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New Auxiliary Principle Technique for General Harmonic Directional Variational Inequalities

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Abstract. This paper explores the utilisation of harmonic variational inequalities to establish the minimum value among two locally Lipschitz continuous harmonic convex functions. This investigation introduces novel classes of harmonic directed variational inequalities, particularly focusing on scenarios like harmonic complementarity and related optimization challenges. The study proposes and analyses various inertial iterative strategies for addressing harmonic directed variational inequalities through the auxiliary principle technique. It examines convergence criteria under specific weak conditions, emphasising the simplicity of the approach compared to other methodologies. The findings presented herein have broad applicability in the context of harmonic variational inequalities and optimization problems, though they are limited to theoretical exploration. Further research is required to implement these strategies numerically.

1. Introduction

Stampacchia's variational inequalities [43] have opened up innovative applications of variational principles, bridging diverse fields such as mathematics, physics, economics, computer science, data analysis, finance, and engineering. Variational inequalities serve as optimality conditions for various convex functions defined on convex sets, offering a versatile framework for addressing linear and nonlinear problems. Harmonic variational inequalities, as demonstrated by Noor and Noor [22], represent the minimum value of differentiable harmonic convex functions within harmonic convex sets. Indeed, this theory offers a natural, direct, straightforward, unified, and efficient framework for addressing a broad spectrum of both linear and nonlinear problems that may initially appear unrelated. A major contribution by Noor and Noor [22] lays down

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an important connection between harmonic variational inequalities and differentiable harmonic convex functions. This relationship consists of a major elongation of variational inequalities, thus improving the fluidity and appropriateness of the theory.

Convexity theory is used in many fields in the pure and applied sciences, where it is very helpful in solving complex and problems with many sides. This theory's concepts and methods very often need an extension, revision, and adaptation in order to make room for new ideas and handle changing problems. The concept of g-convex sets (general convex sets)was first introduced by Noor [20] in relation to an arbitrary operator g. Cristescu et al. [9,11] discusses how important these sets are when solving problems related to optimisation. These applications cover a broad range of fields, such as computer-aided design, and processing of the images for example.

It is important to note that general convex sets and harmonic convex sets are different extensions of older convex sets. Both have proven useful in solving different kinds of issues. This paper presents the understanding of general harmonic convex sets and functions, accepting the possibility of standardisation as well as accounting for arbitrary operators.

This endeavor leads to the exploration of general harmonic directional harmonic variational inequalities, involving the interplay of two bifunctions. The paper meticulously examines significant special cases of these variational inequalities, revealing their intricate structural characteristics.

In the context of solving general harmonic directional inequalities, conventional projection methods and their variants prove to be less effective due to the inherent complexities of these problems. To overcome this challenge, we adopt the auxiliary principle technique, a methodology pioneered by Glowinski et al. [13]. This technique involves the identification of an auxiliary variational inequality and a subsequent demonstration that the solution to this auxiliary problem coincides with the solution to the original problem. Notably, this technique extends its applicability to the derivation of equivalent differentiable optimization problems, enabling the construction of gap (merit) functions. We propose and analyze a range of iterative methods for solving variational inequalities, building upon the foundational work presented in references [1,22,27,36,37]. This paper can be viewed as continuation of our earlier work [1,27]. This paper capitalizes on the auxiliary principle technique, incorporating arbitrary operators to suggest and scrutinize novel classes of inertial iterative methods for solving harmonic directional variational inequalities. The concept of inertial methods, initially introduced by Polyak [41], aims to expedite the convergence of iterative algorithms. Notably, we establish that the convergence of these innovative methods hinges upon the condition of pseudomonotonicity, a requirement that is less restrictive than strict monotonicity. It is imperative to underscore that our results encompass both well-established findings and new contributions pertaining to harmonic variational inequalities, variational inequalities, and closely related optimization problems. These results signify a substantial advancement and refinement of existing knowledge in the realm of harmonic variational inequalities and their various manifestations. The interdisciplinary nature of the subject adds to its allure, as it draws upon numerous branches of both pure and applied sciences, enriching the theory and its practical applications.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $||.||$ respectively. Let C be a nonempty closed convex set in $\mathcal{H}$. Let $j : \mathcal{H} \longrightarrow R$ be a locally Lipschitz continuous function. For the sake of completeness and to convey the main ideas, we include the relevant results from the convex and nonsmooth analysis [6, 8, 41].

Definition 2.1. *[6] Let j be locally Lipschitz continuous at a given point $x \in \mathcal{H}$ and v be any other vector in $\mathcal{H}$. The Clarke's generalized directional derivative of j at x in the direction v, denoted by $j^0(x; v)$, is defined as*

$$f^0(x; v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$

The generalized gradient of j at x, denoted $\partial j(x)$, is defined to be subdifferential of the function $j^0(x; v)$ at 0. *That is*

$$\partial j(x) = \{w \in \mathcal{H} : \langle w, v \rangle \leq j^0(x; v), \quad \forall v \in \mathcal{H}.\}$$

Lemma 2.1. *[6] Let j be a locally Lipschitz continuous at a given point $x \in \mathcal{H}$ with a constant L. Then*

(i). $\partial j(x)$ is a none-empty compact subset of $\mathcal{H}$ and $||\xi|| \leq L$ for each $\xi \in \partial j(x)$.

(ii). For every $v \in \mathcal{H}$, $\quad j^0(x; v) = max\{\langle \xi, c \rangle : \xi \in \partial j(x)\}$.

(iii). The function $v \longrightarrow j^0(x; v)$ is finite, positively homogeneous, subadditive, convex and continuous.

(iv). $j^0(x; -v) = (-j)^0(x; v)$.

(v). $j^0(x; v)$ is upper semicontinuous as a function of $(x; v)$.

