On Cesaro-Hypercyclic Operators

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Abstract. In this paper we characterize some properties of the Cesàro-Hypercyclic and mixing operators. At the same time, we also give a Cesàro-Hypercyclicity criterion and offer an example of this criterion.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space over the scalar field $\mathbb{C}$. As usual, $\mathbb{N}$ is the set of all non-negative integers, $\mathbb{Z}$ is the set of all integers, and $B(\mathcal{H})$ is the space of all bounded linear operators on $\mathcal{H}$. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called hypercyclic if there is some vector $x \in \mathcal{H}$ such that $\text{Orb}(T,x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where such a vector $x$ is said hypercyclic for $T$.

The first example of hypercyclic operator was given by Rolewicz in [12]. He proved that if $B$ is a backward shift on the Banach space $\ell^p$, then $\lambda B$ is hypercyclic if and only if $|\lambda| > 1$.

Let $\{e_n\}_{n \geq 0}$ be the canonical basis of $\ell^2(\mathbb{N})$. If $\{w_n\}_{n \geq 1}$ is a bounded sequence in $\mathbb{C}\setminus\{0\}$, then the unilateral backward weighted shift $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by $T e_n = w_n e_{n-1}$, $n \geq 1$, $T e_0 = 0$, and let $\{e_n\}_{n \in \mathbb{N}}$ be the canonical basis of $\ell^2(\mathbb{Z})$. If $\{w_n\}_{n \in \mathbb{Z}}$ is a bounded sequence in $\mathbb{C}\setminus\{0\}$, then the bilateral weighted shift $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is defined by $T e_n = w_n e_{n-1}$.

The definition and the properties of supercyclic operators were introduced by Hilden and Wallen [10]. They proved that all unilateral backward weighted shifts on a Hilbert space are supercyclic.

A bounded linear operator $T \in B(\mathcal{H})$ is called supercyclic if there is some vector $x \in \mathcal{H}$ such that the projective orbit $\text{C. Orb}(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in $\mathcal{H}$. Such a vector $x$ is said supercyclic for $T$.

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A nice criterion namely Hypercyclicity Criterion, was developed independently by Kitai [8] and, Gethner and Shapiro [7]. The Hypercyclicity Criterion has been widely used to show that many different types of operators are hypercyclic. For instance hypercyclic operators arise in the classes of composition operators [3], adjoints of multiplication operators [7], cohyponormal operators [6], and weighted shifts [13].

For the following theorem, see ([1], [9]).

**Theorem 1.1. (Hypercyclicity Criterion).** Suppose that $T \in B(\mathcal{H})$. If there exist two dense subsets $X_0$ and $Y_0$ in $\mathcal{H}$ and an increasing sequence $n_j$ of positive integers such that:

1. $T^{n_j}x \to 0$ for each $x \in X_0$, and
2. there exist mappings $S_{n_j} : Y_0 \to \mathcal{H}$ such that $S_{n_j}y \to 0$, and $T^{n_j}S_{n_j}y \to y$ for each $y \in Y_0$,

then $T$ is hypercyclic.

In [13] and [14], Salas characterized the bilateral weighted shifts that are hypercyclic and those that are supercyclic in terms of their weight sequence. In [5], N. Feldman gave a characterization of the invertible bilateral weighted shifts that are hypercyclic or supercyclic.

For the following theorem, see [5, Theorem 4.1].

**Theorem 1.2.** Suppose that $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bilateral weighted shift with weight sequence $(w_n)_{n \in \mathbb{Z}}$ and either $w_n \geq m > 0$ for all $n < 0$ or $w_n \leq m$ for all $n > 0$. Then:

1. $T$ is hypercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that $\lim_{k \to \infty} \prod_{j=1}^{n_k} w_j = 0$ and $\lim_{k \to \infty} \prod_{j=1}^{n_k} \frac{1}{w_j} = 0$.
2. $T$ is supercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that $\lim_{k \to \infty} \left( \prod_{j=1}^{n_k} w_j \right) \left( \prod_{j=1}^{n_k} \frac{1}{w_j} \right) = 0$.

