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Characterization of Family of Rayleigh Distribution Through Record Values

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Abstract. An observation that shows greater than all the preceding observations, is called record. The characterization results via conditional expectation based on record values are expressed for family of Rayleigh distribution. Moreover, entropies based on distribution function are discussed and enumerated.

1. Introduction

The records are reported naturally in several fields of study e.g., sports, medical sciences, engineering, and traffic among others. [1] proposed and documented the applications of record values in literature. The detailed treatments on record values have been reported thoroughly, see [2,3] and [4] for instance.

Let $X_1, X_2, \cdots, X_r$ be a sequence of independent and identically distributed (IID) random variables (RVs) having cumulative distribution function (CDF) $F(x)$ and probability density function (PDF) $f(x)$. We consider upper and lower record values in this article.

1.1. **Upper Record:** The joint PDF of the r upper record (UR) values is

$$f(x_1, x_2, \cdots, x_r) = \prod_{i=1}^{r-1} f(x_r) \frac{f(x_i)}{\bar{F}(x_i)}, \quad -\infty < x_1 < x_2 < \cdots < x_r < \infty, \tag{1.1}$$

The marginal PDF of *rth* UR values is given as

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!} \left[-\ln \bar{F}(x)\right]^{r-1} f(x), \quad -\infty < x < \infty \tag{1.2}$$

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while joint PDF of $X_{U(r)}$ and $X_{U(s)}$ is

$$f_{U(r,s)}(x,y) = \frac{1}{\Gamma(r)\Gamma(s-r)}\left[-\ln(\bar{F}(x))\right]^{r-1}\left[-\ln(\bar{F}(y))+\ln(\bar{F}(x))\right]^{s-r-1} \times\frac{f(x)}{\bar{F}(x)}f(y), \quad -\infty < x_r < x_s < \infty. \tag{1.3}$$

Then, the conditional PDF of $(X_{U(s)}|X_{U(r)} = x)$ is mentioned below.

$$f(X_{U(s)}|X_{(U(r)} = x) = \frac{1}{\Gamma(s-r)}\left[-\ln(\bar{F}(y))+\ln(\bar{F}(x))\right]^{s-r-1}\frac{f(y)}{\bar{F}(x)}, \quad \text{for} \quad 1 \le r < s \tag{1.4}$$

Power Rayleigh (PR) distribution was introduced by [5]. A random variable (r.v.) X is said to have PR distribution, if its PDF$_{PR}$ and CDF$_{PR}$ are given respectively by

$$f(x) = \frac{\beta}{\alpha^2}x^{2\beta-1}e^{-\frac{x^{2\beta}}{2\alpha^2}}, \quad \alpha,\beta > 0,\ x > 0. \tag{1.5}$$

$$F(x) = 1-e^{-\frac{x^{2\beta}}{2\alpha^2}}, \quad \alpha,\beta > 0,\ x > 0. \tag{1.6}$$

The k^{th} moments via UR for PR distribution is

$$E(X^k_{U(n)}) = \frac{(2\alpha^2)^{\frac{k}{2\beta}}}{\Gamma(n)}\Gamma\left(\frac{k}{2\beta}+n\right). \tag{1.7}$$

The explorative study on PR distribution, see [5].

1.2. **Lower Record:** Lower record (LR) values have their own significance in specific problems of study. For example, marketing index, freestyle swimming, or marathon walks are interesting information for investors, participants, and observers. The joint PDF of the r LR values is

$$f(x_1,x_2,\cdots,x_r) = \prod_{i=1}^{r-1} f(x_r)\frac{f(x_i)}{F(x_i)}, \quad -\infty < x_1 < x_2 < \cdots < x_r < \infty. \tag{1.8}$$

The marginal PDF of *rth* LR values is given as

$$f_{X_{L(r)}}(x) = \frac{1}{(r-1)!}\left[-\ln F(x)\right]^{r-1} f(x), \quad -\infty < x < \infty, \tag{1.9}$$

and joint PDF of $X_{L(r)}$ and $X_{L(s)}$ is

$$f_{L(r,s)}(x,y) = \frac{1}{\Gamma(r)\Gamma(sr)}\left[-\ln(F(x))\right]^{r-1}\left[-\ln(F(y))+\ln(F(x))\right]^{s-r-1} \times\frac{f(x)}{F(x)}f(y), \quad -\infty < x_r < x_s < \infty. \tag{1.10}$$