(vi). $\forall x \in \mathcal{H}$, there exists a constant $\alpha > 0$ such that

$$|j^0(x; v)| \leq \alpha ||v||, \quad \forall v \in \mathcal{H}.$$

If j is convex on C and locally Lipschitz continuous at $x \in C$, then $\partial j(x)$ coincides with the subdifferential $j'(x)$ of j at x in the sense of convex analysis , and $j^0(x; v)$ coincides with the directional derivative $j'(x; v)$ for each $v \in H$, that is, $j^0(x; v) = \langle j'(x), v \rangle$.

Definition 2.2. *[8, 41] The set $C \subseteq \mathcal{H}$ is said to be a convex set, if*

$$u + \lambda(v - u) \in C, \quad \forall u, v \in C.$$

The ideas and techniques of the convexity are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. In many complicated problems, these concepts have to generalize and extend using some novel ideas and techniques.

Noor [20] introduced and studied the new convex sets by replacing linear structure by the straight-line segment joining two points of a given set by a displaced straight-line segment.

Definition 2.3. *[20] The set $C_g \subseteq \mathcal{H}$ is said to be general (g-convex) convex set with respect to an arbitrary function g, if*

$$u + \lambda(g(v) - u) \in C_g, \quad \forall u, v \in C_g.$$

Cristescu et a. [10] called the general convex (g-convex) set as the Noor-convex set. If $g(v) = mv$, where m i a constant, then the general convex set reduces to m-convex set, introduced and studied by Toader [44]. For the properties and applications of the m-convex sets and m-convex functions, see [40]. The concept of g-convex (Noor-convex) set C_g differs from that of $\mathcal{E}$-convex set introduced by Youness [45]. Cristescu et al. [9–11] have studied the applications of Noor-convex sets in vectorial optimization problems such as ecologic-economic efficiency, railway transport system,image processing. Cristescu et. al. [11] compared these concepts the using the digitization method of the plane $\mathcal{R}^2$ into the grid $\mathcal{Z}^2$.

Definition 2.4. *[8,41] The function ϕ on the convex set C is said to be convex, if*

$$\phi(u + \lambda(v - u)) \leq (1 - \lambda)\phi(u) + \lambda\phi(v), \quad \forall u, v \in C.$$

It is known that the minimum $u \in C$ of the differentiable convex function ϕ is equivalent to fining $u \in C$ such that

$$\langle \phi'(u), v - u \rangle \geq 0, \quad \forall v \in C, \tag{2.1}$$

which is called the variational inequality.

Definition 2.5. *[41] The function ϕ on the convex set C is said to be strongly convex, if there exists a constant $\alpha \geq 0$ such that*

$$\phi(u + \lambda(v - u)) \leq (1 - \lambda)\phi(u) + \lambda\phi(v) \geq \alpha\|v - u\|^2, \quad \forall u, v \in C.$$

For the differentiable strongly convex, we have:

Lemma 2.2. *[41] A differentiable function ϕ is strongly convex, if and only if,*

$$\langle \phi'(v) - \phi'(u), v - u \rangle \geq \alpha\|v - u\|^2, \quad \forall u, v \in C.$$

For the differentiable convex functions, Bregman [5] introduced the distance function

$$\begin{aligned} B(v,u) &= \phi(v) - \phi(u) \geq \langle \phi'(u), v - u \rangle \quad \forall u, v \in C \\ &= \phi(v) - \phi(u) - \langle \phi'(u), v - u \rangle \geq \alpha\|v - u\|^2, \quad \forall u, v \in C, \end{aligned}$$

which is known as the Bregman distance function and has applications in entropy, data analysis, information technology, machine learning and variational inequalities.
Applying the Lemma 2.2, Noor et al. [27] introduced the following new distance function

$$M(v,u) = \langle M(v) - M(u), v - u \rangle, \quad \forall u, v \in C,$$

or equivalently for strongly monotone operator M with constant $\alpha \geq 0$ as

$$M(v,u) = \langle M(v) - M(u), v - u\rangle \geq \alpha \|v - u\|^2, \quad \forall u, v \in C,$$

which is called the modified distance function.
Clearly for $M = \phi'$, both the distance function are equal, that is. $M(.,.) = B(.,.)$ It is an interesting open problem to explore the applications of the modified distance function in information sciences, entropy, machine learning, data analysis and variational inequalities.

Definition 2.6. *[20] The function ϕ on the general convex set C_g is said to be general (g-convex) convex with respect to an arbitrary function g, if*

$$\phi(u + \lambda(g(v) - u)) \leq (1-\lambda)\phi(u) + \lambda\phi(g(v)), \quad \forall u, v \in C_g.$$

Clearly, every convex function is a general convex function, but the converse is not true. For $g(v) = mv$, the general homogenous convex function reduces to:

$$\phi(u + \lambda(mv - u)) \leq (1-\lambda)\phi(u) + \lambda\phi(mv), \quad \forall u, v \in C_m,$$

is called the m- convex function, which is quite different form the concept of Toader [44].
It has been shown [19] that $u \in C_g$ is the minimum of the differentiable generalized convex function ϕ, if and only if, $u \in C_g$ satisfies the inequality

$$\langle \phi'(u), g(v) - u\rangle \geq 0, \quad \forall v \in C_g, \tag{2.2}$$

which is called the general variational inequality. It is worth mentioning that the inequalities (2.1) and (2.2) are quite and distinctly different from each other and have applications in various fields of pure and applied sciences.

Definition 2.7. *[4,14] The set $C_h \subseteq \mathcal{H}$ is said to be a harmonic convex set, if*

$$\frac{uv}{v + \lambda(u - v)} \in C_h, \quad \forall u, v \in C_h, \quad \lambda \in [0,1].$$

For the applications of the harmonic means in circuit theory, numerical analysis, risk analysis and related optimization programming, see [2,4,11,26].

Definition 2.8. *[4,14] The function ϕ on the harmonic convex set C_h is said to be harmonic convex, if*

$$\phi(\frac{uv}{v + \lambda(u - v)}) \leq (1-\lambda)\phi(u) + \lambda\phi(v), \quad \forall u, v \in C_h \quad \lambda \in [0,1].$$

The function ϕ is said to be harmonic concave function, if and only if, $-\phi$ is harmonic convex function.

