2 follows an Ailamujia distribution. Initially, we place n identical units onto a life test, and each of these units is subjected to an initial stress level, S_1. The experiment continues until a fixed time, τ_1. At τ_1, the stress level is switched to S_2, and the experiment carries on until another predetermined time, τ_2. Any lifetimes of units that exceed τ_2 are considered censored data.

We introduce two random variables: N_1, which represents the number of units that fail before τ_1 under normal operating conditions, and N_2, which represents the number of units that fail between τ_1 and τ_2 under accelerated usage conditions. If N_1 equals n, it means that the experiment ends. Otherwise, it is extended until τ_2 is reached.

The data that has been observed follows the pattern of:

$$\{t_{1:n} < \cdots < t_{N_1:n} < \tau_1 \leq t_{N_1+1:n} < \cdots < t_{N_1+N_2:n} \leq \tau_2\}. \qquad (2.1)$$

Our model is based on the following fundamental assumptions:

1) The test relies on SSPALT and involves only two stress levels: S_1 for normal operating conditions and S_2 for accelerated conditions, with the condition that S_1 is less than S_2.
2) The failures of the test items at both stress levels S_1 and S_2 follow the Ailamujia distribution.
3) The lifetime of the examined item in SSPALT adheres to a TRV model.

4) During normal operating conditions, all n units undergo testing. However, if any unit does not fail by the specified time τ_1, it is then subjected to accelerated conditions for further testing.

Through the utilization of Eq. (1.1) to describe the model for SSPALT, the total lifespan, denoted as X, of an item to be:

$$X = \begin{cases} T, & \text{if} \quad 0 < T < \tau_1 \\ \tau_1 + \beta^{-1}(T - \tau_1), & \text{if} \quad \tau_1 \leq T < \tau_2 \end{cases}. \tag{2.2}$$

Assuming $X = \tau_1 + \frac{1}{\beta}(T - \tau_1)$, once we solve for T, we arrive at the expression $T = \tau_1 + \beta(t - \tau_1)$. Subsequently, we can represent the PDF and the CDF for X as follows:

$$g(t) = \begin{cases} g_1(t) = 4\alpha^2 t e^{-2\alpha t}, 0 < t < \tau_1 \\ g_2(t) = 4\alpha^2 \beta\big(\tau_1 + \beta(t - \tau_1)\big) e^{-2\alpha(\tau_1 + \beta(t - \tau_1))}, \tau_1 \leq t < \tau_2 \end{cases}, \tag{2.3}$$

$$G(t) = \begin{cases} G_1(t) = 1 - (1 + 2\alpha t) e^{-2\alpha t}, 0 < t < \tau_1 \\ G_2(t) = 1 - \Big(1 + 2\alpha\big(\tau_1 + \beta(t - \tau_1)\big)\Big) e^{-2\alpha(\tau_1 + \beta(t - \tau_1))}, \tau_1 \leq t < \tau_2 \end{cases}. \tag{2.4}$$

3. Maximum Likelihood Estimation

Given the Type-I censored data described in (2.1), we can derive the likelihood function and subsequently determine the MLEs for the unknown parameters α and β. The likelihood function for the censored data in (2.1) can be expressed as follows:

$$L(\alpha, \beta \mid t) = \frac{n!}{(n - N)!} \left\{ \prod_{i=1}^{N} g(t_{i:n}) \{1 - G(\tau_2)\}^{n-N} \right\},$$

$$0 < t_{1:n} < .. < t_{N:n} < \tau_2, \tag{3.1}$$

see Arnold et al. [22]. In this context, N represents the sum of N_1 and N_2, while $t = (t_{1:n}, \dots, t_{N_1:n}, t_{N_1+1:n}, \dots, t_{N:n})$ stands for the collection of recorded Type-I censored data.

Clearly, the MLE for α is not attainable when N_1 equals 0, and the MLE for β is undefined when N_1 equals n. MLEs for both α and β are only viable when N_1 is equal to or greater than 1, and N_2 is equal to or greater than 1.

When the condition $1 \leq N_1 \leq N - 1$ is satisfied, the likelihood function in Eq. (3.1) transforms to

$$L(\alpha,\beta \mid t)=\frac{n!}{(n-N)!}\left\{\prod_{i=1}^{N_1} g_1(t_{i:n})\right\}\left\{\prod_{i=N_1+1}^{N} g_2(t_{i:n})\right\}\times\{1-G_2(\tau_2)\}^{n-N},$$

$$0<\mathrm{t}_{1:\mathrm{n}}<\cdots<\mathrm{t}_{\mathrm{N}_1:\mathrm{n}}<\tau_1\le \mathrm{t}_{\mathrm{N}_1+1:\mathrm{n}}<\cdots<\mathrm{t}_{\mathrm{N}:\mathrm{n}}<\tau_2, \qquad (3.2)$$

which can be written as:

$$L(\alpha,\beta \mid t)\propto\prod_{i=1}^{N_1}\left\{4\alpha^2 t_{i:n}e^{-2\alpha t_{i:n}}\right\}\times\prod_{i=N_1+1}^{N}\left\{4\alpha^2\beta\big(\tau_1+\beta(t_{i:n}-\tau_1)\big)e^{-2\alpha\left(\tau_1+\beta(t_{i:n}-\tau_1)\right)}\right\}$$

$$\times\left[\Big(1+2\alpha\big(\tau_1+\beta(\tau_2-\tau_1)\big)\Big)e^{-2\alpha\left(\tau_1+\beta(\tau_2-\tau_1)\right)}\right]^{n-N}.$$