The conditional PDF of $(X_{L(s)}|X_{L(r)} = x)$ is given as

$$f(X_{L(s)}|X_{L(r)} = x) = \frac{1}{\Gamma(s-r)}\left[-\ln(F(y))+\ln(F(x))\right]^{s-r-1}\frac{f(y)}{F(x)}, \quad \text{for} \quad 1 \le r < s. \tag{1.11}$$

The powered inverse Rayleigh (PIR) distribution is given by Nashaat [1]

$$f(x,\alpha,\theta) = \frac{2\alpha}{\theta x^{2\alpha+1}}e^{-\frac{1}{\theta x^{2\alpha}}},\ x > 0,\ \alpha,\theta > 0 \tag{1.12}$$

Table 1. The characteristics of means and variances for PRD are based on UR values.

$E(X^k_{U(1)})$							$E(X^k_{U(2)})$						
$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0	$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0
0.5	2.5	0.62	0.71	0.76	0.80	0.82	0.5	1.0	0.93	0.94	0.95	0.96	0.96
1.0	2.0	1.26	1.12	1.07	1.06	1.03	1.0	4.0	1.87	1.49	1.34	1.26	1.21
1.5	4.5	1.89	1.47	1.31	1.24	1.18	1.5	9.0	2.80	1.96	1.65	1.49	1.39
2.0	8.0	2.52	1.78	1.51	1.39	1.30	2.0	16.0	3.73	2.38	1.90	1.67	1.53
2.5	12.5	3.15	2.07	1.69	1.52	1.40	2.5	25.0	4.67	2.76	2.12	1.82	1.64
3.0	18	3.78	2.33	1.85	1.64	1.49	3.0	36.0	5.60	3.11	2.33	1.96	1.75
$E(X^k_{U(3)})$							$E(X^k_{U(4)})$						
$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0	$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0
0.5	1.5	1.17	1.102	1.07	1.05	1.04	0.5	2.0	1.37	1.22	1.16	1.12	1.10
1.0	6.0	2.35	1.74	1.51	1.39	1.31	1.0	8.0	2.74	1.93	1.64	1.48	1.39
1.5	13.5	3.52	2.29	1.85	1.63	1.50	1.5	18	4.11	2.55	2.00	1.74	1.59
2.0	24.0	4.69	2.78	2.13	1.83	1.65	2.0	32	5.48	3.09	2.32	1.96	1.75
2.5	37.5	5.87	3.22	2.39	2.00	1.78	2.5	50	6.85	3.58	2.59	2.14	1.88
3.0	54.0	7.04	3.64	2.61	2.16	1.89	3.0	72	8.22	4.04	2.84	2.30	2.00
$Var(X^k_{U(1)})$							$Var(X^k_{U(2)})$						
$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0	$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.25	0.10	0.07	0.06	0.03	0.03	0.5	0.5	0.14	0.06	0.03	0.02	0.02
1.0	4.0	0.42	0.17	0.11	0.05	0.06	1.0	8.0	0.50	0.17	0.07	0.05	0.03
1.5	20.25	0.94	0.29	0.17	0.06	0.07	1.5	40.5	1.16	0.24	0.08	0.04	0.03
2.0	64	1.66	0.43	0.22	0.08	0.09	2.0	128	2.07	0.34	0.13	0.06	0.04
2.5	156.25	2.60	0.60	0.28	0.09	0.10	2.5	312.5	3.19	0.46	0.18	0.09	0.06
3.0	324	3.75	0.74	0.34	0.10	0.11	3.0	648	4.64	0.58	0.17	0.10	0.06
$Var(X^k_{U(3)})$							$Var(X^k_{U(4)})$						
$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0	$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.75	0.13	0.04	0.04	0.02	0.02	0.5	1	0.12	0.05	0.02	0.02	0.01
1.0	12	0.48	0.16	0.06	0.03	0.02	1.0	16	0.49	0.17	0.05	0.03	0.04
1.5	60.75	1.10	0.21	0.09	0.06	0.04	1.5	81	1.19	0.18	0.11	0.05	0.02
2.0	192	1.97	0.27	0.16	0.07	0.06	2.0	256	1.97	0.31	0.09	0.04	0.02
2.5	468.75	3.04	0.42	0.16	0.09	0.06	2.5	625	2.39	0.37	0.14	0.06	0.05
3.0	972	4.42	0.48	0.20	0.11	0.07	3.0	1296	4.43	0.48	0.16	0.07	0.04

