On KU-Modules Over KU-Algebras

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Abstract. The paper introduces the concept of modules for KU-algebras, named as KU-modules. It presents basic isomorphism theorems for KU-modules and explores their applications, particularly concerning chains of KU-modules. Additionally, it defines and examines exact sequences of KU-modules. The paper discusses various properties of chains of KU-modules and establishes the butterfly lemma in the context of KU-modules.

1. Introduction

KU algebra is a type of logical algebras that was firstly introduced by Prabpayak and Leerawat [6]. Homomorphisms and results for them for KU-algebras are given in ([6], [7]). Next to the introduction of KU-algebras, numerous authors have extensively explored KU-algebras in various research directions, such as fuzzy, neutrosophic, and intuitionistic contexts, as well as in soft and rough senses. For instance, Naveed et al. [9] introduced concepts on cubic KU ideals, whereas fuzzy ideals within this context was given by Mostafa et al. [10]. Additionally, Mostafa et al. [11] delved into the study of Interval-valued fuzzy concepts for KU ideals. Moin and Ali studies and described roughness in a KU algebra [12], while Ali et al. [13] explored the notion of a pseudo-metric for a KU-algebras. Further, Atanassovs intuitionistic fuzzy binormed KU ideals within the framework of KU-algebras was given and studied by Senapati and K.P. Shum [14]. More recently, Ali et al. [3] constructed graphs based on the KU ideals.

The notion of BCK-algebras that is an interesting class of logical algebras and that gives generalization for set-theoretic differences and proportional calculi was introduced by Imai and Iseki [8] as an important tools in logical algebras. This concept shares similarities with the development of Boolean logic based on Boolean algebras. Furthermore, Iseki [5] introduced a superclass of...
BCK-algebras and said that as BCI-algebras that encompasses as a class of BCK-algebras. This article is mainly distributed in six sections.

Section 1 is introduction part for motivation and interest based on KU algebra. Section 2 focuses on general concepts associated with KU-algebras. It initiates by discussing the fundamental concepts of KU-algebras, elucidating their elementary properties. Subsequently, the discussion extends to the exploration of ideals within a KU algebra along with their associated properties. In Section 3, we delve into the exploration of KU-modules, presenting various examples and elucidating their properties. Section 4 is dedicated to the examination of chains of KU-modules. We establish the minimal and maximal conditions for submodules, introduce the Schreier Refinement Theorem, and expound upon the Jordan-Holder Theorem within the context of KU-modules. In Section 5, our focus centers on the study of exact sequences with some of their algebraic properties. Section 6, is investigation of projective and injective KU-modules, outlining their respective properties and implications.

2. Preliminaries

This section is based on related definitions and notations concerning KU-algebras, KU ideals and their posets are discussed.

**Definition 2.1.** [6] The structure \((X, \circ, 1)\) with binary operation \(\circ\) of type \((2, 0)\) is called a KU algebra if for any \(a, b, c \in X\) we have:

\begin{align*}
\text{ku(1)} & \quad (a \circ b) \circ [(b \circ c) \circ (a \circ c)] = 1, \\
\text{ku(2)} & \quad a \circ 1 = 1, \\
\text{ku(3)} & \quad 1 \circ a = a, \\
\text{ku(4)} & \quad a \circ b = b \circ a = 1 \Rightarrow a = b.
\end{align*}

If not specified then, we consider \((X, \circ, 1)\), a KU algebra. \(X\) will indicate a KU algebra throughout the text. 1 is a constant element and fixed for \(X\). The condition \(u \leq v\) indicates a partial order “\(\leq\)” in \(X\) iff \(v \circ u = 1\).

**Lemma 2.1.** [6] The structure \((X, \circ, 1)\) is a KU algebra iff:

\begin{align*}
\text{ku(5)} & \quad a \circ b \leq (b \circ c) \circ (a \circ c), \\
\text{ku(6)} & \quad a \leq 1, \\
\text{ku(7)} & \quad a \leq b, b \leq a \Rightarrow a = b,
\end{align*}

**Lemma 2.2.** In the structure \((X, \circ, 1)\), the given identities are valid [10]:

1. \(c \circ c = 1\),
2. \(c \circ (a \circ c) = 1\),
3. \(a \leq b\) imply \(b \circ c \leq a \circ c\),
4. \(c \circ (b \circ a) = b \circ (c \circ a)\), for all \(a, b, c \in X\),
5. \(b \circ [(b \circ a) \circ a] = 1\).
Example 2.1. [10] Considering $X = \{u, v, w, x, y\}$ we observe that $X$ is a KU algebra with single binary operation $\circ$ that is given by the following table

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Definition 2.2. The following is defined for a KU algebra $X$:

(i) commutativity: $(y \circ x) \circ x = (x \circ y) \circ y$, for all $x, y \in X$.

(ii) Implicativity: $(x \circ y) \circ x = x$, for all $x, y \in X$.

(iii) Boundedness: if there exists an element $1 \in X$ so that for each $x \in X$, $x \leq 1$.

Definition 2.3. [6] A KU ideal is a non-void subset $A$ of $X$ that satisfies:

1. $1 \in A$,
2. $u \circ (v \circ w) \in A, v \in A \Rightarrow u \circ w \in A$, for all $u, v, w \in X$.

Definition 2.4. Maximal ideal in $X$ is a KU ideal $I_M$ of $X$ so that proper ideal $I_M$ of $X$ is not a proper subset of any proper ideal of $X$. Such maximal ideal in KU algebra is called maximal KU ideal.

Definition 2.5. For $|X| \geq 2$, in a bounded KU algebra $X$, there is at least one maximal ideal.