The likelihood function can be simplified as demonstrated below

$$L(\alpha,\beta \mid t)\propto\alpha^{2N}\beta^{N_2}\prod_{i=1}^{N_1}t_{i:n}\prod_{i=N_1+1}^{N}[\tau_1+\beta(t_{i:n}-\tau_1)]$$

$$\times\ e^{-2\alpha\sum_{i=1}^{N_1}t_{i:n}-2\alpha\sum_{i=n_1+1}^{N}[\tau_1+\beta(t_{i:n}-\tau_1)]-2\alpha[\tau_1+\beta(\tau_2-\tau_1)](n-N)}$$

$$\times[1+2\alpha(\tau_1+\beta(\tau_2-\tau_1))]^{n-N}. \qquad (3.3)$$

The expression for the log-likelihood function can be represented as follows:

$$l(\alpha,\beta \mid t)\propto 2Nlog\alpha+N_2log\beta+\sum_{i=1}^{N_1}logt_{i:n}+\sum_{i=N_1+1}^{N}log\ (\tau_1+\beta(t_{i:n}-\tau_1))$$

$$-2\alpha\left[\sum_{i=1}^{N_1}t_{i:n}+\sum_{i=N_1+1}^{N}(\tau_1+\beta(t_{i:n}-\tau_1)+(n-N)\big(\tau_1+\beta(\tau_2-\tau_1)\big)\right]$$

$$+(n-N)log\ (1\ +2\alpha(\tau_1+\beta(\tau_2-\tau_1))). \qquad (3.4)$$

By taking the derivatives of the log-likelihood function with respect to α and β as given in Eq. (3.4), we derive the subsequent set of likelihood equations. Solving these equations is necessary to determine the MLEs for the parameters α and β .

$$\frac{\partial l(\alpha,\beta \mid t)}{\partial\alpha}=\frac{2N}{\alpha}-2\left[\sum_{i=1}^{N_1}t_{i:n}+\sum_{i=N_1+1}^{N}(\tau_1+\beta(t_{i:n}-\tau_1))+(n-N)\big(\tau_1+\beta(\tau_2-\tau_1)\big)\right]$$

$$+\frac{2(n-N)\big(\tau_1+\beta(\tau_2-\tau_1)\big)}{1+2\alpha\big(\tau_1+\beta(\tau_2-\tau_1)\big)}=0. \tag{3.5}$$

$$\frac{\partial l(\alpha,\beta \mid t)}{\partial\beta}=\frac{N_2}{\beta}+\sum_{i=N_1+1}^{N}\frac{t_{i:n}-\tau_1}{\tau_1+\beta(t_{i:n}-\tau_1)}+\frac{2\alpha(n-N)(\tau_2-\tau_1)}{1+2\alpha\big(\tau_1+\beta(\tau_2-\tau_1)\big)}$$

$$-2\alpha\left[\sum_{i=N_1+1}^{N}(t_{i:n}-\tau_1)+(n-N)(\tau_2-\tau_1)\right]=0. \tag{3.6}$$

The process of estimation, involving Eq.s (3.5) and (3.6), cannot be solved analytically. As a result, these equations can be solved simultaneously using a numerical approach like the Newton-Raphson method or other similar techniques.

Theorem 3.1 If we have observed Ailamujia's lifetimes to failure under the SSPALT model with Type-I censoring, then the MLEs for the parameters α and β both exist and are unique.

Proof: First, we prove that MLE$(\alpha)=\hat{\alpha}$ exists uniquely. By employing Eq. (3.5), we obtain

$$\frac{2N}{\alpha}+\frac{2(n-N)\big(\tau_1+\beta(\tau_2-\tau_1)\big)}{1+2\alpha\big(\tau_1+\beta(\tau_2-\tau_1)\big)}$$
$$=2\left[\sum_{i=1}^{N_1}t_{i:n}+\sum_{i=N_1+1}^{N}(\tau_1+\beta(t_{i:n}-\tau_1))+(n-N)\big(\tau_1+\beta(\tau_2-\tau_1)\big)\right],$$

or equivalently

$$H(\alpha)=A,$$

where $H(\alpha)=\frac{d}{\alpha}+\frac{e}{1+\alpha f}$, such that $d=2N$, $e=2(n-N)\big(\tau_1+\beta(\tau_2-\tau_1)\big)$ and $f=2\big(\tau_1+\beta(\tau_2-\tau_1)\big)$.