The CDF of PIR distribution is

$$F(x,\alpha,\theta) = e^{-\frac{1}{\theta x^{2\alpha}}},\ x > 0,\ \alpha,\theta > 0. \tag{1.13}$$

The k^{th} moments via LR for PIR distribution is

$$E(X^k_{L(n)}) = \frac{(\theta)^{\frac{k}{2\alpha}-1}}{(n-1)!}\Gamma\left(\frac{k}{2\alpha}+n\right). \tag{1.14}$$

Table 2. The characteristics of means and variances for PIRD based on LR values at $\theta = 1$.

$E(X_{L(n)})$								$Var(X_{L(n)})$							
$\alpha\backslash n$	1	2	3	4	5	6	7	$\alpha\backslash n$	1	2	3	4	5	6	7
1	0.89	1.33	1.66	1.94	2.18	2.40	2.60	1	0.21	0.23	0.24	0.24	0.25	0.24	0.24
2	0.91	1.13	1.28	1.38	1.47	1.54	1.60	2	0.06	0.05	0.02	0.03	0.019	0.028	0.04
3	0.93	1.08	1,18	1.24	1.29	1.33	1.37	3	0.03	0.02	0.09	0.002	0.005	0/071	0.003
4	0.94	1.06	1.13	1.17	1.21	1.23	1.27	4	0.03	0.006	0.003	0.01	0.006	0.027	0.012

2. Characterizations

Characterization properties confirm a distribution is mathematically sound and informative and support practicing researchers in deciding on the applicability of the model in introducing it to a population from which samples are available. This important method spread in various directions. e.g., recurrence relation, truncation, hazard rate function, conditional expectation, and many more. Several authors have paid attention to characterizing the distribution via a conditional approach. A list of contributors is given to the readers for detailed review, see [7]- [17]. The following relations between PDFs and CDFs are satisfied.

$$f(x) = \frac{\beta}{\alpha^2}x^{2\beta-1}\bar{F}(x). \tag{2.1}$$

$$f(x) = \frac{2\alpha}{\theta x^{2\alpha+1}}F(x) \tag{2.2}$$

Theorem 2.1. *Let $X \sim PRD(\alpha, \beta)$ with CDF_{PR} and PDF_{PR}. Then for $1 \leq r < s$*

$$E[\Psi(X_{U(s)}|X_{U(r)} = x)] = x^{2\beta} + 2(s-r)\alpha^2, \quad x \in (0, \infty), \tag{2.3}$$

where $\varphi = y^{2\beta}$.

Proof. From (1.3), we have

$$E[\varphi(X_{U(s)}|X_{U(r)} = x)] = \frac{1}{\Gamma(s-r)}\int_x^\infty \varphi\left[-\ln(\bar{F}(y)) + \ln(\bar{F}(x))\right]^{s-r-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{2.4}$$

Using PDF_{PR} (1.5) and CDF_{PR} (1.6) in (2.4), we have

$$= \frac{1}{\Gamma(s-r)}\int_x^\infty y^{2\beta}\left[\frac{y^{2\beta}}{2\alpha^2} - \frac{x^{2\beta}}{2\alpha^2}\right]^{s-r-1}\frac{\beta y^{2\beta-1}}{\alpha^2}e^{-\left[\frac{y^{2\beta}}{2\alpha^2}-\frac{x^{2\beta}}{2\alpha^2}\right]}dy. \tag{2.5}$$

Letting $t = \frac{y^{2\beta}}{2\alpha^2} - \frac{x^{2\beta}}{2\alpha^2}$. Then (2.5) reduces to

$$= \frac{1}{\Gamma(s-r)} \int_0^\infty \left[2\alpha^2 t + x^{2\beta}\right] t^{s-r-1} e^{-t} dt. \tag{2.6}$$

Simplifying (2.6), it leads to (2.3).