Definition 2.6. A proper commutative ideal of a bounded KU algebra is called maximal commutative ideal containing $I_M$.

Definition 2.7. Any implicative KU algebra (that is proper commutative) ideal of a bounded KU algebra is called maximal commutative ideal containing $I_M$.

Example 2.2. [12] Let $X = \{1, b, c, u, v, w\}$ in which $\circ$ is defined by the following table:

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We see that $(X, \circ, 1)$ is a KU algebra. Further we can show that $A = \{1, b\}$ is a KU ideal and $B = \{1, b, c, u, v\}$ is KU ideal of $X$.

The following conditions for a non-void set with a single binary operation construct a KU algebra.
Proposition 2.1. \((X, \circ, 1)\) is a KU algebra iff:

i. \((b \circ c) \circ ((c \circ a) \circ (b \circ a)) = 1\) for all \(a, b, c \in X\);
ii. \((b \circ 1) \circ a = a\) for all \(a, b \in X\);
iii. For all \(a, b, c \in X\) such that \(a \circ b = 1, b \circ a = 1 \Rightarrow a = b\).

Proof. Considering direct part and saying \((X, \circ, 1)\) is a KU algebra, we see that i. follows from ku(1). Further, ii. follows from fourth condition of a KU algebra. iii. is directly followed from ku(2) and ku(3) as \((b \circ 1) \circ a = 1 \circ a = a\).

Considering indirect part, if \((X, \circ, 1)\) satisfies all three conditions above, then ku(1) and ku(4) holds true from i. and ii.. Now, replace \(b\) by \(a\), \(a\) by 1 and \(c\) by 1 in i. and use ii. we find that,\((a \circ 1) \circ [(1 \circ 1) \circ (a \circ 1)] = 1 \Rightarrow (1 \circ 1) \circ (a \circ 1) = 1 \Rightarrow a \circ 1 = 1\) that proves ku(2). Further, using \(a \circ 1 = 1\) in ii. we have, 1 \(\circ a = a\) for every \(a \in X\). Hence this proves that \((X, \circ, 1)\) is a KU algebra.

\[\square\]

Definition 2.8. A prime ideal \(P\) of \(X\) where \(X\) is a KU algebra is defined as, if \((b \ast a) \ast a \in P \Rightarrow a \in P\) or \(b \in P\).

Definition 2.9. A KU algebra \(X\) is called a bounded KU algebra if for an \(e\) exists in \(X\) so that \(x \leq e\) for any \(x \in X\). The element \(e\) is called unit of \(X\). In a bounded KU algebra, we write \(x \ast e\) for \(\mathfrak{A}(X)\).

Theorem 2.1. If \(X\) is a bounded KU algebra having greatest element 1, then for any \(x, y \in X\):
1. \(\mathfrak{A}1 = 0\) and \(\mathfrak{A}0 = 1\).
2. \(\mathfrak{A}y \ast \mathfrak{A}x \leq x \ast y\).
3. \(y \leq x \Rightarrow \mathfrak{A}x \leq \mathfrak{A}y\).

Theorem 2.2. For a bounded KU algebra \(X\), it is commutative iff \(x \land y = (y \ast x) \ast x, x \lor y = \mathfrak{A}(\mathfrak{A}x \land \mathfrak{A}y)\).

We have the following theorem that we shall use later on:

Theorem 2.3. We have given assertions that hold true:

i. If \(X\) is implicative, then it is commutative.
ii. If \(X\) is commutative, then it forms a lattice with \(x \lor y = \mathfrak{A}(\mathfrak{A}x \lor \mathfrak{A}y)\) and \(x \land y = (x \ast y) \ast y\).
iii. Bounded implicative KU algebra and Boolean algebra are equivalent.

Lemma 2.3. If \(X\) is a commutative KU algebra, then \((X, \leq)\) is a lower KU-semilattice. If \(X\) is bounded and commutative, then \((X, \leq)\) is a KU-lattice.

Lemma 2.4. If \(I\) is an ideal of a KU algebra \(X\), we get \(I_1 \lor I_2 \in I\), for any \(I_1, I_2 \in I\).

Theorem 2.4. If \((X, \ast, \land, \lor, 0, 1)\) is a bounded implicative KU algebra, then we have, for all \(u, v \in X\):

i. \(\mathfrak{A}u = u\).
ii. \(\mathfrak{A}u \lor \mathfrak{A}v = \mathfrak{A}(u \lor v)\) and \(\mathfrak{A}u \land \mathfrak{A}v = \mathfrak{A}(u \land v)\).
iii. \(\mathfrak{A}u \ast \mathfrak{A}v = v \land u\).
iv. \(u \land \mathfrak{A}u = 0\).
v. \( u \lor \mathcal{R}_u = 1 \).
vi. \( (v \ast u) \ast u = \mathcal{R}_v \ast u = \mathcal{R}_v \lor u = v \ast u \).
vii. \( \mathcal{R}\mathcal{R}_u = u \).

3. KU-Modules

In this section we connect KU algebra with module theory concepts defining KU-modules and studying properties for them.

**Definition 3.1.** For a KU algebra \((X, \ast, 0)\), an abelian group \(A\) under usual addition and a mapping \(X \times A \rightarrow A\) defined by \((x, m) \rightarrow x \times m\) such that

1. \((x \land y)m = x(ym)\)
2. \(x(A_1 + A_2) = xA_1 + xA_2\)
3. \(0m = 0\) for all \(x, y \in X\) and \(A_1, A_2 \in A\) where \(x \land y = (x \ast y) \ast y\). is called a left KU-module. In a similar fashion right KU-module is defined. In case \(X\) is bounded then for some \(m \in A\), \(1m = m\), \(A\) is called a unitary KU-module.