Let $M_n(T)$ denote the arithmetic mean of the powers of $T \in B(\mathcal{H})$, that is

$$M_n(T) = \frac{1 + T + T^2 + \ldots + T^{n-1}}{n}, \quad n \in \mathbb{N}^*.$$ 

If the arithmetic means of the orbit of $x$ are dense in $\mathcal{H}$ then the operator $T$ is said to be Cesàro-hypercyclic. In [11], Fernando León-Saavedra proved that an operator is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{n^{-1}T^n x\}_{n \geq 1}$ is dense in $\mathcal{H}$ and characterized the bilateral weighted shifts that are Cesàro-hypercyclic.

For the following proposition, see [11, Proposition 3.4].

**Proposition 1.1.** Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral weighted shift with weight sequence $(w_n)_{n \in \mathbb{Z}}$. Then $T$ is Cesàro-hypercyclic if and only if there exists an increasing sequence $n_k$ of positive integers such that for any integer $q$,

$$\lim_{k \to \infty} \prod_{i=1}^{n_k} \frac{w_{i+q}}{n_k} = \infty \quad \text{and} \quad \lim_{k \to \infty} \prod_{i=0}^{n_k-1} \frac{w_{i+1}}{n_k} = 0.$$
In this paper we will characterize some properties of the Cesàro-Hypercyclic and mixing operators. Furthermore, we give a Cesàro-Hypercyclicity criterion and offer an example of this criterion.

2. Main results

Suppose \( \{n^{-1}T^n : n \geq 1\} \) is a sequence of bounded linear operators on \( \mathcal{H} \)

**Definition 2.1.** An operator \( T \in B(\mathcal{H}) \) is Cesàro-hypercyclic if and only if there exists a vector \( x \in \mathcal{H} \) such that the orbit \( \{n^{-1}T^n x\}_{n \geq 1} \) is dense in \( \mathcal{H} \)

The following example gives an operator which is Cesàro-hypercyclic but not hypercyclic.

**Example 2.1.** [11] Let \( T \) be the bilateral backward shift with the weight sequence \( w_n = \begin{cases} 1 & \text{if } n \leq 0, \\ 2 & \text{if } n \geq 1. \end{cases} \)

Then \( T \) is not hypercyclic, but it is Cesàro-hypercyclic.

Now, we will give an example of a hypercyclic and supercyclic operator which is not Cesàro-hypercyclic.

**Example 2.2.** Let \( T \) be the bilateral backward shift with the weight sequence \( w_n = \begin{cases} 2 & \text{if } n < 0, \\ \frac{1}{2} & \text{if } n \geq 0. \end{cases} \)

Then \( T \) is not Cesàro-hypercyclic, but it is hypercyclic and supercyclic.

**Definition 2.2.** We say that \( T \in B(\mathcal{H}) \) is Cesàro-topologically transitive if for every nonempty open subsets \( U \) and \( V \) of \( \mathcal{H} \) there exists \( n \geq 1 \) such that \( (n^{-1}T^n)(U) \cap V \neq \emptyset \).

**Definition 2.3.** We say that \( T \in B(\mathcal{H}) \) is Cesàro-mixing if for every nonempty open subsets \( U \) and \( V \) of \( \mathcal{H} \) there exists \( m \geq 1 \) such that \( (n^{-1}T^n)(U) \cap V \neq \emptyset \), \( \forall n \geq m \).

The set of Cesàro-hypercyclic vectors for \( T \) is denoted by \( CH(T) \).

**Theorem 2.1.** Let \( T \) be a cesàro-hypercyclic operator. Then

\[
CH(T) = \bigcap_{k \geq 1} \bigcup_{n \geq 1} (n^{-1}T^n)^{-1}(B_k),
\]

where \( (B_k)_{k \geq 1} \) is a countable open basis for \( \mathcal{H} \).