For the sufficiency part, we began with,

$$g_{s|r}(x) = x^{2\beta} + 2(s-r)\alpha^2 \tag{2.7}$$

$$\frac{1}{\Gamma(s-r)} \int_x^\infty \varphi\left[-\ln(\bar{F}(y)) + \ln(\bar{F}(x))\right]^{s-r-1} f(y)dy = g_{s|r}(y)\bar{F}(x). \tag{2.8}$$

Differentiating above expression w.r.t. x,

$$\frac{1}{\Gamma(s-r-1)} \int_x^\infty \varphi\left[-\ln(\bar{F}(y)) + \ln(\bar{F}(x))\right]^{s-r-2} \frac{f(y)}{\bar{F}(x)} f(y)dy = g'_{s|r}(x)\bar{F}(x) + f(x)g_{s|r}(x)$$

$$\begin{aligned} g_{s|r+1}(x) &= g'_{s|r}(x)\frac{\bar{F}(x)}{f(x)} + g_{s|r}(x) \\ \frac{f(x)}{\bar{F}(x)} &= \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = \frac{\beta}{\alpha^2} x^{2\beta-1} \\ \bar{F}(x) &= e^{-x^{\frac{2\beta}{2\alpha^2}}}. \end{aligned}$$

Theorem 2.1 is completed. □

Corollary 2.1. *As the conditions mentioned in Theorem 2.1 and for* $1 \le r \le s < t$

$$E[\varphi(X_{U(t)} - X_{U(s)}|X_{U(r)} = x)] = 2(t-s)\alpha^2$$

if and only if (1.6) *holds.*

Proof. Proof is easy, hence omitted. □

Theorem 2.2. *Let* $X \sim PIRD(\alpha, \theta)$ *with* CDF_{PIR} $F(x)$ *and* PDF_{PIR} $f(x)$*. Then for* $1 \le r < s$

$$E[\psi(X_{U(s)}|X_{U(r)} = x)] = x^{-2\alpha} + (s-r)\theta,\ x \in (0, \infty). \tag{2.9}$$

where $\psi = y^{-2\alpha}$.

Proof. From (1.11), we have

$$E[\psi(X_{L(s)}|X_{L(r)} = x)] = \frac{1}{\Gamma(s-r)} \int_0^x \psi\left[-\ln(F(y)) + \ln(F(x))\right]^{s-r-1} \frac{f(y)}{F(x)} dy. \tag{2.10}$$

Using PDF_{PIR} (1.12) and CDF_{PIR} (1.13) in (2.10), we have

$$= \frac{1}{\Gamma(s-r)} \int_0^x y^{-2\alpha} \left[\frac{1}{\theta y^{2\alpha}} - \frac{1}{\theta x^{2\alpha}}\right]^{s-r-1} \frac{2\alpha}{\theta y^{2\alpha+1}} e^{-\left[\frac{1}{\theta y^{2\alpha}} - \frac{1}{\theta x^{2\alpha}}\right]} dy. \tag{2.11}$$

Letting $t = \frac{1}{\theta y^{2\alpha}} - \frac{1}{\theta x^{2\alpha}}$, then (2.11) reduces to

$$= \frac{1}{\Gamma(s-r)} \int_0^\infty \left(t + \frac{1}{\theta x^{2\alpha}}\right) t^{s-r-1} e^{-t} dt. \tag{2.12}$$

Simplifying (2.12), it leads to (2.9).