**Definition 3.2.** Consider \(A_1\) and \(A_2\) are KU-modules. A homomorphism is a mapping \(f : A_1 \rightarrow A_2\) for any \(A_1, A_2 \in A_1\) satisfying the conditions:

1. \(f(A_1 + A_2) = f(A_1) + f(A_2)\).
2. \(f(xA_1) = xf(A_1)\) \(\forall x \in X\).

\(\text{Ker}(f) = \{x \in A_1 : f(x) = 0\}\) is kernel for \(f\) and \(\text{Im}(f) = \{f(A_1) : A_1 \in A_1\}\) is image for \(f\).

\(\text{Ker}(f)\) is a submodule of \(A_1\) and \(\text{Im}(f)\) is a submodule of \(A_2\). Also \(f\) is monomorphism iff \(\text{Ker}(f)\) is 0. Now we have the following isomorphism theorem:

**Definition 3.3.** Let \(A_1\) and \(A_2\) are KU-modules and \(f : A_1 \rightarrow A_2\) be an epimorphism. If \(B\) is a submodule of \(A_2\) and \(A' = f^{-1}(B)\) then \(A'/A'\) is isomorphic to \(A'_2/B\). Moreover, if \(B = \{0\}\), then \(A'/\text{Ker}(f) \geq A'_2\).

Let \(A\) be a KU-module and \(A_1, A_2, A_3\) are submodules of \(A\) then \(A_1 + A_2/A_3\) is isomorphic to \(A_1/A_2 \cap A_2\). Next if \(A_3 \subset A_2 \subset A_1\), then \(A_2/A_3\) is submodule of \(A_3/A_1\), we have \((A_1/A_3)/(A_2/A_3)\) is isomorphic to \(A_1/A_2\).

4. Chains of KU-Modules

Based on the concept of chains, connections and relationship between KU ideals and module theory will be established.

**Theorem 4.1.** Ideals of a bounded implicative KU algebra \(X\) is a submodule.

*Proof.* We define \(x_1 + x_2 = \{(x_1 \ast x_2) \land (x_2 \ast x_1)\}\) in a bounded implicative KU algebra \(I\). By ideal property of KU-algebras we have \(x_2 \ast x_1 \in I\) and \(x_1 \ast x_2 \in I\). Further in a bounded implicative KU algebra KU ideals are lattice ideals. Hence \(x_1 + x_2 \in I\). Further, \(x_1 + 0 = 0 + x_1 = x_1\) for all \(x_1 \in I\). That means 0 is the identity element in addition. Furthermore, \(x_1 \in I, 0 \in I \Rightarrow x_1 + x_1 = 0\). That
is \( x_1 \) is inverse of \( x_1 \). That shows \((I,+)\) is a subgroup of bounded implicative KU algebra. Next \( x_1 \in X, x_2 \in I \) then \( x_1 \land x_2 = x_1 x_2 \in I \). This means \( I \) is a submodule of \( X \).

\[ \text{Definition 4.1.} \quad \text{Let } A \text{ is a KU-module. } A \text{ will satisfy minimal(maximal) conditions for submodules if each non-void collections of submodules of } A \text{ contains a maximal(resp. minimal) member of } A. \]

\[ \text{Definition 4.2.} \quad \text{Let } A \text{ be KU-module of a commutative KU algebra. Then } A \text{ will satisfy the descending chain condition of submodules if for any chain of submodules, e.g. } A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \text{ is terminated in the sense that } A_k = A_k + 1 \text{ for some } k \geq 0. \text{ Ascending chain condition is defined in a similar fashion.} \]

\[ \text{Definition 4.3.} \quad \text{Let } A \text{ is a module of a commutative KU algebra. Then we say that } A \text{ will satisfy maximal(minimal) conditions for submodules if every non-empty collections of submodules contains a maximal(minimal) conditions for submodules if every non-empty collections of submodules contains a maximal(minimal) member.} \]

\[ \text{Proposition 4.1.} \quad \text{If } A \text{ is a KU-module then following are equivalent:} \]

(a) \( A \) satisfies the minimal(maximal) condition for submodules.

(b) Ascending (descending) chain condition are satisfied by \( A \) for its submodules.

\[ \text{Proof.} \quad (a) \Rightarrow (b). \text{ Let } A \text{ satisfies the maximal condition for submodules of a KU-module.} \]

Further let \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \) be given chain of submodules of \( A \). Next, \( A = \{A_i \mid i \in I \} \) where \( A_i \) repeats in the sequence. According to our assumption \( A \) has maximal element, say \( A_\mu \), then \( A_m = A_\mu \) for all \( m \geq \mu \). For a minimal condition similar fashion will work.

\[ (b) \Rightarrow (a). \text{ We consider } A \text{ satisfies the ascending chain condition for submodules of KU-module.} \]

Let \( \Omega \) be a non void set of submodules of \( M \). We claim that \( \Omega \) must contain a maximal member. Let us assume that \( \Omega \) has no maximal element. Let \( A_1 \in \Omega \) then since \( A_1 \) is not maximal in \( \Omega \), we can find \( A_2 \in \Omega \) such that \( A_1 \subset A_2 \). But \( A_2 \) is not maximal in \( \Omega \). Consequently; there exists \( A_3 \in \Omega \), for which \( A_2 \subset A_3 \) and so on. In this way a strictly increasing sequence of submodules \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \) is generated, contrary to our hypothesis about \( (b) \). For a descending chain condition similar fashion will work.

\[ \square \]

The following theorem can be established on the lines of the proof of Proposition 2.4. Butterfly Lemma that can be considered as another isomorphism theorem.