**Proof.** Let \( (B_k)_{k \geq 1} \) is a countable open basis for \( \mathcal{H} \). We have \( x \in CH(T) \) if and only if \( \{n^{-1}T^n x : n \geq 1\} \) is dense in \( \mathcal{H} \) if and only if for each \( k \geq 1 \), there exist \( n \geq 1 \) such that \( n^{-1}T^n x \in B_k \) if and only if \( x \in \bigcap_{k \geq 1} \bigcup_{n \geq 1} (n^{-1}T^n)(B_k) \).

□
Corollary 2.1. If \( T \) is cesàro-topologically transitive operator, then \( \text{CH}(T) \) is a dense set in \( \mathcal{H} \).

Proof. For every non-empty open \( U \) of \( \mathcal{H} \) and for all \( k \geq 1 \), there exist \( n \geq 1 \) such that the set \( (n^{-1}T^n)^{-1}(U) \cap B_k \) is nonempty and open. Hence the set

\[
A_k = \bigcup_n \left( (n^{-1}T^n)^{-1}(B_k) \right)
\]

is nonempty and open. Furthermore, \( U \cap A_k \neq \emptyset \) for all \( k \geq 1 \). Thus each \( A_k \) is dense in \( \mathcal{H} \) and so by the Baire category theorem and theorem 2.1 \( \text{CH}(T) \) is also dense in \( \mathcal{H} \). \( \square \)

Theorem 2.2. (Cesàro-Hypercyclicity Criterion). Suppose that \( T \in B(\mathcal{H}) \). If there exist two dense subsets \( M_0 \) and \( M_1 \) in \( \mathcal{H} \) and an increasing sequence \( (n_j) \) of positive integer such that:

1. \( \frac{1}{n_j}T_n x \to 0 \) for each \( x \in M_0 \), and
2. there exist mappings \( S_{n_j} : M_1 \to \mathcal{H} \) such that \( S_{n_j} y \to 0 \) and \( \frac{1}{n_j}S_{n_j} y \to y \) for each \( y \in M_1 \),

then \( T \) is Cesàro-hypercyclic.

Proof. Let \( U \) and \( V \) be non-empty open subsets of \( \mathcal{H} \). By topologically transitive, it is enough to prove that there exist \( n \geq 1 \) such that

\[
(n^{-1}T^n)^{-1}(U) \cap V \text{ is non-empty.}
\]

Since \( M_0 \) and \( M_1 \) are dense in \( \mathcal{H} \), there exist \( x \in M_0 \cap V, y \in M_1 \cap U \). And since \( U \) and \( V \) are nonempty open subsets, there exists \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subseteq V \) and \( B(y, \varepsilon) \subseteq U \). By assumption, there exist \( (n_k) \) such that

\[
||n_k^{-1}T^n x|| \leq \frac{\varepsilon}{2}, \quad ||S_{n_k} y|| \leq \frac{\varepsilon}{2} \quad \text{and} \quad ||n_k^{-1}T^n S_{n_k} y - y|| \leq \frac{\varepsilon}{2}.
\]

Define \( u = x + S_{n_k} y \). We know that \( u \in \mathcal{H} \) and \( u \in V \), since \( ||u - x|| = ||S_{n_k} y|| \leq \frac{\varepsilon}{2} \). Since

\[
||n_k^{-1}T^n u - y|| = ||n_k^{-1}T^n x|| + ||n_k^{-1}T^n S_{n_k} y - y|| < \varepsilon,
\]

we have that \( n_k^{-1}T^n u \in U \). Then \( (n_k^{-1}T^n)^{-1}(U) \cap V \neq \emptyset \) and \( T \) is cesàro-hypercyclic. \( \square \)

Suppose \( T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) be a unilateral weighted shift given by \( T e_n = w_n e_{n-1} \), \( n \geq 1 \), \( T e_0 = 0 \). Let \( \{e_n\}_{n \geq 0} \) be the canonical basis of \( \ell^2(\mathbb{N}) \). Define the sequence of linear mappings \( S_k \) as \( S_k e_j = n_k \left( \prod_{i=1}^{n} w_{i+j} \right)^{-1} e_{j+n_k} \).