We recall that the minimum of a differentiable harmonic convex function on the harmonic convex set C_h can be characterized by the variational inequality. This is result is due to Noor and

Noor [24]. For the differentiable harmonic convex function ϕ, $u \in C_h$ is a minimum of ϕ, if and only if, $u \in C_h$ satisfies the inequality

$$\langle \phi'(u), \frac{uv}{u-v} \rangle \geq 0, \quad \forall v \in C_h. \tag{2.3}$$

The inequality of type (2.3) is called the harmonic variational inequality. For the motivation, applications, numerical results and other aspects of the harmonic variational inequalities, see [1, 15, 24–26, 30, 32]

From the above discussion, we note that the convex sets and convex functions have been generalized in different way to tackle the complicated problems. All these concepts are different and distinct from each other. It is natural to unify these concepts. We introduce and study some new classes of harmonic convex sets and harmonic convex functions, which is the main aim of this paper.

Definition 2.9. *The set $C_{hg} \subseteq \mathcal{H}$ is said to be general harmonic convex set with respect to an arbitrary function g, if*

$$\frac{ug(v)}{g(v) + \lambda(u - g(v)} \in C_{hg}, \quad \forall u, v \in C_{hg}, \quad \lambda \in [0, 1].$$

Definition 2.10. *The function ϕ on the general harmonic convex set C_{hg} is said to be general harmonic convex with respect to an arbitrary function g, if*

$$\phi(\frac{ug(v)}{g(v) + \lambda(u - g(v))}) \leq (1 - \lambda)\phi(u) + \lambda\phi(g(v)), \quad \forall u, v \in C_{hg}, \quad \lambda \in [0, 1].$$

The function ϕ is said to be general harmonic concave function, if and only if, $-\phi$ is a general harmonic convex function.

We prove that the minimum of the locally Lipschitz continuous general harmonic function on the general harmonic convex set C_{hg} can be characterized by the variational inequality.

Theorem 2.1. *Let ϕ be a locally Lipschitz harmonic convex function on the general harmonic convex set C_{hg}. Then $u \in C_{hg}$ is a minimum of ϕ, if and only if, $u \in C_{hg}$ satisfies the inequality*

$$\phi'(u, \frac{ug(v)}{u - g(v)}) \geq 0, \quad \forall v \in C_{hg}, \tag{2.4}$$

which is called the general harmonic directional variational inequality.

Proof. Let $u \in C_{hg}$ is a minimum of a locally Lipschitz continuous general harmonic convex function ϕ. Then

$$\phi(u) \leq \phi(g(v)), \quad \forall v \in C_{hg}. \tag{2.5}$$

Since C_{hg} is a general harmonic convex set, so $\forall u, v \in C_{hg}, \quad g(v)_\lambda = \frac{ug(v)}{u+\lambda(u-g(v))} \in C_{hg}$. Replacing $g(v)$ by $g(v)_\lambda)$ in (2.5) and diving by λ and taking limit as $\lambda \to 0$, we have

$$0 \leq \frac{\phi(\frac{ug(v}{g(v)_\lambda+\lambda(u-g(v))}) - \phi(u)}{\lambda} = \phi'(u, \frac{ug(v)}{u-g(v)}).$$

the required result (2.4).
Conversely, let the function ϕ be a general harmonic convex function on the general harmonic convex set C_{hg}. Then

$$\frac{ug(v)}{g(v)+\lambda(u-g(v))} \leq (1-\lambda)\phi(u) + \lambda\phi(g(v)) = \phi(u) + \lambda(\phi(g(v)) - \phi(u)),$$

which implies that

$$\phi(g(v)) - \phi(u) \quad \geq \quad \lim_{\lambda \to 0} \frac{\phi(\frac{ug(v)}{g(v)+\lambda(u-g(v))}) - \phi(u)}{\lambda} = \phi'(u, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \text{using} \quad (2.4).$$

Consequently, it follows that

$$\phi(u) \leq \phi(g(v)), \quad \forall v \in C_{hg}.$$

This shows that $u \in C_h$ is the minimum of a locally Lipschitz continuous harmonic convex function.

□

We would like to mention that Theorem 2.1 implies that the locally Lipschitz continuous harmonic optimization programming problem can be studied via the general harmonic directional variational inequality (2.4).
Using the ideas and techniques of Theorem 2.1, we can derive the following result.

Theorem 2.2. *Let ϕ be a locally Lipschitz continuous harmonic convex functions on the general harmonic convex set C_{hg}. Then*

$$(i). \qquad \phi g(v)) - \phi(u) \geq \phi'(u, \frac{ug(v)}{u-g(v)}), \quad \forall u, v \in C_{hg}.$$

$$(ii). \qquad \phi'(u, \frac{ug(v)}{v-g(u)}) + \phi'(g(v), \frac{ug(v)}{g(v)-u}), \leq 0, \quad \forall u, v \in C_{hg}.$$

Motivated by Theorem 2.1 and Theorem 2.2, we introduce some new concepts.

Definition 2.11. *A bifunction $B(.,.)$ is said to be a general harmonic monotone bifunction with respect to an arbitrary operator g, if and only if,*

$$B(u, \frac{ug(v)}{u-g(v)}) + B(g(v), \frac{ug(v)}{u-g(v)}), \leq 0 \quad \forall u, v \in \mathcal{H}.$$

Definition 2.12. *A bifunction $B(.,.)$ is said to a general harmonic pseudomonotone bifunction with respect to the bifunction $W(.,.)$, if*

$$B(g(v), \frac{ug(v)}{u-g(v)}) + W(g(v), \frac{ug(v)}{u-g(v)}) \leq 0, \quad \forall v \in \mathcal{H}.$$

A general harmonic monotone bifunction is a general harmonic pseudomonotone bifunction, but the converse is not true.

