Int. J. Anal. Appl. (2024), 22:73

International Journal of Analysis and Applications

Weak (τ_1, τ_2)-Continuity for Multifunctions

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Abstract. This paper is concerned with the concept of weakly (τ_1, τ_2)-continuous multifunctions. Moreover, several characterizations of weakly (τ_1, τ_2)-continuous multifunctions are investigated.

1. Introduction

The concept of weakly continuous functions was introduced by Levine [12]. Furthermore, Levine [11] introduced the notion of semi-continuous functions. Neubrunnová [14] showed that semi-continuity is equivalent to quasi-continuity due to Marcus [13]. Popa and Stan [19] introduced and studied the concept of weakly quasi-continuous functions. Weak quasi-continuity is implied by both quasi-continuity and weak continuity which are independent of each other. Rose [20] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Noiri [15] studied properties of some weak forms of continuity. In 2002, Popa and Noiri [16] introduced the concept of weakly (τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of weakly (τ, m)-continuous functions. Popa and Noiri [17] introduced and investigated the notion of weakly M-continuous functions as functions from a set satisfying some minimal conditions into a set satisfying some minimal conditions. In particular, several characterizations of pairwise weakly M-continuous functions were presented in [8]. Ekici et al. [9] introduced a new class of functions called weakly λ-continuous functions which is weaker than λ-continuous functions and studied some fundamental properties of weakly λ-continuous

Received: Feb. 29, 2024.

2020 *Mathematics Subject Classification.* 54C08, 54C60, 54E55.

Key words and phrases. $\tau_1\tau_2$-open set; weakly (τ_1, τ_2)-continuous multifunction; weak continuity.

https://doi.org/10.28924/2291-8639-22-2024-73

ISSN: 2291-8639

functions. In [3], the present author introduced the concept of weakly $\star$-continuous functions and investigated the relationships between weak $\star$-continuity and $\theta(\star)$-continuity. Moreover, some characterizations of $\beta(\star)$-continuous multifunctions were studied in [5]. Popa and Noiri [18] introduced the concept of weakly m-continuous multifunctions and discussed the relationships between almost m-continuity and weak m-continuity. Laprom et al. [10] introduced and investigated the notion of almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [21] introduced and studied the concept of weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Furthermore, several characterizations of weakly $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were established in [6] and [4], respectively. In this paper, we introduce the concept of weakly (τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of weakly (τ_1, τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [7] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [7] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [7] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [7] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*
(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*
(5) *$\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [21] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed*, $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [21] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [21] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [21] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [21] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 2.2. [21] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-$Cl(A) = (\tau_1, \tau_2)\theta$-$Cl(A)$.*
(2) *$(\tau_1, \tau_2)\theta$-$Cl(A)$ is $\tau_1\tau_2$-closed in X.*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$.

3. Weakly (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notion of weakly (τ_1, τ_2)-continuous multifunctions. Moreover, some characterizations of weakly (τ_1, τ_2)-continuous multifunctions are discussed.

Definition 3.1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-$Cl(V_1)$ and $\sigma_1\sigma_2$-$Cl(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$.*

Theorem 3.1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is weakly (τ_1, τ_2)-continuous;*
(2) *$F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2$-$Int(F^+(\sigma_1\sigma_2$-$Cl(V_1)) \cap F^-(\sigma_1\sigma_2$-$Cl(V_2)))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y;*
(3) *$\tau_1\tau_2$-$Cl(F^-(\sigma_1\sigma_2$-$Int(K_1)) \cup F^+(\sigma_1\sigma_2$-$Int(K_2))) \subseteq F^-(K_1) \cup F^+(K_2)$ for every $\sigma_1\sigma_2$-closed sets K_1, K_2 of Y;*
(4)
$$\begin{aligned}&\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_1))) \cup F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_2))))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(B_2))\end{aligned}$$
for every subsets B_1, B_2 of Y;
(5) *$F^+(\sigma_1\sigma_2$-$Int(B_1)) \cap F^-(\sigma_1\sigma_2$-$Int(B_2)) \subseteq \tau_1\tau_2$-$Int(F^+(\sigma_1\sigma_2$-$Cl(B_1)) \cap F^-(\sigma_1\sigma_2$-$Cl(B_2)))$ for every subsets B_1, B_2 of Y;*
(6) *$\tau_1\tau_2$-$Cl(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\sigma_1\sigma_2$-$Cl(V_1)) \cup F^+(\sigma_1\sigma_2$-$Cl(V_2))$ for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y.*