Example 2.3. Taking \( n_j = n \geq 1 \) and suppose \( \lim_{n \to \infty} \prod_{i=1}^{n} w_{i+n} = \infty \) and \( \lim_{n \to \infty} \prod_{i=0}^{n-1} \frac{w_{i+j}}{n} = 0 \). Let \( M_0 = M_1 = \text{span}\{e_j : j \in \mathbb{N}\} \) and \( S_n = S^n \), where \( S_n \) is the right inverse of \( n^{-1}T^n \). So we get

\[
\frac{T^n}{n} e_j = \prod_{i=0}^{n-1} \frac{w_{i+j}}{n} e_{j+n} \to 0 \quad \text{forall } j \in \mathbb{N}.
\]

Furthermore, we have

\[
S_n e_j = S^n e_j = \frac{n}{\prod_{i=1}^{n} w_{j+i}} \to 0.
\]
Hence \( \frac{T^n}{n} S_n e_j - e_j \rightarrow 0 \) for all \( j \in \mathbb{N} \). Thus \( T \) satisfies the Cesàro-Hypercyclicity Criterion with respect to \( (n_j) = (n) \).

**Theorem 2.3.** Let \( T \in B(\mathcal{H}) \). Then the following (1) and (2) are equivalent:

1. \( T \) satisfies Cesàro-Hypercyclic Criterion.
2. (Outer Cesàro-Hypercyclic Criterion) There exist an increasing sequence \( (n_k) \) of positive integer, a dense linear subspace \( Y \subseteq \mathcal{H} \) and, for each \( y \in Y \), a dense linear subspace \( X_0 \) of \( \mathcal{H} \) such that:
   a. There exists a sequence of mappings \( S_{n_k} : Y_0 \rightarrow \mathcal{H}, \ k \in \mathbb{N} \) such that \( (n_k^{-1} T^{n_k} \circ S_{n_k}) y \rightarrow y \), for each \( y \in Y_0 \) and
   b. \( ||n_k^{-1} T^{n_k} x|| ||S_{n_k} y|| \rightarrow 0 \) For each \( y \in Y_0 \) and \( x \in X_0 \).

**Proof.** It is obvious that any operator satisfying the Cesàro-Hypercyclicity Criterion also satisfy the criteria of (2). It suffices to show that (2) implies (1). Let \( U_i, V_i \subseteq \mathcal{H} \) non-empty open sets with \( i = 1, 2 \). The same argument as in the proof of [2, Theorem 3.2] and [4, Theorem 2.5] can be used to show that there exist \( (n_k) \) of positive integer such that
   \[ (n_k^{-1} T^{n_k})^{-1} (U_i) \cap V_i \neq \emptyset, \text{ for } i = 1, 2. \]

Then we can know that \( (T \oplus T) \) is cesàro-hypercyclic for \( \mathcal{H} \oplus \mathcal{H} \) and \( (x, y) \) is cesàro-hypercyclic vector for \( (T \oplus T) \). In particular, \( x \) is cesàro-hypercyclic vector for \( T \) and \( CH(T) \) is a dense \( G_0 \) of \( \mathcal{H} \).

Let \( (U_k) \) be a base of 0-neighborhoods in \( \mathcal{H} \). Then there exist \( (n_k) \) of positive integer such that
   \[ n_k^{-1} T^{n_k} x \in U_k \text{ and } n_k^{-1} T^{n_k} y \in x + U_k \text{ for all } k \geq 1. \]

This implies that \( n_k^{-1} T^{n_k} x \rightarrow 0 \) and \( n_k^{-1} T^{n_k} y \rightarrow x \). Let \( M_0 = M_1 = \text{Orb}(T, x) \), which is dense in \( \mathcal{H} \). Also for all \( k \geq 1 \) define
   \[ S_{n_k} (n^{-1} T^n x) = n^{-1} T^n y. \]