\[ \text{Theorem 4.2.} \quad \text{Let } N, N', K \text{ and } K' \text{ are submodules of a KU-module so that } N' \subset N \text{ and } K' \subset K. \text{ Then } [N' + (N \cap K)] \cap [N' + N \cap K'] \equiv [K' + (K \cap N)] \cap [(K' + (K \cap N'))]. \]

\[ \text{Proof.} \quad \text{Let } A_1 = N' + (N \cap K'), A_2 = N \cap K. \text{ Then } A_1 + A_2 = N' + N \cap K. \text{ Now, we have } A_1 \cap A_2 = [N' + (N \cap K')] \cap (N \cap K) = N' \cap K + N \cap K'. \text{ By Isomorphism theorem, we have } A_1 + A_2 / A_1 \equiv A_2 / A_1 \cap A_2 [N' + N \cap K] / N' + (N \cap K') \equiv [N \cap K] / [N \cap K' + (N' \cap K)]. \text{ By the symmetry of the above expression, it shows that } [K' + (K \cap N)] / [K' + (K \cap N')] \equiv [K \cap N] / [N \cap K' + (N' \cap K)]. \quad \square \]
For a given module $M$, the chain of submodules, length of a chain and its refinement is defined in a similar way. By simple module, we mean a module, which has no non-trivial submodules. A simple module is the module that has no non-trivial submodules.

**Schreir Refinement Theorem in terms of KU-module**

**Theorem 4.3.** Any two chains of KU-modules say $N = A_0 \subset A_1 \subset \cdots \subset A_t = M$ and $N' = A'_0 \subset A'_1 \subset \cdots \subset A'_s = M$ can be easily refined so that the output chains have the same length and a factor module $A_{t+1}/A_t$ of the first chain is isomorphic to some factor module $M_{i+1}/M_i$ of the second chain.

**Definition 4.4.** Let $A$ be a non-zero KU-module, then a finite descending chain of submodules of $A$ starting with $A$ and ending with $(0)$ is called a KU-composition series, i.e. $M = A_0 \supset A_1 \supset \cdots \supset A_m = (0)$ in such a way that their successive quotient $A_i/A_{i+1}$ are simple for all $i, 0 < i < m - 1$. Here $A$ is called length of the series.

**Jordan-Holder Theorem in case of KU-module**

In a non-zero KU-module any two composition series are equivalent if they have the same length and same simple quotients up to the order and isomorphisms. Then we have

(i.) $m = n$ and

(ii.) for each $i, 0 < i < m - 1$, there exists $j = j(i)$ for $0 \leq j \leq n - 1$ such that $A_i/A_{i+1} \cong N_j/N_{j+1}$.

### 5. Exact Sequences in KU-Module

Homology of KU-modules through exact sequences is mentioned in this section. Some homological properties are studied.

**Definition 5.1.** Let $f_1 : A_1 \to A_2$ and $f_2 : A_2 \to A_3$ be homomorphism of KU-modules. Then $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ is called exact sequence if $\text{Im}(f_1) = \text{Ker}(f_2)$. It is called semi exact sequence if $\text{Im}(f_1) \supset \text{Ker}(f_2)$. We can extend this concept up to $A_n$ exact KU-modules.

**Remark 5.1.** (i.) If $f_1$ is one-one at $A_1$ then $0 \to A_1 \xrightarrow{f_1} A_2$ is exact sequence.

(ii.) If $f_1$ is onto at $A_2$ then $A_1 \xrightarrow{f_1} A_2 \to 0$ is exact sequence.

**Theorem 5.1.** In the homology of a KU-module with an exact sequence $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$, then the below conditions are equivalent:

(i.) $f_1$ is an epimorphism.

(ii.) $f_2$ is the trivial homomorphism.

(iii.) $f_3$ is monomorphism.

**Theorem 5.2.** Let $f_1 : A_1 \to A_2$ and $g : A_2 \to A_3$ be two homomorphism of KU-module. Then it is trivial if and only if $\text{Im}(f_1) \subseteq \text{Ker}(f_2)$.

**Proof.** We first prove direct part.

Let $f_3$ is trivial homomorphism and let $A_2$ be any element in $\text{Im}(f_1)$. We have that there exists
an element $A_1 \in A_1$ such that $f_1(A_1) = A_2$ and hence $f_2(A_2) = f_2(f_1(A_1)) = f_3(A_1) = 0$. Thus $A_2 \in \text{Ker}(h_2)$. This proves that $\text{Im}(f_1) \supset \text{Ker}(f_2)$.

Now we consider indirect part.

Consider $\text{Im}(f_1) \subset \text{Ker}(f_2)$ and let $A_1$ be any element of $A_1$. Then we have $f_3(A_1) = f_2(f_1(A_1))$. Since $f_1(A_1) \in \text{Im}(f_1) \supset \text{Ker}(f_2)$. Then we have $f_2(f_1(A_1)) = 0$ $\Rightarrow$ that $f_3(A_1) = 0$ for all $A_1 \in A_1$. This proves that $f_3$ is a trivial homomorphism. □

**Lemma 5.1.** Considering $A_1, A_2, A_3$ be KU-modules and let $f_3 : A_1 \rightarrow A_2$ be an epimorphism, and let $f_2 : A_1 \rightarrow A_3$ be a homomorphism. If $\text{Ker}(f_3) \subseteq \text{Ker}(f_2)$, then there is a unique homomorphism $f_1 : A_2 \rightarrow A_3$ satisfying $f_1 \circ f_2 = f_3$.