Also, $A=2\big[\sum_{i=1}^{N_1}t_{i:n}+\sum_{i=N_1+1}^{N}(\tau_1+\beta(t_{i:n}-\tau_1))+(n-N)\big(\tau_1+\beta(\tau_2-\tau_1)\big)\big]$, it's worth noting that A is a positive constant. Moreover, it can be inferred that:

$$\lim_{\alpha\to0^+}H(\alpha)=\infty,$$

$$\lim_{\alpha\to\infty}H(\alpha)=0.$$

See Amleh and Raqab [23]. The diagram below illustrates the coupling behaviour of $H(\alpha)$, where it intersects with $y = A$ at a single point.

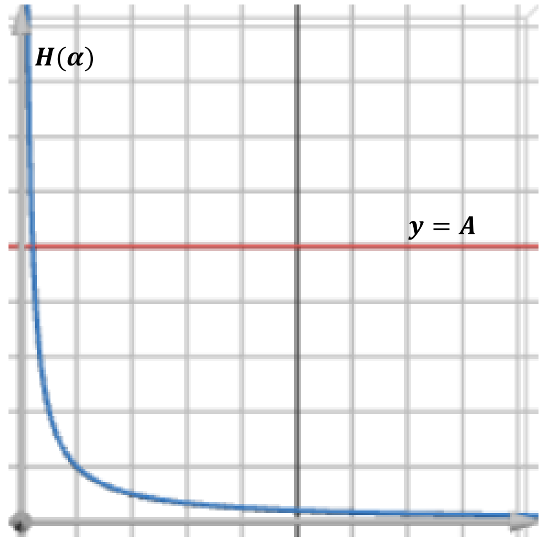

Fig. 5: The plot of $H(\alpha) = A$ shows a single intersection point.

Therefore, $\mathrm{MLE}(\alpha) = \hat{\alpha}$ exists uniquely.

Now, we prove that $\mathrm{MLE}(B) = \hat{B}$ exists uniquely. By utilizing Eq. (3.6) and following the identical procedure as before, we find that:

$$\frac{N_2}{\beta} + \sum_{i=N_1+1}^{N} \frac{t_{i:n} - \tau_1}{\tau_1 + \beta(t_{i:n} - \tau_1)} + \frac{2\alpha(n-N)(\tau_2 - \tau_1)}{1 + 2\alpha\big(\tau_1 + \beta(\tau_2 - \tau_1)\big)}$$
$$= 2\alpha\left[\sum_{i=N_1+1}^{N} (t_{i:n} - \tau_1) + (n-N)(\tau_2 - \tau_1)\right],$$

or equivalently

$$K(\beta) = C,$$

where $K(\beta)$ is defined as $K(\beta) = \frac{N_2}{\beta} + \sum_{i=N_1+1}^{N} \frac{t_{i:n}-\tau_1}{\tau_1+\beta(t_{i:n}-\tau_1)} + \frac{2\alpha(n-N)(\tau_2-\tau_1)}{1+2\alpha\left(\tau_1+\beta(\tau_2-\tau_1)\right)}$, and C is given by:

$$C = 2\alpha\left[\sum_{i=N_1+1}^{N} (t_{i:n} - \tau_1) + (n-N)(\tau_2 - \tau_1)\right],$$

it is clear that C is a positive constant. Furthermore, it can be concluded that:

$$\lim_{\beta \to 0^+} K(\beta) = \infty,$$

$$\lim_{\beta \to \infty} H(\beta) = 0.$$

The diagram below illustrates the coupling behaviour of $K(\beta)$, where it intersects with $y = C$ at a single point.

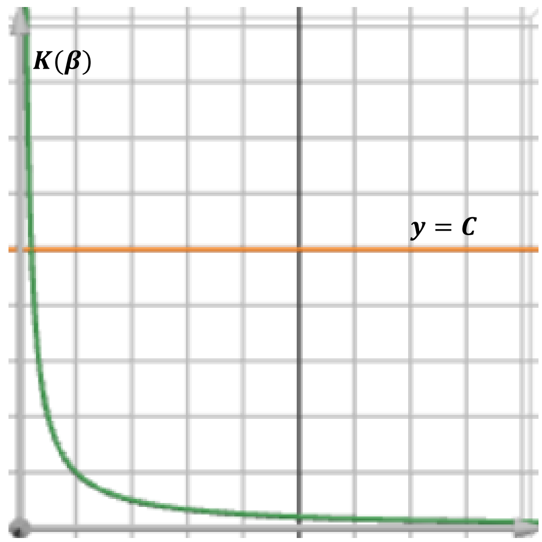

Fig. 6: The plot of $K(\beta) = C$ shows a single intersection point.

Therefore, MLE$(\beta) = \hat{\beta}$ exists uniquely.