Note that
   \[ n_k^{-1} T^n S_{n_k} (n^{-1} T^n x) = n_k^{-1} T^n (n_k^{-1} T^n y) = n^{-1} T^n (n_k^{-1} T^n y) \rightarrow n^{-1} T^n x. \]

Hence (1) holds. We complete the proof.\( \square \)

**Proposition 2.1.** Let \( T \in B(\mathcal{H}) \). The following statements are equivalent.

1. \( T \in CH(\mathcal{H}) \).
2. \( T \) is cesàro-topologically transitive.
3. For each \( x, y \in \mathcal{H} \), there exist sequences \( (x_k) \) in \( \mathcal{H} \), \( (n_k) \) in \( \mathbb{N} \), such that \( x_k \rightarrow x \) and \( n_k^{-1} T^{n_k} x_k \rightarrow y \).
4. For each \( x, y \in \mathcal{H} \), and each neighborhood \( W \) of the zero in \( \mathcal{H} \), there exist \( z \in \mathcal{H} \), \( n \geq 1 \) such that \( x - z \in W \) and \( n^{-1} T^n z - y \in W \).

**Proof.**

1 \( \Leftrightarrow \) 2: By Theorem 2.1 and Corollary 2.1.
2 \Rightarrow 3: \text{Let } x, y \in \mathcal{H}, \text{ and let } B(x, \frac{1}{k}), B(y, \frac{1}{k}) \text{ for all } k \geq 1. \text{ Then, there exist } (n_k) \text{ in } \mathbb{N}^* \text{ and } (x_k) \text{ in } \mathcal{H} \text{ such that } x_k \in B(x, \frac{1}{k}) \text{ and } n_k^{-1}T^n x_k \in B(y, \frac{1}{k}) \text{ for all } k \geq 1. \text{ Then } ||x_k - x|| < \frac{1}{k} \text{ and } ||n_k^{-1}T^n x_k - y|| < \frac{1}{k} \text{ for all } k \geq 1.

3 \Rightarrow 4: \text{Follows immediately from part (3).}

4 \Rightarrow 2: \text{Let } U \text{ and } V \text{ be two non-empty open subset of } \mathcal{H}. \text{ Let } W \text{ be a neighborhood for zero, pick } x \in U \text{ and } y \in V, \text{ so there exist } z \in \mathcal{H}, n \geq 1 \text{ such that } x - z \in W \text{ and } n^{-1}T^n z - y \in W. \text{ It follows immediately that } z \in U \text{ and } n^{-1}T^n z \in V.

\□

**Definition 2.4.** Let \( T \in B(\mathcal{H}) \). For every \( x_0 \in \mathcal{H} \) and \( n \geq 1 \) the sets

\[
L_{\text{mix}}(x_0) := \{ x_1 \in \mathcal{H} : n^{-1}T^n x_0 \to x_1 \}
\]

\[
J_{\text{mix}}(x_0) = \{ x_1 \in \mathcal{H} : \text{for every neighborhood } V_0, V_1 \text{ of } x_0, x_1 \text{ respectively, there exists } m \geq 1 \text{ such that } (n^{-1}T^n)(V_0) \cap V_1 \neq \emptyset \text{ for every } n \geq m \}
\]

will be called the cesàro-mixing limit set of \( x_0 \) under \( T \) and cesàro-mixing extended limit set of \( x_0 \) under \( T \) respectively.

**Proposition 2.2.** An equivalent definition for the set \( J_{\text{mix}}(x_0) \) is the following:

\[
J_{\text{mix}}(x_0) = \{ x_1 \in \mathcal{H} : \text{there exists a sequence } (x_n)_{n \geq 1} \text{ in } \mathcal{H} \text{ such that } x_n \to x_0 \text{ and } n^{-1}T^n x_0 \to x_1 \}
\]

**Proof.** Let us prove that

\[
J_{\text{mix}}(x_0) \subset \{ x_1 \in \mathcal{H} : \text{there exists a sequence } (x_n)_{n \geq 1} \text{ in } \mathcal{H} \text{ such that } x_n \to x_0 \text{ and } n^{-1}T^n x_0 \to x_1 \}
\]