**Proof.** Let $A_2 \in A_2$. Since $f_3$ is an epimorphism hence $A_1 \in A_1$ and $A_2 \in f_3(A_1)$. We have the following mapping equations:

$$A_1 \xrightarrow{f_3} A_2 \xrightarrow{f_1} A_3 \text{ and } A_1 \xrightarrow{f_3} A_3 \quad \ldots (5.1)$$

$$A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \text{ and } A_2 \xrightarrow{f} A_3 \quad \ldots (5.2)$$

$$A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \rightarrow 0 \text{ and } A_2 \xrightarrow{f} A_4 \text{ and } A_3 \xrightarrow{f} A_4 \quad \ldots (5.3)$$

$$A_4 \xrightarrow{f} A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \text{ and } 0 \xrightarrow{f} A_1 \text{ and } A_4 \xrightarrow{f} A_1 \quad \ldots (5.4)$$

$$A_1 \xrightarrow{=} A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \text{ and } A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \xrightarrow{\gamma} A_3 \text{ and } A_2 \xrightarrow{\beta} A_2 \quad \ldots (5.5)$$

$$A_1 \xrightarrow{=} A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \text{ and } A_1 \xrightarrow{f} A_2 \xrightarrow{f} A_3 \xrightarrow{\gamma} A_3 \text{ and } A_2 \xrightarrow{\beta} A_2 \quad \ldots (5.6)$$

Define $f_1 : A_2 \rightarrow A_3$ by $f_1(A_2) = f_2(A_1)$, where $A_2 = f_3(A_1)$.

Next, $f_1 \circ f_3 = f_2$. Let $f_1 : A_2 \rightarrow A_3$ so that $f_1 f_3 = f_1' f_3$. Then, $f_1 = f_1'$. Hence $f_1$ is unique. □

**Proposition 5.1.** Let $A_1, A_2, A_3$ be KU-modules and let $f_2 : A_1 \rightarrow A_3$ is a homomorphism and $f_3 : A_2 \rightarrow A_3$ is a monomorphism with $\text{Im}(f_2) \subset \text{Im}(f_3)$. Then there is a homomorphism $f_1 : A_1 \rightarrow A_2$ that is unique and satisfy $f_2 = f_3 \circ f_1$ i.e. equation (2) commutes.

**Proof.** For every $A_1 \in A_1, f_2(A_1) \in A_3 \text{ and hence } f_2(A_1) \in \text{Im}f_2$. Since $\text{Im}(f_2) \subseteq \text{Im}(f_3)$ and $f_3$ is a monomorphism, therefore there exists a unique $A_2 \in A_2$ so that $f_3(A_2) = f_2(A_1)$. Therefore, there is a function $f_1 : A_1 \rightarrow A_2, A_1 \rightarrow A_2$ so that $f_2 = f_1 \circ f_3$. Hence $f_3$ is a unique homomorphism so that $f_2 = f_1 \circ f_3$. □

**Theorem 5.3.** In a KU-module with $f_1$ and $f_2$ as exact homomorphism with Equation (5.6), and $f_1 \circ f_3 = 0$, we have a unique homomorphism $f_4 : A_3 \rightarrow A_4$ so that $f_2 \circ f_4 = f_3$. 
Proof. Since \( f_1 \circ f_3 = 0 \), we get that \( \text{Ker} (f_2) = \text{Im}(f_1) \subset \text{Ker}(f_3) \). By using Proposition 5.5, there is a unique homomorphism \( f_4 : A_4 \to A_1 \) satisfying \( k \circ f_1 = f_3 \). \( \square \)

**Theorem 5.4.** Considering the Equation 4 of homomorphism of KU-module, \( f_1 \) and \( f_2 \) are exact homomorphisms and \( f_3 \circ f_2 = 0 \), then there is a unique homomorphism \( f_4 : A_4 \to A_1 \) that satisfy \( f_4 \circ f_1 = f_3 \).

Proof. \( f_3 \circ f_2 = 0 \), so we have \( \text{Im}(f_3) \subset \text{Ker}(f_2) = \text{Im}f \). By Proposition 5.5, there is a unique homomorphism \( f_4 : A_4 \to A_1 \) satisfying \( k \circ f_1 = f_3 \). \( \square \)

**Theorem 5.5.** Consider \( A_1, A_2, A_3 \) and \( A'_1, A'_2, A'_3 \) be KU-modules over \( X \), in Equation (5.6) given homomorphism commutes and are exact. Next if \( \alpha, \gamma \) and \( f' \) are monomorphism, then \( \beta \) is also monomorphism.

Proof. We see that all homomorphisms commutes, hence \( f'\alpha = \beta f \) and \( g'\beta = \gamma g \) and it is exact. Thus \( \text{Im}(f) = \text{Ker}(g) \) and \( \text{Im}(f') = \text{Ker}(g') \).

Further let \( A_2 \in \text{Ker}(\beta) \). Since they are commutative, we have \( \gamma g(A_2) = g'\beta(A_2) = 0, g(A_2) = 0 \), because \( \gamma \) is one to one. Therefore, \( A_2 \subseteq \text{Ker}g = \text{Im}f \). \( \square \)

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{\alpha_3} A'_3 \xrightarrow{\beta_3} A_3 \quad \text{and} \quad A_1 \xrightarrow{\alpha_1} A'_1 \xrightarrow{\beta_1} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \quad \ldots \quad (5.7)
\]

\[
A_2 \xrightarrow{\alpha_2} A'_2 \xrightarrow{\beta_2} A_2 \quad \text{and} \quad A'_1 \xrightarrow{f'_1} A'_2 \xrightarrow{f'_2} A'_3. \quad \ldots \quad (5.8)
\]

The exactness of the Equation (5.7) and by the fact that the Equation (5.7) and the Equation (5.8) commutes, \( f_1 \) is a monomorphism, all these imply that \( A_2 = 0 \). This proves that \( \text{ker} \beta = 0 \).