The desired conclusion is obtained by combining the results on α and β.■

The quantile function for the Ailamujia distribution is important to generate data that will be used in simulation. In fact, it cannot be obtained in closed form. However, it can handled by taking the inverse of the function $G(t)$ defined in Eq. (1.3). To proceed, we use the following well-known theorem in mathematical statistics.

Theorem 3.2 Probability Integral Transform.

1. If a random variable X is continuous with CDF $F(x)$, then the random variable $Y = F(X)$ follows a uniform distribution over $[0, 1]$, i.e., $Y \sim U(0, 1)$.

2. Conversely, if Y follows a uniform distribution over $[0, 1]$, i.e., $Y \sim U(0, 1)$, then the random variable $X = F^{-1}(Y)$ has a CDF $F(x)$.

The algorithm for generating the data and calculating the MLEs for the parameters α and β is carried out using the subsequent procedure.

First step: Create a set of randomly chosen values with a total of n elements, following a Uniform distribution $U(0,1)$. Then, derive the order statistics from these values:

$$U_{1:n} < U_{2:n} < \cdots < U_{n:n},$$

Second step: Determine the random variable N_1 for which

$$U_{N_1} < P(T \leq \tau_1) = G_1(\tau_1) \leq U_{\mathrm{N_1+1:n}},$$

where T symbolizes the time of failure, leading to the following :

$$U_{N_1} < 1-(1+2\alpha\tau_1)e^{-2\alpha\tau_1} \leq U_{N_1+1:n},$$

Third step: Create the required censored sample by solving the following two equations using the order statistics $U_{\mathrm{i:n}}$.

$$1+2\alpha t_i = (1-U_{\mathrm{i:n}})e^{2\alpha t_i}, i = 1,2,\dots,N_1 \tag{3.7}$$

$$1+2\alpha[\tau_1+\beta(t_i-\tau_1)] = (1-U_{\mathrm{i:n}})e^{2\alpha[\tau_1+\beta(t_i-\tau_1)]}, i = N_1+1,\dots,N \tag{3.8}$$

Fourth step: Calculate the MLEs for α and β using Eq.s (3.5) and (3.6), relying on the censored data $t_{1:n}, t_{2:n}, \dots, t_{N_1:n}, t_{N_1+1:n}, \dots, t_{N:n}$, as described in Eq.s (3.7) and (3.8). For more details, see Alkhalfan [24], Amleh and Raqab [25] and Amleh [26].

Within this section, we establish approximate CIs for the parameters α and β . This is accomplished by leveraging the asymptotic distributions of the MLEs. The asymptotic distribution of the MLEs for α and β can be expressed as follows:

$$\left[(\hat{\alpha}-\alpha),(\hat{\beta}-\beta)\right] \to N\left(0, I^{-1}(\alpha,\beta)\right),$$

where $I^{-1}(\alpha,\beta)$ signifies the variance-covariance matrix for the parameter $\delta = (\alpha,\beta)$. This approximation can be made using the components of the observed Fisher information matrix denoted as $I_{ij}(\alpha,\beta)$, where i and j take on values of 1 and 2. Specifically, we can use $I_{ij}(\hat{\alpha},\hat{\beta})$ for this estimation, where:

$$I_{ij}(\hat{\delta}) = \frac{\partial^2 l(\delta \mid t)}{\partial\delta_i \partial\delta_j} \mid_{\delta=\hat{\delta}}. \tag{3.9}$$

Hence, the asymptotic Fisher-information matrix I can be expressed as follows:

$$I=\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}, \tag{3.10}$$

the following equations can be used to represent the elements of the aforementioned matrix I:

$$I_{11}=\frac{\partial^2 l(\delta \mid t)}{\partial \alpha^2}=\frac{-2N}{\alpha^2}-\frac{4(n-N)\left(\tau_1+\beta(\tau_2-\tau_1)\right)^2}{\left(1+2\alpha\left(\tau_1+\beta(\tau_2-\tau_1)\right)\right)^2}. \tag{3.11}$$

$$I_{12}=\frac{\partial^2 l(\delta \mid t)}{\partial \alpha \partial \beta}=-2\left[\sum_{i=N_1+1}^{N}(t_i-\tau_1)+(n-N)(\tau_2-\tau_1)\right]$$
$$+\frac{2(n-N)(\tau_2-\tau_1)\left(1+2\alpha\left(\tau_1+\beta(\tau_2-\tau_1)\right)\right)-4\alpha(n-N)(\tau_2-\tau_1)\left(\tau_1+\beta(\tau_2-\tau_1)\right)}{\left(1+2\alpha\left(\tau_1+\beta(\tau_2-\tau_1)\right)\right)^2}. \tag{3.12}$$

$$I_{22}=\frac{\partial^2 l(\delta \mid t)}{\partial \beta^2}=\frac{-N_2}{\beta^2}-\sum_{i=N_1+1}^{N}\frac{(t_i-\tau_1)^2}{\left(\tau_1+\beta(t_i-\tau_1)\right)^2}-\frac{4\alpha^2(n-N)(\tau_2-\tau_1)^2}{\left(1+2\alpha\left(\tau_1+\beta(\tau_2-\tau_1)\right)\right)^2}. \tag{3.13}$$

You can formulate the two-sided CIs at the approximate confidence level of $100(1-\gamma)\%$ for α and β in the following manner:

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{V_{11}}, \tag{3.14}$$

$$\hat{\beta} \pm z_{\gamma/2}\sqrt{V_{22}}, \tag{3.15}$$

where $z_{\gamma/2}$ denotes the upper percentile at $100(\gamma/2)^{th}$ for the standard normal distribution, while V_{11} and V_{22} signify the elements along the main diagonal of the inverse of matrix I. See Amleh and Raqab [27].