Let \( x_1 \in J_{\text{mix}}(x_0) \) and consider the open balls

\[
V_0 = B(x_0, \frac{1}{n}), V_1 = B(x_1, \frac{1}{n}) \text{ centered at } x_0, x_1 \in \mathcal{H} \text{ and with radius } 1/n \text{ for } n \geq 1. \text{ Then there exists } m \geq 1 \text{ so that } (n^{-1}T^n)(V_0) \cap V_1 \neq \emptyset \text{ for every } n \geq m. \text{ Hence there exists } x_n \in V_0 = B(x_0, \frac{1}{n}) \text{ such that } n^{-1}T^n(x_n) \in V_1. \text{ Therefore there exists a sequence } (x_n) \text{ in } \mathcal{H} \text{ such that } x_n \to x_0 \text{ and } Tx_n \to x_1. \text{ The converse is obvious.}
\□

**Proposition 2.3.** Let \( T \in B(\mathcal{H}) \). For every \( x_0 \in \mathcal{H}, J_{\text{mix}}(x_0) = X. \text{ Then } T \text{ is cesàro-mixing.}

**Proof.** Let \( V_0, V_1 \) the nonempty open. Consider \( x_1 \in V_1 \). Since \( J_{\text{mix}}(x_0) = X. \text{ There exists } m \geq 1 \text{ such that } (n^{-1}T^n)(V_0) \cap V_1 \neq \emptyset \text{ for every } n \geq m. \text{ By definition } T \text{ is cesàro-mixing.}
\□

**Theorem 2.4.** Let \( S \in B(\mathcal{H}) \) and \( S^n = n^{-1}T^n \). If \( S \) is power bounded then \( J_{\text{mix}}(x_0) = L_{\text{mix}}(x_0) \) for every \( x_0 \in \mathcal{H} \).

**Proof.** Since \( S \) is power bounded, there exists a positive number \( M \) such that \( ||S^n|| \leq M \) for every positive integer \( n \geq 1 \). Let \( x_0 \in \mathcal{H} \). If \( J_{\text{mix}}(x_0) = \emptyset \) there is nothing to prove. Therefore assume that
Since the inclusion $L^{mix}(x_0) \subset J^{mix}(x_0)$ is always true, it suffices to show that $J^{mix}(x_0) \subset L^{mix}(x_0)$. Take $x_1 \in J^{mix}(x_0)$. There exist a sequence $(x_n)$ in $\mathcal{H}$ such that $x_n \to x_0$ and $S^n x_n \to x_1$. Then we have
\[
\|S^n x_0 - x_1\| \leq \|S^n x_0 - S^n x_n\| + \|S^n x_n - x_1\| \leq M\|x_0 - x_n\| + \|S^n x_n - x_1\|
\]
and letting $n$ goes to infinity to the above inequality, we get that $x_1 \in L^{mix}(x_0)$.

**Theorem 2.5.** Let $T$ and $S$ in $B(\mathcal{H})$ and $T \oplus S$ is cesàro-mixing operator, then $T$ and $S$ are cesàro-mixing operators, respectively.

**Proof.** Let $U_1$, $U_2$, $V_1$ and $V_2$ be open sets in $\mathcal{H}$, then $U_1 \oplus V_1$ and $U_2 \oplus V_2$ are open in $\mathcal{H} \oplus \mathcal{H}$. So there exists an $n_0 \geq 1$ such that
\[
\left(n^{-1} (T \oplus S)\right)^{-1} (U_1 \oplus V_1) \cap (U_2 \oplus V_2) \neq \emptyset.
\]
Then
\[
\left(n^{-1}T^n\right)^{-1} (U_1) \cap U_2 \neq \emptyset, \quad \left(n^{-1}S^n\right)^{-1} (V_1) \cap V_2 \neq \emptyset.
\]
Therefore, $T$ and $S$ are cesàro-mixing operators, respectively. \qed

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**References**