**Theorem 5.6.** Let \( A_1, A_2, A_3 \) and \( A'_1, A'_2, A'_3 \) be KU-modules over \( X \), as shown in the Equation (5.6) where each part commutes and \( \alpha, \beta, \gamma \) are isomorphism. If the second part of the Equation (5.6) is exact then the third part, is also exact.

Proof. Given that each part of the Equation (5.6) are commuting so \( f'_1\alpha = \beta f_1 \) and \( f'_2\beta = \gamma f_2 \). As first two parts are exact, therefore \( \text{Im}f'_1 = \text{ker} f'_2 \). By using these information and the fact that \( \alpha, \beta, \gamma \) are isomorphisms, we can deduce that \( \text{Im}f_1 = \text{ker} f_2 \). This shows that the last part is also exact. \( \square \)

**Theorem 5.7.** Let the Equation (5.7) and (5.8) be equations related to KU-modules and homomorphism with \( \beta_i \alpha_i = I_i \), where \( I_i \)’s denote associated identity correspondence, from \( i = 1, 2, 3 \). If the first part of the Equation (5.8) is exact, then \( A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \) is also exact.

Proof. Since the Equations (5.7) and (5.8) commutes, hence

\[
f'_1\alpha_1 = \alpha_2 f_1 \quad \text{and} \quad f'_2\alpha_2 = \alpha_3 f_2
\]

\[
f_1\beta_1 = \beta_2 f'_1 \quad \text{and} \quad f_2\beta_2 = \beta_3 f'_2
\]

As the middle row \( A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \) is exact, therefore \( \text{Im}f'_1 = \text{ker} f'_2 \). In order to prove first part of the Equation (5.7) and last part of the Equation (5.8) are also exact, we have to show that
$lm(f_1) = \ker(f_2)$. Let $A_i \in A_i$, and consider $\alpha_3 f_2 f_1(A_1) = f_2'(\alpha_2) f_1(A_1) = f_2' f_1' \alpha_1(A_1)$ because $\alpha_2 f_1 = f_1' \alpha_1$ (by Eq. 5.9). Since $\text{Ker } f_2' = \text{Im} f_1, f_2' f_1'(\alpha_1(A_1)) = 0$, and hence $\alpha_3 f_2 f_1(A_1) = 0$. This \(\Rightarrow\) that $\alpha_3 f_2 f_1 = 0$ and hence $\beta_3 \alpha_3 f_2 f_1 = 0$. This shows that $\text{Im} f_1 \subseteq \text{Ker } f_2$. For the converse part, consider $x \in \text{Ker } f_2$, we get then $f_2(x) = 0$ and hence $\alpha_3 f_2(x) = 0$. As $f_2' \alpha_2 = f_2 \alpha_2$, therefore, $\alpha_2(x) \in \ker\ f_2'$ and hence $\alpha_2(x) \in \ker\ f_2' = \text{Im} f_1'$, therefore $\alpha_2(x) \in \text{Im} f_1'$ and hence $\alpha_2(x) = f_1'(m'_1)$ for some $m'_1 \in A'_1$.

$$0 \to A'_1 \xrightarrow{\alpha'} A_1 \xrightarrow{\beta'} A_1'' \to 0 \text{ and } 0 \to A'_2 \xrightarrow{\alpha''} A_2 \xrightarrow{\beta''} A_2'' \to 0 \text{ and } 0 \to A'_3 \xrightarrow{\alpha'''} A_3 \xrightarrow{\beta'''} A_3'' \to 0$$

\(\ldots\) (5.10)

and

$$0 \to A'_1 \xrightarrow{\beta'} A'_2 \xrightarrow{\beta''} A'_3 \to 0 \text{ and } 0 \to A_1 \xrightarrow{\beta} A_2 \xrightarrow{\beta} A_3 \to 0 \text{ and } 0 \to A'_1 \xrightarrow{\beta'} A'_2 \xrightarrow{\beta''} A'_3 \to 0$$

\(\ldots\) (5.11)

$\beta_2 \alpha_2(x) = \beta_2 f'(m'_1)$. As $\beta_2 \alpha_2 = I_2$, therefore $x = \beta_2 f'(m'_1) = f(\beta_1(m'_1))$ by Eq. 5.9.

This shows that $x \in \text{Im } f$. Therefore $\text{Ker } g \subseteq \text{Im } f$ that $\Rightarrow \text{Ker } g = \text{Im } f$ and hence required sequence is exact. \(\square\)

**Theorem 5.8.** Let $A_1, A'_1, A''_1, A_2, A'_2, A''_2, A_3, A'_3, A''_3$ be KU-modules over $X$ as in the Equation (5.10) and (5.11). If all three parts of the Equation (5.11) are exact, and if all parts of the Equation (5.10) are exact, then there exists unique homomorphism $\alpha'' : A'_3 \to A_3$ and $\beta'' : A_3 \to A''_3$ such that the sequence $0 \to A'_3 \xrightarrow{\alpha''} A_3 \xrightarrow{\beta''} A''_3 \to 0$ is semi-exact and the complete diagram commutes.