For the sufficiency part, we consider.

$$g_{s|r}(x) = x^{-2\alpha} + (s-r)\theta \tag{2.13}$$

$$\frac{1}{\Gamma(s-r)} \int_0^x \psi \left[-\ln(F(y)) + \ln(F(x))\right]^{s-r-1} f(y)dy = g_{s|r}(x)F(x). \tag{2.14}$$

Differentiating (2.14) w.r.t. x, we obtain,

$$-\frac{1}{\Gamma(s-r-1)} \int_0^x \psi \left[-\ln(F(y)) + \ln(F(x))\right]^{s-r-2} \frac{f(y)}{F(x)} f(y)dy = g'_{s|r}(x)F(x) + f(x)g_{s|r}(x)$$

$$\begin{aligned} g_{s|r+1}(x) &= g'_{s|r}(x)\frac{F(x)}{f(x)} + g_{s|r}(x) \\ \frac{f(x)}{F(x)} &= \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = \frac{2\alpha}{\theta x^{2\alpha+1}}. \end{aligned}$$

which is (2.2) hence, Theorem 2.2 is completed. □

Corollary 2.2. *As the conditions mentioned in Theorem 2.2 and for* $1 \le r \le s < t$,

$$E[\Psi(X_{L(t)} - X_{L(s)}|X_{L(r)} = x)] = (t-s)\theta$$

if and only if $X \sim PIRD(\alpha, \theta)$.

3. Entropy

[18] introduced the concept of entropy based on the PDF. There are several extensions of Shannon entropy reported in the literature. There are some situations where Shannon entropy is not suitable to calculate the uncertainty for the event. To counter this problem, an alternative measure of entropy based on the CDF are reported, which is mentioned below.

3.1. **Cumulative Residual Entropy.** The cumulative residual entropy (CRE) based on survival function is defined as (see, [19]).

$$CRE(X) = -\int_0^\infty \bar{F}(x) \ln \bar{F}(x) dx.$$

Theorem 3.1. *Let* $X \sim PRD(\alpha, \beta)$ *has the finite mean. Then the CRE of*X *can also be expressed in terms of UR values as follows*

$$CRE(X) = E(X_{U(2)}) - \mu, \tag{3.1}$$

where

$$\mu = (2\alpha^2)^{\frac{1}{2\beta}} \Gamma\left(\frac{1}{2\beta} + 1\right).$$

Proof. The CRE is

$$\begin{aligned} CRE(X) &= -\int_0^\infty \bar{F}(x)\ln \bar{F}(x)dx \\ &= -\bar{F}_{PR}(x)\ln \bar{F}_{PR}(x)x\Big|_0^\infty - \int_0^\infty x\left[\ln \bar{F}_{PR}(x) f_{PR}(x) + f_{PR}(x)\right]dx \\ &= -\int_0^\infty x\left[\ln \bar{F}_{PR}(x)\right] f_{PR}(x)dx - \int_0^\infty x f_{PR}(x)dx \end{aligned} \tag{3.2}$$

Upon simplification (3.2), we obtain (3.1). □

Table 3. CRE based on specific values of α and β.

$\alpha\backslash\beta$	0.5	1.0	1.5	2.0	2.5	3.0
0.5	0.50	0.30	0.23	0.19	0.16	0.14
1.0	2.00	0.61	0.37	0.27	0.20	0.18
1.5	4.50	0.91	0.49	0.34	0.25	0.21
2.0	8.00	1.23	0.60	0.39	0.28	0.23
2.5	12.50	1.53	0.70	0.43	0.30	0.24
3.0	18.00	1.82	0.78	0.48	0.32	0.26

3.2. **Cumulative Entropy.** The cumulative entropy (CE) based on distribution function is defined as (see, [20]).

$$CE(X) = -\int_0^\infty F(x)\ln F(x)dx.$$

Therefore, CE for PIRD is given as

$$CE(X) = \frac{\theta^{\frac{1}{2\alpha}-1}}{2\alpha}\Gamma\left(\frac{1}{2\alpha}+1\right).$$

Table 4. Values of CE for different parameters.

$\alpha\backslash\theta$	1	2	3	4	5
1	0.445	0.315	0.258	0.223	0.200
2	0.227	0.136	0.099	0.079	0.068
3	0.155	0.086	0.062	0.048	0.040
4	0.117	0.064	0.044	0.035	0.028

Conclusion

The family of Rayleigh distribution is successfully characterized by conditional expectation based on upper-lower record values. The entropies are also tabulated for these distributions. It is expected the proposed technique will attract more consideration in the field of mathematical statistics.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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