Proof. (1) ⇒ (2): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then, $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. By (1), there exists a $\sigma_1\sigma_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V_1)$ and $\sigma_1\sigma_2$-Cl$(V_2) \cap F(z) \neq \emptyset$ for each $z \in U$. Thus,

$x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V_1)) \cap F^-(\sigma_1\sigma_2\text{-Cl}(V_2))$ and hence
$x \in \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V_1)) \cap F^-(\sigma_1\sigma_2\text{-Cl}(V_2)))$. Therefore,

$$F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V_1)) \cap F^-(\sigma_1\sigma_2\text{-Cl}(V_2))).$$

(2) ⇒ (3): Let K_1, K_2 be any $\sigma_1\sigma_2$-closed sets of Y. Then $Y - K_1$ and $Y - K_2$ are $\sigma_1\sigma_2$-open sets in Y. By (2), we have

$$\begin{aligned} X - (F^-(K_1) \cup F^+(K_2)) &= (X - F^-(K_1)) \cap (X - F^+(K_2)) \\ &= F^+(Y - K_1) \cap F^-(Y - K_2) \\ &\subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - K_1)) \cap F^-(\sigma_1\sigma_2\text{-Cl}(Y - K_2))) \\ &= \tau_1\tau_2\text{-Int}((X - F^-(\sigma_1\sigma_2\text{-Int}(K_1))) \cap (X - F^+(\sigma_1\sigma_2\text{-Int}(K_2)))) \\ &= \tau_1\tau_2\text{-Int}(X - (F^-(\sigma_1\sigma_2\text{-Int}(K_1)) \cup F^+(\sigma_1\sigma_2\text{-Int}(K_2)))) \\ &= X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K_1)) \cup F^+(\sigma_1\sigma_2\text{-Int}(K_2))) \end{aligned}$$

and hence $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K_1)) \cup F^+(\sigma_1\sigma_2\text{-Int}(K_2))) \subseteq F^-(K_1) \cup F^+(K_2)$.

(3) ⇒ (4): Let B_1, B_2 be any subsets of Y. Then $\sigma_1\sigma_2\text{-Cl}(B_1)$ and $\sigma_1\sigma_2\text{-Cl}(B_2)$ are $\sigma_1\sigma_2$-closed in Y and by (3),

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1))) \cup F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_2)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(B_2)).$$

(4) ⇒ (5): Let B_1, B_2 be any subsets of Y. By (4), we have

$$\begin{aligned} &F^-(\sigma_1\sigma_2\text{-Int}(B_1)) \cap F^+(\sigma_1\sigma_2\text{-Int}(B_2)) \\ &= X - (F^+(\sigma_1\sigma_2\text{-Cl}(Y - B_1)) \cup F^-(\sigma_1\sigma_2\text{-Cl}(Y - B_2))) \\ &\subseteq X - \tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B_1))) \cup F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B_2)))) \\ &= X - \tau_1\tau_2\text{-Cl}(F^+(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_1))) \cup F^-(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_2)))) \\ &= X - \tau_1\tau_2\text{-Cl}((X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_1)))) \cup (X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_2))))) \\ &= X - \tau_1\tau_2\text{-Cl}(X - (F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B_1))) \cap F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_2))))) \\ &= \tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_1))) \cap F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B_2)))). \end{aligned}$$

Thus, $F^-(\sigma_1\sigma_2\text{-Int}(B_1)) \cap F^+(\sigma_1\sigma_2\text{-Int}(B_2)) \subseteq \tau_1\tau_2\text{-Int}(F^-(\sigma_1\sigma_2\text{-Cl}(B_1)) \cap F^+(\sigma_1\sigma_2\text{-Cl}(B_2)))$.

(5) ⇒ (2): This is obvious.