Definition 2.13. *An operator T is said to be strongly monotone, if there exist a constant $\xi \geq 0$, such that*

$$\langle Tv - Tu, v - u \rangle \geq \xi \|v - u\|^2, \quad \forall u, v \in \mathcal{H}.$$

and Lipschitz continuous, if there exists a constant $\zeta \geq 0$, such that

$$\|Tv - Tu\| \leq \zeta \|v - u\|, \quad \forall u, v \in \mathcal{H}.$$

Consider the energy (virtual) functional

$$I[v] = F(v) - \phi(v), \quad \forall v \in \mathcal{H}. \tag{2.6}$$

where $F(v)$ and $\phi(v)$ are two harmonic convex functions. The problem (2.6) is called the difference of two harmonic functions. One can explore the characterization and applications of this problem in various branches of pure and applied sciences such as stock exchange, machine learning, data analysis and information technology.
We now consider the optimality conditions of the energy function $I[v]$ defined by (2.6) using the technique of Theorem 2.1.

Theorem 2.3. *Let F and ϕ be locally Lipschitz continuous harmonic convex functions on the general harmonic convex set C_{hg}. If $u \in C_{hg}$ is the minimum of the functional $I[v]$ defined by (2.6), then*

$$F'(u, \frac{ug(v)}{u - g(v)}) - \phi'(u, \frac{ug(v)}{u - g(v)}) \geq 0, \quad \forall v, u \in C_{hg}. \tag{2.7}$$

In many applications, the inequalities of the type (2.7) may not arise as the minimum of the sum of the two locally Lipschitz continous harmonic convex functions. These facts motivated us to consider more general harmonic variational inequality, which contains the inequalities (2.7) as a special case.

For given nonlinear continuous bifunctions $B(.,.), W(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$, and operator $g : \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $u \in C_{hg}$ such that

$$B(u, \frac{ug(v)}{u - g(v)}) + W(u, \frac{ug(v)}{u - g(v)}) \geq 0, \quad \forall v \in C_{hg}, \tag{2.8}$$

which is called the general harmonic directional variational inequality.

Special Cases. We now discuss some new and known classes of general harmonic directional variational inequalities(GHDVI) and related optimization problems.

(I). if $B(u, \frac{ug(v)}{u-g(v)}) = \langle Tu, \frac{ug(v)}{u-g(v)}) \rangle$ and $W(u, \frac{ug(v)}{u-g(v)}) = \langle A(u), \frac{ug(v)}{u-g(v)} \rangle$, then the problem (2.8) reduces to finding $u \in C_{hg}$ such that

$$\langle Tu, \frac{ug(v)}{u-g(v)}\rangle + \langle A(u), \frac{ug(v)}{u-g(v)}\rangle \geq 0, \quad \forall v \in C_{hg}, \tag{2.9}$$

which is called the general harmonic variational inequality .

(II). If $\langle A(u), \frac{ug(v)}{u-g(v)}\rangle = \phi'(u, \frac{ug(v)}{u-g(v)})$, where $\phi'(u)$ denotes directional derivative of the general harmonic convex function $\phi(u)$ in the direction $\frac{ug(v)}{u-g(v)}$, then problem (2.9) reduces to finding $u \in C_{hg}$, such that

$$\langle Tu, \frac{ug(v)}{u-g(v)}\rangle + \phi'(u, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg}, \tag{2.10}$$

which is also called the harmonic directional variational inequality.

(III). For $\langle A(u), \frac{ug(v)}{u-g(v)}\rangle = J^0(u, \frac{ug(v)}{u-g(v)})$, where $J^0(.,.)$ is the directional derivative of the nonlinear Lipschitz continuous harmonic function of j, the problem (2.8) reduces to finding $u \in C_{hg}$ such that

$$B(u, \frac{ug(v)}{u-g(v)}) + J^0(u, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg}, \tag{2.11}$$

which is known as general harmonic directional hemivariational inequality. One can discuss the applications of harmonic harmonic hemivariational inequalities in structural analysis of elasticity and structural analysis applying the techniques discussed in [38–40].

(IV). If $C^{\star}_{hg} = \{u \in \mathcal{H} : \langle u, \frac{ug(v)}{u-g(v)}\rangle \geq 0, \quad \forall v \in C_{hg}\}$ is a polar harmonic convex cone of the harmonic convex C_{hg}, then problem (2.9) is equivalent to fining $u \in \mathcal{H}$, such that

$$\frac{ug(v)}{u-g(v)} \in C_{hg}, \quad Tu + A(u) \in C^{\star}_{hg}, \quad \langle Tu + A(u), \frac{ug(v)}{u-g(v)}\rangle = 0, \quad \forall v \in C_{hg}, \tag{2.12}$$

is called the general harmonic complementarity problem. For the applications, numerical methods and other aspects of complementarity problems, see [6,14–17,21,32,37] and the references therein.

(V). If $C_{hg} = \mathcal{H}$, then problem (2.9) is equivalent to fining $u \in \mathcal{H}$, such that

$$\langle Tu + A(u), \frac{ug(v)}{u-g(v)}\rangle = 0, \quad \forall v \in \mathcal{H}, \tag{2.13}$$

which is called the weak formulation of the harmonic boundary value problem.

(VI). For $Au = A|u|$, the problem (2.13) reduces to finding $u \in \mathcal{H}$ such that

$$\langle Tu + A|u|, \frac{ug(v)}{u-g(v)}\rangle = 0, \quad \forall v \in \mathcal{H}, \tag{2.14}$$

which is called the system of absolute value harmonic equations.

(VII). If $W(A(u), \frac{ug(v)}{u-g(v)}) = 0$, then problem (2.8) reduces to finding $u \in C_{hg}$ such that

$$B(u, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg}, \tag{2.15}$$

which is called the general harmonic directional variational inequality.
For different and suitable choice of the bifunctions, operators and the spaces, one can obtained several new and known classes of the harmonic variational inequalities and related optimization problems.