**Proof.** Since all parts of the Equation (5.11) are exact, we have $\text{Im}(f'_1) = \ker(f'_2)$, $\text{Im}(f_1) = \ker f_2$, $\text{Im}(f''_1) = \ker(f''_2)$ (2) Also all parts of the Equation (5.10) are exact, we have $\text{Im}(\alpha') = \ker(\beta')$ and $\text{Im}(\alpha) = \ker(\beta)$ (3) Since both Equations commute, we have $\alpha f'_1 = f_1 \alpha'$, $f_1 \beta = f''_1 \beta'$ (4) We intend to show that $\text{Im}(\alpha'') \subseteq \ker(\beta'')$. Observe that in this diagram $\alpha'$, $\alpha$ and $f'_1, f_1, f''_1, f''_2$ are one to one. Similarly $f''_2, f_2, f''_2$ and $\beta', \beta$ are onto. Now we have $f_2 \alpha f'_1 = f_2 f_1 \alpha' = 0 \alpha = 0 (\text{Im} f = \ker f_2)$. First we prove the existence of $\alpha''$ and $\beta''$. Define a mapping $\alpha'' : A'_3 \to A_3$ by $\alpha''(m'_3) = A_3$ where $f_2 \alpha(m'_2) = A_3$ and $f_2'(m'_2) = (m'_3)$ ($f'_2$ is an epimorphism). Then $\alpha''$ is well-defined and indeed homomorphism. Moreover, $f_2 \alpha(m'_2) = \alpha'' f'_2(m'_2)$ and hence $f_2 \alpha = \alpha'' f'_2$. Now we prove the uniqueness of $\alpha''$.

Since $\alpha'' f'_2 = f_2 \alpha$, suppose there exists another homomorphism such that $\alpha_0 : A'_3 \to A_3$ such that $f_2 \alpha = \alpha_0 f'_2$. We will show that $\alpha_0 = \alpha''$.

Let $m'_3 \in A_3$. Then as $f'_2$ is onto, we have $m'_3 = f'_2(m'_2)$ for some $m'_2 \in A_2$. Thus, $\alpha_0(m'_3) = \alpha_0 f'_2(m'_2)$. Similarly $\alpha''(m'_3) = \alpha'' f'_2(m'_2)$. Since $f_2 \alpha = \alpha_0 f'_2 = \alpha_0 f'_2$ and $f_2 \alpha(m'_2) = \alpha'' f'_2(m'_2)$ $f_2 \alpha(m'_2) = \alpha_0 f'_2(m'_2)$ which $\Rightarrow$ that $\alpha_0(m'_3) = \alpha''(m'_3)$ for all $m'_3 \in A_3$. Thus $\alpha_0 = \alpha''$. This proves the uniqueness of $\alpha''$.

Define $\beta'' : A_3 \to A''_3$ by $\beta''(A_3) = m''_3$, where $f''_2 \beta(A_2) = m''_3$ and $f_2(A_2) = A_3$. Then $\beta''$ is well-defined and indeed homomorphism. Moreover, $f''_2 \beta(A_2) = \beta'' f_2(A_2)$ (by (3)) therefore, $f''_2 \beta = \beta'' f_2$. The uniqueness of $\beta''$ can be established on the lines similar to $\alpha''$.

Now for semi-exactness, let $m'_2 \in \ker(\alpha'')$, since $f'_2$ is surjective, we see that there exists $m'_2 \in M'_2$ such
that $f'_2(m'_2) = m'_3$. Thus, by definition of $\alpha'$, $0 = \alpha''(m'_3) = \alpha'' f'_2(m'_2) = f_2 \alpha(m'_2)$. This shows that $\alpha(m'_2) \in \text{ker}(f_2) = \text{lm}(f_1)$ (by the sequence). Therefore $\alpha(m'_2) = f_1(A_1)$ for some $A_1 \in A_1$. By the exactness of middle row $\beta \alpha(m'_2) = 0$. Since $0 = \beta \alpha(m'_2) = \beta f(A_1) = f'_1 \beta'(A_1)$ (by (3)). Therefore, $\beta'(A_1) \in \text{ker}(f'_1) = \{0\}$ and hence $A_1 \in \text{ker}(\beta') = \text{lm}(\alpha')$ (by the exactness of the sequence) which $\Rightarrow$ that $A_1 = \alpha'(m'_1)$ for some $m'_1 \in A'_1$. Therefore, we have $\alpha(m'_2) = f_1(A_1) = f_1 \alpha'(m'_1) = \alpha f'_1(m'_1)$. Since $\alpha$ is a monomorphism, $m'_2 = f'_1(m'_1)$, it follows that $m'_3 = f'_2(m'_2) = f'_2 f'_1(m'_1) = 0$ (lm$f'' = \text{ker} f'_2$) and so $\text{ker}(\alpha'') = \{0\}$ i.e. $\alpha''$ is monomorphism.

Next, since $\beta'' f_2 = f'' \beta$ and since $f''$, $\beta$ are each surjective, so $\beta'' f_2$ is surjective, which indeed $\Rightarrow$ that $\beta''$ is surjective. Finally, we have $\beta'' \alpha'' f_2 = f'' \beta \alpha = f'' 0 = 0(\text{ker} \beta = \text{lm} \alpha)$. Since $f'_2$ is surjective, then indeed $f'_2(A'_2) = A'_3$ and $\beta'' \alpha'' f'_2(A'_2) = \beta'' \alpha'' (A'_3) = 0$. This means that $\beta'' \alpha''$ is zero mapping, therefore $\beta'' \alpha'' = 0$. This shows that $\text{lm}(\alpha'') \subseteq \text{ker}(\beta'')$. □

6. Projective and Injective KU-Modules

In this section, we discuss Projective KU-modules and Injective KU-modules. We find that these modules are dual to each other.