4. Least Square Estimations

Swain et al. [28] proposed the LS method for parameter estimation in the Beta distribution. Utilizing SSPALT, this method is to be applied to derive estimations for the unknown parameter and the acceleration factor of the Ailamujia distribution in the presence of Type-I censored data. The least squares estimates (LSEs) for the unknown parameters α and β, referred to as $\hat{\alpha}_{Ls}$ and $\hat{\beta}_{Ls}$, are acquired by minimizing the subsequent function concerning α and β.

$$Ls=\sum_{i=1}^{N_1}\left[G\left(t_{i:N_1}\right)-\frac{i}{N_1+1}\right]^2+\sum_{j=1}^{N_2}\left[G\left(t_{j:N_2}\right)-\frac{j}{N_2+1}\right]^2, \tag{4.1}$$

the Eq. (4.1) can be written as

$$Ls=\sum_{i=1}^{N_1}\left[\frac{N_1+1-i}{N_1+1}-\left(1+2\alpha t_{i:N_1}\right)e^{-2\alpha t_{i:N_1}}\right]^2$$

$$+\sum_{j=1}^{N_2}\left[\frac{N_2+1-j}{N_2+1}-\left(1+2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)\right)e^{-2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)}\right]^2. \tag{4.2}$$

The LSEs can be achieved by solving the following two equations simultaneously.

$$\frac{\partial Ls}{\partial\alpha}=\sum_{i=1}^{N_1}8\alpha t^2e^{-2\alpha t_{i:N_1}}\left[\frac{N_1+1-i}{N_1+1}-\left(1+2\alpha t_{i:N_1}\right)e^{-2\alpha t_{i:N_1}}\right]$$

$$+\sum_{j=1}^{N_2}8\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)^2e^{-2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)}$$

$$\left[\frac{N_2+1-j}{N_2+1}-\left(1+2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)\right)e^{-2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)}\right]=0, \tag{4.3}$$

$$\frac{\partial Ls}{\partial\beta}=\sum_{j=1}^{N_2}8\alpha^2\left(t_{j:N_2}-\tau_1\right)\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)e^{-2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)}$$

$$\left[\frac{N_2+1-j}{N_2+1}-\left(1+2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)\right)e^{-2\alpha\left(\tau_1+\beta\left(t_{j:N_2}-\tau_1\right)\right)}\right]=0. \tag{4.4}$$

Eq.s (4.3) and (4.4) cannot be obtained in closed forms. Therefore, we should resort to numerical methods to find their solutions.

5. Simulation Study and Data Analysis

In this section, a comprehensive simulation study is utilized to evaluate the efficiency of the estimation techniques presented in the previous sections. Furthermore, a genuine dataset is employed to showcase the precision and practicality of the various estimation methods introduced in this paper.

5.1 Simulation Study

Because there were no prior estimation outcomes related to the discussed methods employing Type-I censoring, we conducted a Monte Carlo simulation to investigate their characteristics and compare the performance of the two estimators.

We performed a simulation study to calculate the model parameters using the maximum likelihood method, as well as the LS method, and determine their associated bias and mean square errors (MSEs). The bias and MSE of an estimator $\hat{\theta}$ of the parameter θ are defined as follows, respectively

$$Bias(\hat{\theta}) = \frac{1}{M}\sum_{K=1}^{M}(\hat{\theta}_k - \theta),$$

$$MSE(\hat{\theta}) = \frac{1}{M}\sum_{K=1}^{M}(\hat{\theta}_k - \theta)^2.$$

Moreover, we obtain the approximate CIs by using Eq.s (3.14) and (3.15).

Type-I censored samples were created following the procedure outlined in Section 3, with specified sample sizes, censoring schemes, and parameter values. A comparative study is conducted based on the following schemes:

Scheme 1: $\alpha = 0.4$, $\beta = 1.5$, $\tau_1 = 2$ and $\tau_2 = 5$.
Scheme 2: $\alpha = 1.7$, $\beta = 0.3$, $\tau_1 = 0.5$ and $\tau_2 = 3$.
We create Type-I censored samples from the Ailamujia model, employing these two schemes and repeating the process $M = 1000$ times in the simulation. Using these samples, we calculate MLEs and LSEs for the model's parameters. We perform these calculations using R software and present the results in Table 1. Additionally, Table 2 provides information for 95% CIs.