3. Iterative Methods and Convergence Analysis

Several techniques, such as projection, resolvent, and descent methods, have been extensively employed to address variational inequalities and their various formulations. However, when it comes to solving harmonic variational inequalities, none of these conventional techniques are applicable. To expedite the convergence analysis of iterative methods, inertial-type iterative techniques were introduced by Polyak [42]. Alvarez [3] conducted a comprehensive analysis of the weak convergence properties of the relaxed and inertial hybrid projection-proximal point algorithm. This analysis focused on maximal monotone operators in Hilbert space. The practical applications of inertial-type methods for solving variational inequalities, variational inclusions, and their variant forms can be found in references [1,3,17,19,20,22,24,25,27,32,33] and the relevant literature.

In response to the limitations posed by conventional methods, the auxiliary principle technique, pioneered by Glowinski et al. [13] and further developed by Lions et al. [16], has emerged as a viable alternative for addressing variational inequalities and their variant forms. The utility of this technique extends to a wide array of problems, as evidenced by its application in references [13, 16, 17, 19–21, 32, 35–37, 46]. In the context of solving the problem (2.8), we make use of the auxiliary principle technique, taking into account an arbitrary operator. With the help of this method, which is flexible and uniting in nature, we can efficiently approximate solutions. With careful choice of the arbitrary operator, one can find several known variations of this flexible method.

For a given $u \in C_{hg}$ satisfying (2.8), consider the problem of finding $w \in C_{hg}$ such that

$$\begin{aligned} \rho B(w + \eta(u-w)), \frac{g(v)w}{w-g(v)}) &+ \langle M(w) - M(u)u, v-w \rangle \\ &+ \rho W(w + \eta(u-w), \frac{g(v)w}{g(v)-w}) \geq 0, \forall v \in C_{hg}, \end{aligned} \tag{3.1}$$

where $\rho > 0, \eta \in [0,1]$ are constants and $M : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator.
Inequality of type (3.1) is called the auxiliary harmonic directional variational inequality.
If $w = u$, then w is a solution of (2.8). This simple observation enables us to suggest the following iterative method for solving (2.8).

Algorithm 3.1. *For a given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\rho B(u_{n+1}+\eta(u_n-u_{n+1})), \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}) &+ \langle M(u_{n+1})-M(u_n), v-u_{n+1}\rangle \\ &\geq -\rho W(u_{n+1}+\eta(u_n-u_{n+1})), \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}\rangle, \forall v \in C_{hg}.\end{aligned}$$

Algorithm 3.1 is called the hybrid proximal point algorithm for solving (2.8).

Special Cases

We now consider some cases of Algorithm 3.1.

(I). For $\eta = 0$, Algorithm 3.1 reduces to:

Algorithm 3.2. *For a given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\rho B(u_{n+1}, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}) &+ \langle M(u_{n+1})-M(u_n), v-u_{n+1}\rangle \\ &\geq -\rho W(u_{n+1}, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}\rangle, \quad \forall v \in C_{hg}. \qquad (3.2)\end{aligned}$$

(II). If $\eta = 1$, then Algorithm 3.1 reduces to:

Algorithm 3.3. *For a given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(u_n, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}) + \langle M(u_{n+1})-M(u_n), v-u_{n+1}\rangle \geq -\rho W(u_n, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}\rangle, \quad \forall v \in C_{hg}.$$

(III). If $\eta = \frac{1}{2}$, then Algorithm 3.1 collapses to:

Algorithm 3.4. *For a given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\rho B(\frac{u_{n+1}+\eta u_n}{2}), \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}) &+ \langle M(u_{n+1})-M(u_n), v-u_{n+1}\rangle \\ &\geq -\rho W(\frac{u_{n+1}+\eta(u_n}{2}, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}\rangle, \quad \forall v \in C_{hg}.\end{aligned}$$

which is called the mid-point proximal method for solving the problem (2.8).

(IV). If $W(.;.) = 0$, then Algorithm 3.1 reduces to the following

Algorithm 3.5. *For a given $u_0 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho B(u_{n+1}+\eta(u_n-u_{n+1})), \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}) + \langle M(u_{n+1})-N(u_n), v-u_{n+1}, \rangle \geq 0, \quad \forall v \in C_{hg},$$

for solving harmonic directional variational inequality.

We now consider the convergence criteria of Algorithm 3.2.

Theorem 3.1. *Let $u \in C_{hg}$ be a solution of (2.8) and let u_{n+1} be the approximate solution obtained from Algorithm 3.2. Let the bifunction $B(.,.)$ be pseudomonotone with respect to the bifunction$W(.,.)$. If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi \| u_n - u_{n+1} \| \leq \zeta \| u - u_n \|. \tag{3.3}$$

Proof. Let $u \in C_{hg}$ be a solution of (2.8). Then

$$-B(v, \frac{ug(v)}{u - g(v)}) - W(v, \frac{ug(v)}{u - g(v)}) \geq 0, \quad \forall v \in C_{hg}. \tag{3.4}$$

since the bifunction $B(.,.)$ is pseudomonotone with respect to the bifunction $W(.,.)$.
Now taking $v = u_{n+1}$ in (3.4), we have

$$-B(u_{n+1}, \frac{ug(u_{n+1})}{u - g(u_{n+1})}) - W(u_{n+1}, \frac{ug(u_{n+1})}{u - g(u_{n+1}}) \geq 0. \tag{3.5}$$

Taking $v = u$ in (3.2), we get

$$\rho B(u_{n+1}, \frac{ug(u_{n+1})}{u - u_{n+1}}) + \langle M(u_{n+1}) - M(u_n), u - u_{n+1} \rangle + \rho W(u_{n+1}, \frac{ug(u_{n+1})}{u - g(u_{n+1})}) \geq 0,$$

which can be written as

$$\langle M(u_{n+1}) - M(u_n), u - u_{n+1} \rangle \geq -\rho B(u_{n+1}, \frac{ug(u_{n+1})}{u - g(u_{n+1})}) - \rho W(u_{n+1}, \frac{ug(u_{n+1})}{u - g(u_{n+1}}) \geq 0, \tag{3.6}$$

where we have used (3.5).
From the equation (3.6), we have

$$\begin{aligned} 0 &\leq \langle M(u_{n+1}) - M(u_n), u - u_{n+1} \rangle \\ &= \langle M(u_{n+1}) - M(u_n), u - u_n + u_n - u_{n+1} \rangle \\ &= \langle M(u_{n+1}) - M(u_n), u - u_n \rangle + \langle M(u_{n+1} - M(u_n), u_n - u_{n+1} \rangle, \end{aligned}$$

from which it follows that

$$\langle M(u_{n+1} - M(u_n), u_{n+1} - u_n \rangle \leq \langle M(u_{n+1}) - M(u_n), u - u_n \rangle.$$