**Definition 6.1.** A KU-module $Q$ is called projective if for given any Equations (6.1) to (6.8) in which the first part of the Equation is exact, there exists an $X$-homomorphism $f_3 : Q \rightarrow M_1$ such that $f_3 g_1 = f_1$ i.e., the Equations commute.

The following proposition follows from the above definition.

**Proposition 6.1.** Consider the Equation (6.2) has $M_1, M_2, M_3$ KU-modules and $Q$ is projective, $M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3$ is exact and $g_2 f_1 = 0$. Then there exists a KU-homomorphism $f_3 : Q \rightarrow M_1$ such that $f_3 g_1 = f_1$.

**Corollary 6.1.** Let the Equation (6.3) shows commutativity in which $M_1, M_2, M_3$ and $Q, R, S$ are KU-modules and $Q$ is projective, $f_6 f_5 = 0$ and $M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3$ is exact. Consequently, there exists an KU-homomorphism $f_3 : Q \rightarrow M_1$ for which the Equations commute.

**Proof.** Since the Equation commutes, $g_2 h_1 = h_2 f_2$. Now consider the Equation (6.4). From the Equation, we have $g_2 h_1 f_3 = (h_2 f_6) f_5 = h_2(f_6 f_5) = 0$ ($f_6 f_5 = 0$).

$$Q \xrightarrow{f_1} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} 0 \text{ and } Q \xrightarrow{f_1} M_2. \quad \ldots (6.1)$$

$$Q \xrightarrow{f_1} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3 \text{ and } Q \xrightarrow{f_1} M_2. \quad \ldots (6.2)$$

$$Q \xrightarrow{f_1} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3 \text{ and } Q \xrightarrow{f_1} R \xrightarrow{f_2} S \xrightarrow{h_2} M_3 \text{ and } T \xrightarrow{h_1} M_2 \quad \ldots (6.3)$$

$$Q \xrightarrow{f_1} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3 \text{ and } Q \xrightarrow{h_1 f_1} M_2. \quad \ldots (6.4)$$

$$0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_3} Q \text{ and } M_1 \xrightarrow{f_2} Q, \xrightarrow{g} A_3 \text{ and } A_2 \xrightarrow{\beta} A_2. \quad \ldots (6.5)$$
\[ M_1 \xrightarrow{f_1} M_2 \xrightarrow{g_1} M_3 \xrightarrow{f_3} Q \quad \text{and} \quad M_2 \xrightarrow{g_2} Q. \quad \ldots (6.6) \]

\[ M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} Q \quad \text{and} \quad M_1 \xrightarrow{h_1} L \xrightarrow{g_1} M \xrightarrow{g_2} Q \quad \text{and} \quad M_2 \xrightarrow{h_2} M. \quad \ldots (6.7) \]

\[ M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} R \quad \text{and} \quad M_2 \xrightarrow{g_2h_2} R. \quad \ldots (6.8) \]

Therefore \( g_2h_1f_1 = 0 \). Since \( Q \) is projective and \( g_2h_1f_1 = 0 \), therefore by proposition 4.2, there exists a homomorphism \( f_3 : Q \to M_1 \) such that \( f_3g_1 = h_1f_1 \). Consequently, all the squares are commutative. □

**Definition 6.2.** A KU-module \( R \) is said to be injective if for an Equation say (6.5) with \( 0 \xrightarrow{f_1} M_1 \to M_3 \) is an exact sequence there exists homomorphism \( f_3 : M_3 \to R \) such that \( f_3g_1 = h_1f_1 \).

By using the injectivity and certain natural homomorphism, we establish the following proposition.

**Proposition 6.2.** Consider Equation (6.6) in which \( M_1, M_2, M_3 \) are KU-modules and \( R \) is injective module, \( M_1 \xrightarrow{f_1} M_2 \xrightarrow{g_1} M_3 \) is exact and \( g_2f_1 = 0 \). Then there exists a homomorphism \( f_3 : M_3 \to R \) such that \( g_1f_3 = g_2 \).

**Corollary 6.2.** Consider the Equation (6.7) is commutative where \( R \) is an injective module, \( g_2g_1 = 0 \) and \( M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \) is an exact sequence. Then there exists a KU-homomorphism \( f_3 : M_3 \to R \) for which the diagram, commutes.

**Proof.** Since the Equation commutes, hence \( g_1h_1 = h_2f_1 \). Now consider the Equation (6.8). By this equation, we get that \( g_2h_2f_1 = g_2(h_2f_1) = g_2(g_1h_1) = (g_2g_1)h_1 \). Since \( g_2g_1 = 0 \) therefore \( g_2h_2f_1 = 0 \). Also \( R \) is injective and \( g_2h_2f_1 = 0 \), therefore by proposition 4.5, therefore there exists a homomorphism \( f_3 : M_3 \to R \) such that \( f_3f_2 = g_2h_2 \). Consequently, all parts of the Equations are commutative. □

As an open Problem with this study we can say Theorem 4.1 provides a minor relationship of ideal theory of KU algebra and its Modules. It can be consider as ideal-theory of KU-algebras and its module theory. In special case we can say bounded implicative KU-algebras. It would be interesting to check and provide some general relationship between ideal theory and module theory of KU-algebras or other newly defined logical algebras.

**7. Conclusion**

In this article a discussion based on the Chains of KU-modules of KU-algebras are given. Basic isomorphism theorems for KU-modules, projective and injective modules are provided and explored with some of theirs applications. Schreir Refinement Theorem and Jordan-Holder Theorem in terms of KU-module is also studied. Additionally, it defines and examines exact sequences of KU-modules. Various properties of chains of KU-modules are established and the
butterfly lemma in the context of KU-modules are given. As a future scope on this article, some further classical properties can be extended for a KU-module with chain conditions.

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