The following observations can be deduced from the values presented in the tables:

1) As the sample size, n, increases, the MSEs of both estimates follow a decreasing trend.
2) LSEs generally exhibit smaller biases and MSEs compared to those obtained through MLEs in most cases. As a result, the LSE method excels over the MLE method.
3) Regarding interval estimation, the true parameter values reside within all the examined CIs, and these intervals become narrower as the sample size, n, increases.

Table 1: MLE and LSE with their associated bias and MSE.

Scheme 1: $\alpha = 0.4$, $\beta = 1.5$, $\tau_1 = 2$ and $\tau_2 = 5$.							
n		MLE			LSE		
		Estimate	Bias	MSE	Estimate	Bias	MSE
60	α	0.3769	-0.0230	0.0039	0.4189	0.0189	0.0021
	β	1.7738	0.2738	0.1937	1.5673	0.0673	0.0071
90	α	0.3751	-0.0248	0.0029	0.4166	0.0166	0.0015
	β	1.7343	0.2343	0.1575	1.5595	0.0595	0.0065
120	α	0.3782	-0.0217	0.0016	0.4211	0.0211	0.0012
	β	1.7414	0.2414	0.1049	1.5596	0.0596	0.0063
Scheme 2: $\alpha = 1.7$, $\beta = 0.3$, $\tau_1 = 0.5$ and $\tau_2 = 3$.							
n		MLE			LSE		
		Estimate	Bias	MSE	Estimate	Bias	MSE
60	α	1.5853	-0.1146	0.0867	1.7691	0.0691	0.0622
	β	0.3604	0.0604	0.0126	0.4349	0.1349	0.0264
90	α	1.5810	-0.1189	0.0516	1.7688	0.0688	0.0348
	β	0.3571	0.0571	0.0067	0.4204	0.1204	0.0231
120	α	1.5889	-0.1110	0.0353	1.7911	0.0911	0.0285
	β	0.3458	0.0457	0.0054	0.4228	0.1228	0.0217

Table 2: 95% CIs of the censored lifetimes.

Scheme 1: $\alpha = 0.4$, $\beta = 1.5$, $\tau_1 = 2$ and $\tau_2 = 5$.		
n		95% CIs
60	α	(0.2403, 0.5114)
	β	(0.9775, 2.5666)
90	α	(0.2731, 0.4805)
	β	(1.1326 , 2.3579)
120	α	(0.2903, 0.4629)
	β	(1.2299, 2.2516)

Scheme 2: $\alpha = 1.7$, $\beta = 0.3$, $\tau_1 = 0.5$ and $\tau_2 = 3$.		
n		95% CIs
60	α	(1.1070, 2.0582)
	β	(0.1817, 0.5214)
90	α	(1.1953, 1.9459)
	β	(0.2250, 0.4892)
120	α	(1.2468, 1.9010)
	β	(0.2360, 0.4657)

5.2 Data Analysis

In this subsection, we analyze a real data set supplied by Murthy et al. [29] to demonstrate the performance of the various methods introduced in this chapter, grounded in the times between breakdowns for a repairable system. In fact, Kamal et al. [30] have previously utilized this particular data set. We utilize SSPALT to analyze the provided data, specifying τ_1 as 1.2 and τ_2 as 2.5. Subsequently, we determine the total count of failures, with N being 20 out of a total of n, which comprises 25 observations. Consequently, the failure data, derived from the AT-II PHC actual data provided in Table 3 under both normal and stress conditions, is presented as follows:

Table 3: The AT-II PHC data set that was generated.

Normal Condition	0.11 0.30 0.40 0.45 0.59 0.63 0.70 0.71 0.74 0.94 1.17
Stress Condition	1.23 1.24 1.43 1.49 1.82 1.86 1.97 2.23 2.37

To demonstrate the accuracy of our model, specifically the Ailamujia TRV model, we have graphed the actual cdf of lifetimes in Fig. 7, along with the cdf derived from the MLEs. In particular, the Kolmogorov-Smirnov test has been employed to assess the goodness-of-fit of the data to the Ailamujia TRV model. The test statistic, indicating the distance between the fitted and experimental distribution functions, is 0.1226, and the associated p-value is close to 1. Consequently, utilizing the Ailamujia distribution in the TRV model is deemed appropriate and justified for fitting this dataset.