Using the strongly monotonicity and Lipschitz continuity of the operator M, we obtain

$$\xi \| u_n - u_{n+1} \|^2 \leq \zeta \| u_n - u_{n+1} \| \| u - u_n \|.$$

This implies that

$$\xi \| u_n - u_{n+1} \| \leq \zeta \| u - u_n \|,$$

the required result (3.3). □

Theorem 3.2. *Let H be a finite dimensional space and all the assumptions of Theorem 3.1 hold. Then the sequence $\{u_n\}_0^\infty$ given by Algorithm 3.2 converges to a solution u of (2.8).*

Proof. Let $u \in C_{hg}$ be a solution of (2.8). From (3.3), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\frac{\xi}{\zeta}\sum_{n=0}^{\infty}\|u_{n+1} - u_n\| \leq \|u_0 - u\|,$$

which implies that

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.7}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^{\infty}$; a subsequence $\{u_{n_j}\}_1^{\infty}$ of $\{u_n\}_0^{\infty}$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (3.2), taking the limit $n_j \longrightarrow \infty$ and using (3.7), we have

$$B(\hat{u}, \frac{\hat{u}g(v)}{\hat{u} - g(v)}) + W(\hat{u}, \frac{\hat{u}g(v)}{\hat{u} - g(v)}) \geq 0, \quad \forall v \in C_{hg},$$

which implies that $\hat{u}$ solves the problem (2.8) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^{\infty}$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty}(u_n) = \hat{u}.$$

the required result. □

We again consider the auxiliary principle technique to suggest some hybrid inertial proximal point methods for solving the problem (2.8).

For a given $u \in C_{hg}$ satisfying (2.8), consider the problem of finding $w \in C_{hg}$ such that

$$\begin{aligned} B(w + \eta(u - w)), \frac{g(v)w}{g(v) - w}) \quad &+ \quad \langle M(w) - M(u) + \alpha(u - u), v - w)\rangle \\ &+ \quad W((w + \eta(w - u))), \frac{g(v)w}{w - g(v)}) \geq 0, \quad \forall v \in C_{hg}, \end{aligned} \tag{3.8}$$

where $\rho > 0, \alpha, \eta, \in [0, 1]$ are constants.

Clearly, for $w = u,$ w is a solution of (2.8). This fact motivated us to to suggest the following hybrid inertial iterative method for solving (2.8).

Algorithm 3.6. *For given $u_0, u_1 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho B(u_{n+1} + \eta(u_n - u_{n+1})), \frac{g(v)u_{n+1}}{g(v) - u_{n+1}}) + \langle M(u_{n+1}) - M(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\geq -W((u_{n+1} + \eta(u_n - u_{n+1})), \frac{g(v)u_{n+1}}{g(v) - u_{n+1}}), \quad \forall v \in C_{hg}. \end{aligned}$$

which is known as the inertial iterative method.

Note that for $\alpha = 0$, Algorithm 3.6 is exactly the Algorithm 3.1. Using essentially the technique of Theorem 3.1 and Noor [11], one can study the convergence analysis
of Algorithm 3.6.
If $\eta = \frac{1}{2}, \eta = 0, \eta = 1$, then Algorithm 3.6 reduces to:

Algorithm 3.7. *For given $u_0, u_1 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho B(\frac{u_{n+1}+u_n}{2}), \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}\rangle \quad &+ \quad \langle M(u_{n+1}) - M(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\geq \quad -\rho W(\frac{u_{n+1}+u_n}{2}), \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}), \quad \forall v \in C_{hg}.\end{aligned}$$

which is known as the hybrid inertial mid-point iterative method.

Algorithm 3.8. *For given $u_0, u_1 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\rho B(u_{n+1}, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}) \quad &+ \quad \langle M(u_{n+1}) - M(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\geq \quad -\rho W(u_{n+1}, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}), \quad \forall v \in C_{hg}.\end{aligned}$$

which is known as the implicit inertial mid-point iterative method.

Algorithm 3.9. *For given $u_0, u_1 \in C_{hg}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\rho B(u_n, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}) \quad &+ \quad \langle M(u_{n+1}) - M(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\geq \quad -\rho W(u_n, \frac{g(v)u_{n+1}}{g(v)-u_{n+1}}), \quad \forall v \in C_{hg}.\end{aligned}$$

which is known as the explicit inertial mid-point iterative method.

Remark 3.1. *For different and appropriate values of the parameters η, α, the operators g, A, the bifunctions $B(.,.), W(.,.)$, the convex set C_{hg} and the space $\mathcal{H}$, one can obtain a wide class of inertial type iterative methods for solving the harmonic variational inequalities and related optimization problems.*

4. Generalizations and future research

We note that some of the results obtained in this paper may be extended for trifunction general harmonic variational inequalities involving trifunctions and nonlinear operators.

For given nonlinear trifunctions $B(.,.,.), \Phi(.,.,.) : C_{hg} \times C_{hg} \times C_{hg} : \mathcal{H} \longrightarrow \mathcal{R}$ and nonlinear continuous operators $T, A, g : C_{hg} \longrightarrow \mathcal{H}$, consider the problem of finding $u \in C_{hg}$ such that

$$B(u, Tu, \frac{ug(v)}{u-g(v)}) + \Phi(u, A(u), \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg}. \tag{4.1}$$

The problem (4.1) is called the trifunction general harmonic variational inequality.
We would like to mention that one can obtain various classes of general harmonic variational

inequalities for appropriate and suitable choices of the trifunctions, operators T, A, g and harmonic convex sets.

(I). If $T = 0, A = 0$ and $B(.,.,.) = B(.,.), \Phi(.,.,.) = \Phi(.,.)$, and a nonlinear operator $g : \mathcal{H} \longrightarrow \mathcal{H}$, then the problem (4.1) reduces to finding $u \in C_{hg}$ such that

$$B(u, \frac{ug(v)}{u-g(v)}) + \Phi(u, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg}, \tag{4.2}$$

which is called the general harmonic direction variational inequality (2.8).
(II). For $B(.,.,.) = B(.,.), \Phi(.,.,.) = \Phi(.,.)$, and nonlinear operators $T, A, g : \mathcal{H} \longrightarrow \mathcal{H}$, the problem (4.1) reduces to finding $u \in C_{hg}$ such that

$$B(Tu, \frac{ug(v)}{u-g(v)}) + \Phi(Au, \frac{ug(v)}{u-g(v)}) \geq 0, \quad \forall v \in C_{hg}. \tag{4.3}$$

The problem (4.3) is called the bifunction general harmonic variational inequality, which appears to be a new one.
(III). For $g = I$, the identity operator, problem (4.1) collapses to find $u \in C_h$ such that

$$B(u, Tu, \frac{uv}{u-v}) + \Phi(u, A(u), \frac{uv}{u-v}) \geq 0, \quad \forall v \in C_h. \tag{4.4}$$

The problem (4.4) is called the trifunction harmonic variational inequality.

Remark 4.1. *The theory of harmonic variational inequalities has yet to reach a level of development that provides a comprehensive framework for the thorough investigation of these problems. There exists a substantial need for further research in all these domains to establish a robust foundation for practical applications. Notably, areas requiring more in-depth exploration include well-posedness, sensitivity analysis, and shape optimization concerning harmonic variational inequalities. It's worth mentioning that these aspects have garnered significant attention in the context of variational inequalities, as documented in references [22, 36, 37].*

From both mathematical and engineering perspectives, investigating the sensitivity properties of a variational inequality problem can yield valuable insights into the problem under consideration. Additionally, it has the potential to stimulate innovative problem-solving approaches. The application of fuzzy set theory has found relevance in numerous branches of mathematical and engineering sciences, spanning artificial intelligence, computer science, control engineering, management science, operations research, and variational inequalities, as elaborated in references [36,37]. Exploring the fuzzy aspects of harmonic variational inequalities and their applications represents a promising avenue for future research and potential interdisciplinary collaborations.

Conclusion: The study's contributions are outlined in this section, all the while zooming in on the addition of new categories of harmonic directed variational inequalities. Important issues are covered, with examples being systems of harmonic absolute value problems and harmonic complementarity problems. In order to create inertial iterative techniques for solving these inequalities without the use of projection or resolvent operators, the paper describes the application of the auxiliary principle technique with arbitrary operators. Convergence analysis is conducted under less stringent conditions, though the discussion remains theoretical. The research underscores the need for further work to develop practical numerical algorithms for solving harmonic variational inequalities. The paper suggests that the applications of harmonic variational inequalities extend to various pure and applied fields. It encourages future research to explore practical implementations and investigate aspects like well-posedness, sensitivity analysis, and shape optimization of harmonic variational inequalities. Additionally, the study hints at the potential utilisation of fuzzy set theory in the context of harmonic variational inequalities.

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References

[1] A.A. AlShejari, M.A. Noor, K.I. Noor, Inertial Algorithms for Bifunction Harmonic Variational Inequalities, Int. J. Anal. Appl. 22 (2024), 46. https://doi.org/10.28924/2291-8639-22-2024-46.

[2] F. Al-Azemi, O. Calin, Asian Options With Harmonic Average, Appl. Math. Inform. Sci. 9 (2015), 2803–2811.

[3] F. Alvarez, Weak Convergence of a Relaxed and Inertial Hybrid Projection-Proximal Point Algorithm for Maximal Monotone Operators in Hilbert Space, SIAM J. Optim. 14 (2004), 773–782. https://doi.org/10.1137/s1052623403427859.

[4] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Generalized Convexity and Inequalities, J. Math. Anal. Appl. 335 (2007), 1294–1308. https://doi.org/10.1016/j.jmaa.2007.02.016.

[5] L.M. Bregman, The Relaxation Method of Finding the Common Point of Convex Sets and Its Application to the Solution of Problems in Convex Programming, USSR Comput. Math. Math. Phys. 7 (1967), 200–217. https://doi.org/10.1016/0041-5553(67)90040-7.

[6] F.H. Clarke, Optimization and Nonsmooth Analysis, John Wiley & Sons, New York, 1983.

[7] R.W. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problem, SIAM, Philadelphia, 1992.

[8] G. Cristescu, L. Lupsa, Non-Connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, 2002.

[9] G. Cristescu, M. Gianu, Detecting the Non-Convex Sets With Youness and Noor Types Convexities, Bul. Stiint. Univ. Politeh. Timis. Ser. Mat.-Fiz. 55 (2010), 20–27.

[10] G. Cristescu, G. Mihail, Shape Properties of Noor's g-Convex Sets, in: Proceeding of the 12th Symposium of Mathematics and Its Applications, Timisoara, Romania, (2009), 1-13.

[11] G. Cristescu, G. Mihail, Computer Aided Optimization Techniques Based Generalized Convex Maps, in: Annals of the Oradea University, Fascicle of Management and Technological Engineering, 2011.

[12] F. Giannessi, A. Maugeri, M.S. Pardalos, Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models, Kluwer Academic, Dordrecht, 2001.

[13] R. Glowinski, J.L. Lions, R. Tremolieres, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.

[14] I. Iscan, Hermite-Hadamard Type Inequalities for Harmonically Convex Functions, Hacettepe J. Math. Stat. 43 (2014), 935–942.

[15] A.G. Khan, M.A. Noor, M. Pervez, K.I. Noor, Relative Reciprocal Variational Inequalities, Honam Math. J. 40 (2018), 509–519.