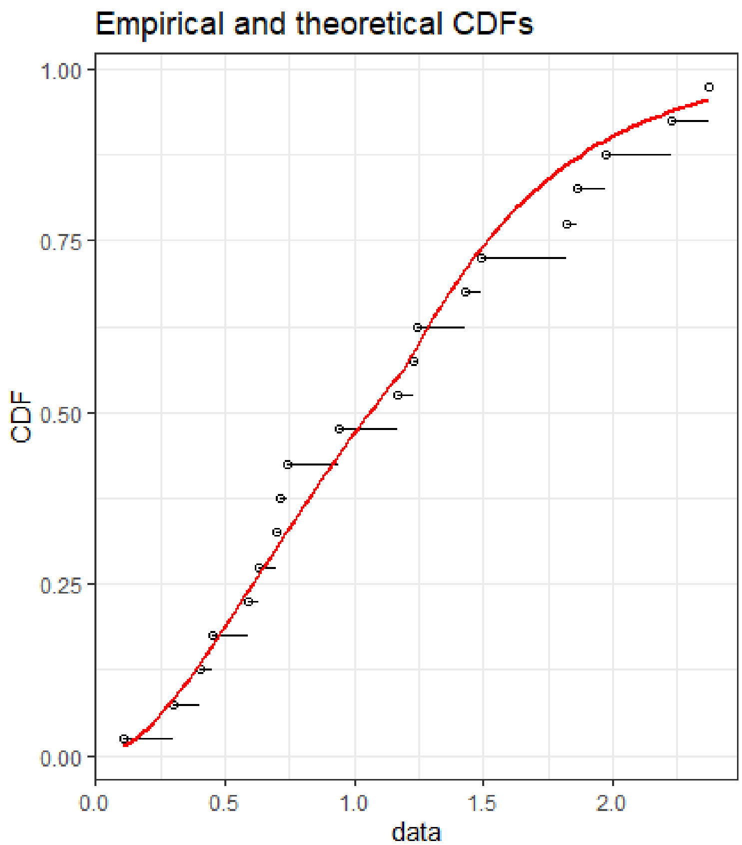

Fig. 7: The empirical CDF (dots); and the estimated CDF of Ailamujia TRV model based on MLE (solid line).

Based on the AT-II PHC data provided in Table 3, we have conducted parameter estimations for the Ailamujia distribution within the framework of the SSPALT model. The results are presented in Table 4 below:

Table 4: MLE and LSE using the SSPALT for the AT-II PHC data.

	MLE	LSE
α	0.4042	0.3156
β	2.2871	2.7791

It is noticeable that the MLEs and LSEs are close to each other.

6. Conclusions

This paper involves the estimation of Ailamujia distribution parameters within the context of PALT with Type-I censored data. Both maximum likelihood and LS methods are applied, and a Monte Carlo simulation study is conducted to compare the various estimation methods, taking into consideration biases and MSEs. In conclusion, it is noted that the performance of the LSE method surpasses that of the MLE method.

It is important to note that although the study primarily addresses Type-I censoring, the techniques discussed can also be adapted for other censoring schemes such as Type-II, hybrid or progressive censoring.

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References

[1] W.B. Nelson, Accelerated Testing: Statistical Models, Test Plans, and Data Analysis, John Wiley & Sons, New York, 1990.

[2] A.F. Attia, A.A. Abdel-Ghaly, M.M. Abdel-Ghani, The Estimation Problem of Partially Accelerated Life Tests for Weibull Distribution by Maximum Likelihood Method With Censored Data, In: Proceeding of the 31st Annual Conference of Statistics, Computer Sciences and Operation Research, ISSR, Cairo University, Egypt, (1996), 128-138.

[3] M.M. Abdel-Ghani, Investigation of Some Lifetime Models Under Partially Accelerated Life Tests, Ph.D. Thesis, Department of Statistics, Faculty of Economics and Political Science, Cairo University, Giza, Egypt, 1998.

[4] A.A. Abdel-Ghaly, E.H. El-Khodary, A.A. Ismail, Maximum Likelihood Estimation and Optimal Design in Step Partially Accelerated Life Tests for the Pareto Distribution With Type-I Censoring, In: Proceedings of the 14th annual conference on Statistics and Computer Modeling in Human and Social Sciences, Faculty of Economics and Political Science, Cairo University, Egypt, (2002), 16-29.

[5] A.A. Abdel-Ghaly, E.H. El-Khodary, A.A. Ismail, Estimation and Optimal Design in Step Partially Accelerated Life Tests for The Compound Pareto Distribution Using Type II Censoring, In: Proceeding of the 15st Annual Conference on Statistics and Computer Modeling in Human and Social Sciences, Faculty of Economics and Political Science, Cairo University, Egypt, (2003), 35-47.

[6] M.M. Abdel-Ghani, The Estimation Problem of the Log Logistic Parameters in Step Partially Accelerated Life Tests Using Type-I Censored Data, Nat. Rev. Soc. Sci. 41 (2004), 1-19.

[7] A.A. Ismail, Inference for a Step-Stress Partially Accelerated Life Test Model With an Adaptive Type-II Progressively Hybrid Censored Data From Weibull Distribution, J. Comp. Appl. Math. 260 (2014), 533–542. https://doi.org/10.1016/j.cam.2013.10.014.

[8] A.A. Ismail, Likelihood and Bayesian Estimations for Step-Stress Life Test Model Under Type-I Censoring, Hacettepe J. Math. Stat. 44 (2015), 1235-1245.