[16] J.L. Lions, G. Stampacchia, Variational inequalities, Commun. Pure Appl. Math. 20 (1967), 493–519. https://doi.org/10.1002/cpa.3160200302.

[17] M. Aslam Noor, On Variational Inequalities, PhD Thesis, Brunel University, London, 1975.

[18] M.A. Noor, General Variational Inequalities, Appl. Math. Lett. 1 (1988), 119–122. https://doi.org/10.1016/0893-9659(88)90054-7.

[19] M.A. Noor, General Algorithm for Variational Inequalities, J. Optim. Theory Appl. 73 (1992), 409–413. https://doi.org/10.1007/bf00940189.

[20] M.A. Noor, Differentiable Non-Convex Functions and General Variational Inequalities, Appl. Math. Comput. 199 (2008), 623–630. https://doi.org/10.1016/j.amc.2007.10.023.

[21] M.A. Noor, New Approximation Schemes for General Variational Inequalities, J. Math. Anal. Appl. 251 (2000), 217–229. https://doi.org/10.1006/jmaa.2000.7042.

[22] M. Aslam Noor, Some Developments in General Variational Inequalities, Appl. Math. Comput. 152 (2004), 199–277. https://doi.org/10.1016/s0096-3003(03)00558-7.

[23] M.A. Noor, K.I. Noor, Higher Order Generalized Variational Inequalites and Noncovex Optimization, U.P.B. Sci. Bull., Ser. A, 85 (2023), 77–88.

[24] M.A. Noor, K.I. Noor, Harmonic Variational Inequalities, Appl. Math. Inf. Sci. 10 (2016), 1811–1814. https://doi.org/10.18576/amis/100522.

[25] M.A. Noor, K.I. Noor, Some Implicit Methods for Solving Harmonic Variational Inequalities, Int. J. Anal. Appl. 12 (2016), 10–14.

[26] M.A. Noor, K.I. Noor, Some New Classes of Harmonic Hemivariational Inequalities, Earthline J. Math. Sci. 13 (2023), 473–495.

[27] M. A. Noor and K. I. Noor, New Iterative Schemes for General Harmonic Variational Inequalities, J. Math. Anal. In Press.

[28] M.A. Noor, K.I. Noor, Generalized Biconvex Functions and Directional Bivariational Inequalities, Transylvanian J. Math. Mech. 15 (2023), 79–95.

[29] M.A. Noor, K.I. Noor, Dynamical System Technique for Solving Quasi Variational Inequalities, U.P.B. Sci. Bull., Ser. A. 84 (2022), 55–66.

[30] M.A. Noor, K.I. Noor, M.U. Awan, Exponentially Harmonic General Variational Inequalities, Montes Taurus J. Pure Appl. Math. 6 (2024), 110–118.

[31] M.A. Noor, K.I. Noor, S. Iftikhar, Integral Inequalities for Differentiable Relative Harmonic Preinvex Functions (Survey), TWMS J. Pure Appl. Math. 7 (2016), 3–19.

[32] M.A. Noor, A.G. Khan, A. Pervaiz, K.I. Noor, Solution of Harmonic Variational Inequalities by Two-Step Iterative Scheme, Turk. J. Inequal. 1 (2017), 46–52.

[33] M.A. Noor, K.I. Noor, S. Iftikhar, Strongly Generalized Harmonic Convex Functions and Integral Inequalities, J. Math. Anal. 7 (2016), 66–77.

[34] M.A. Noor, K.I. Noor, S. Iftikhar, Some Characterizations of Harmonic Convex Functions, Int. J. Anal. Appl. 15 (2017), 179–187.

[35] M.A. Noor, K.I. Noor, Higher Order Generalized Variational Inequalites and Noncovex Optimization, U.P.B. Sci. Bull., Ser. A, 85 (2023), 77–88.

[36] M.A. Noor, K.I. Noor, M.T. Rassias, New Trends in General Variational Inequalities, Acta Appl. Math. 170 (2020), 981–1064. https://doi.org/10.1007/s10440-020-00366-2.

[37] M.A. Noor, K.I. Noor, T.M. Rassias, Some Aspects of Variational Inequalities, J. Comput. Appl. Math. 47 (1993), 285–312. https://doi.org/10.1016/0377-0427(93)90058-j.

[38] M.A. Noor, W. Oettli, On General Nonlinear Complementarity Problems and Quasi Equilibria, Le Math. 49 (1994), 313–331.

[39] M. Patriksson, Nonlinear Programming and Variational Inequalities: A Unified Approach, Kluwer Academic Publishers, Dordrecht, 1999.

[40] Z. Pavic, M. Avci Ardic, The Most Important Inequalities of m-Convex Functions, Turk. J. Math. 41 (2017), 625–635. https://doi.org/10.3906/mat-1604-45.

[41] J.E. Pecaric, F. Proschan, Y.L. Tong, Convex Functions and Statistical Applications, Academic Press, New York, 1992.

[42] B.T. Polyak, Some methods of speeding up the convergence of iteration methods, USSR Comput. Math. Math. Phys. 4 (1964), 1–17. https://doi.org/10.1016/0041-5553(64)90137-5.

[43] G. Stampacchia, Formes Bilineaires Coercitives sur les Ensembles Convexes, C. R. Acad. Sci. Paris. 258 (1964), 4413–4416.

[44] G. Toader, Some Generalizations of the Convexity, In: Proc. Colloq. Approx. Optim. Cluj-Napoca, Romania, Univ. Cluj-Napoca, (1985), 329–338.

[45] E.A. Youness, E-Convex Sets, E-Convex Functions, and E-Convex Programming, J. Optim. Theory Appl. 102 (1999), 439–450. https://doi.org/10.1023/a:1021792726715.

[46] D.L. Zhu, P. Marcotte, Co-Coercivity and Its Role in the Convergence of Iterative Schemes for Solving Variational Inequalities, SIAM J. Optim. 6 (1996), 714–726. https://doi.org/10.1137/s1052623494250415.