[9] M.A. Fawzy, H. Athar, Inferences from Ailamujia Distribution based on Progressive Type-II Censored Data, J. Stat. Appl. Prob. 9 (2020), 483-493. https://doi.org/10.18576/jsap/090305.

[10] P.K. Goel, Some Estimation Problems in the Study of Tampered Random Variables, Technical Report No. 50, Department of Statistics, Carnegie-Mellon University, Pittsburgh, Pennsylvania, 1971.

[11] M.H. Degroot, P.K. Goel, Bayesian Estimation and Optimal Designs in Partially Accelerated Life Testing, Naval Res. Log. 26 (1979), 223-235. https://doi.org/10.1002/nav.3800260204.

[12] J.F. Lawless, Statistical Models and Methods for Lifetime Data, 2nd ed., John Wiley & Sons, Hoboken, New Jersey, 2003.

[13] A.C. Cohen, B.J. Whitten, Parameter Estimation in Reliability and Life Span Models, Marcel Dekker /CRC Press, New York, 1988.

[14] A.C. Cohen, Truncated and Censored Samples: Theory and Applications, Marcel Dekker/CRC Press, New York, 1991.

[15] H.Q. Lv, L.H. Gao, C.L. Chen, Ailamujia Distribution and Its Application in Supportability Data Analysis, J. Acad. Armored Force Eng. 16 (2002), 48-52.

[16] G.T. Pan, B.H. Wang, C.L. Chen, Y.B. Huang, M.T. Dang, The Research of Interval Estimation and Hypothetical Test of Small Sample of Distribution, Appl. Stat. Manage. 28 (2009), 468-472.

[17] B. Long, Bayesian Estimation of Parameter on Ailamujia Distribution Under Different Prior Distribution, Math. Practice Theory, 4 (2015), 186-192.

[18] L.P. Li, Minimax Estimation of the Parameter of Ailamujia Distribution Under Different Loss Functions, Sci. J. Appl. Math. Stat. 4 (2016), 229-235.

[19] A.A. Rather, C. Subramanian, S. Shafi, K.A. Malik, P.J. Ahmad, B.A. Para, T.R. Jan, A New Size Biased Distribution with Applications in Engineering and Medical Science, Int. J. Sci. Res. Math. Stat. Sci. 5 (2018), 75-85.

[20] A. Aijaz, A. Ahmad, R. Tripathi, Inverse Analogue of Ailamujia Distribution with Statistical Properties and Applications, Asian Res. J. Math. 16 (2020), 36-46. https://doi.org/10.9734/arjom/2020/v16i930218.

[21] U. Jan, K. Fatima, S.P. Ahmad, On Weighted Ailamujia Distribution and Its Applications to Lifetime Data, J. Stat. Appl. Prob. 6 (2017), 619-633. https://doi.org/10.18576/jsap/060317.

[22] B.C. Arnold, N. Balakrishnan, H.N. Nagaraja, A First Course in Order Statistics, John Wiley & Sons, New York, 1992.

[23] M.A. Amleh, M.Z. Raqab, Prediction of Censored Weibull Lifetimes in a Simple Step-Stress Plan with Khamis-Higgins Model, Stat. Optim. Inf. Comp. 10 (2021), 658-677. https://doi.org/10.19139/soic-2310-5070-1069.

[24] L. Alkhalfan, Inference For A Gamma Step-Stress Model Under Censoring, PhD Thesis, McMaster University, Canada, 2012.

[25] M.A. Amleh, M.Z. Raqab, Inference for Step-Stress Plan With Khamis-Higgins Model Under Type-II Censored Weibull Data, Qual. Reliab. Eng. Int. 39 (2023), 982-1000. https://doi.org/10.1002/qre.3275.

[26] M.A. Amleh, Prediction of Future Lifetimes for a Simple Step-Stress Model with Type-II Censoring and Rayleigh Distribution, WSEAS Trans. Math. 21 (2022), 131-143. https://doi.org/10.37394/23206.2022.21.20.

[27] M.A. Amleh, M.Z. Raqab, Inference in Simple Step-Stress Accelerated Life Tests for Type-II Censoring Lomax Data, J. Stat. Theory Appl. 20 (2021), 364-379. https://doi.org/10.2991/jsta.d.210406.001.

[28] J. J. Swain, S. Venkatraman, J.R. Wilson, Least Squares Estimation of Distribution Function in Johnsons Translation System, J. Stat. Comp. Simul. 29 (1988), 271-297. https://doi.org/10.1080/00949658808811068.

[29] D. P. Murthy, M. Xie & R. Jiang, Weibull models, John Wiley & Sons, Hoboken, New Jersey, 2004.

[30] M. Kamal, A. Rahman, S. Zarrin, H. Kausar, Statistical Inference Under Step Stress Partially Accelerated Life Testing for Adaptive Type-II Progressive Hybrid Censored Data, J. Reliab. Stat. Stud. 14 (2021), 585-614. https://doi.org/10.13052/jrss0974-8024.14211.