(2) ⇒ (1): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. Then, $x \in F^+(V_1) \cap F^-(V_2)$. By (2), we have
$F^+(V_1) \cap F^-(V_2) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V_1)) \cap F^-(\sigma_1\sigma_2\text{-Cl}(V_2)))$. Then, there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V_1)) \cap F^-(\sigma_1\sigma_2\text{-Cl}(V_2))$. Therefore, $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V_1)$ and $\sigma_1\sigma_2\text{-Cl}(V_2) \cap F(z) \neq \emptyset$ for every $z \in U$. This shows that F is weakly (τ_1, τ_2)-continuous.

(4) ⇒ (6): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y. By (4), we have

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(F^-(V_1) \cup F^+(V_2)) &\subseteq \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2))))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(V_2)).\end{aligned}$$

(6) ⇒ (2): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y. By (6), we have

$$\begin{aligned}F^+(V_1) \cap F^-(V_2) &\subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cap F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))\\ &= X - (F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V_2))))\\ &\subseteq X - \tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V_1)) \cup F^+(Y - \sigma_1\sigma_2\text{-Cl}(V_2)))\\ &= \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V_1)) \cap F^-(\sigma_1\sigma_2\text{-Cl}(V_2))).\end{aligned}$$

□

Definition 3.2. [2] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\tau_1\tau_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_1$-Cl$(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.1. [2] *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly (τ_1, τ_2)-continuous;*
(2) *$f^{-1}(V) \subseteq \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;*
(5) *$f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(B)))$ for every subset B of Y;*
(6) *$\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Theorem 3.2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is weakly (τ_1, τ_2)-continuous;*
(2)

$$\begin{aligned}&\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B_1))) \cap F^+(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B_2))))\\ &\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B_1)) \cup F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B_2))\end{aligned}$$

for every subsets B_1, B_2 of Y;

(3)

$$\begin{aligned}&\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_1))) \cup F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B_2))))\\ &\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B_1)) \cup F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B_2))\end{aligned}$$

for every subsets B_1, B_2 of Y;

(4)

$$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))) \cup F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$$

for every $\sigma_1\sigma_2$-open sets V_1, V_2 of Y;

(5)

$$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))) \cup F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$$

for every $(\sigma_1, \sigma_2)p$-open sets V_1, V_2 of Y;

(6) *$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int(K_1)) \cup F^+(\sigma_1\sigma_2\text{-}Int(K_2))) \subseteq F^-(K_1) \cup F^+(K_2)$ for every $(\sigma_1, \sigma_2)r$-closed sets K_1, K_2 of Y.*

Proof. (1) ⇒ (2): Let B_1, B_2 be any subset of Y. Then $(\sigma_1, \sigma_2)\theta\text{-Cl}(B_1)$ and $(\sigma_1, \sigma_2)\theta\text{-Cl}(B_2)$ are $\sigma_1\sigma_2$-closed in Y. By Theorem 3.1, we have

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B_1))) \cup F^+(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B_2))))$$
$$\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B_1)) \cup F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B_2)).$$

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(B) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ for every subset B of Y.

(3) ⇒ (4): This is obvious since $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1, \sigma_2)\theta\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(4) ⇒ (5): Let V_1, V_2 be any $(\sigma_1, \sigma_2)p$-open sets of Y. Since $V_i \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_i))$, we have $\sigma_1\sigma_2\text{-Cl}(V_i) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_i)))$ for $i = 1, 2$. Now, put $U_i = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_i))$, then U_i is $\sigma_1\sigma_2$-open in Y and $\sigma_1\sigma_2\text{-Cl}(U_i) = \sigma_1\sigma_2\text{-Cl}(V_i)$. Then by (4), we have

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(V_2)).$$

(5) ⇒ (6): Let K_1, K_2 be any $(\sigma_1, \sigma_2)r$-closed sets of Y. Then $\sigma_1\sigma_2\text{-Int}(K_1)$ and $\sigma_1\sigma_2\text{-Int}(K_2)$ are $(\sigma_1, \sigma_2)p$-open in Y and by (5),

$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K_1)) \cup F^+(\sigma_1\sigma_2\text{-Int}(K_2)))$$
$$= \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K_1)))) \cup F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K_2)))))$$
$$\subseteq F^-(K_1) \cup F^+(K_2).$$

(6) ⇒ (1): Let V_1, V_2 be any $\sigma_1\sigma_2$-open sets of Y. Then $\sigma_1\sigma_2\text{-Cl}(V_1)$ and $\sigma_1\sigma_2\text{-Cl}(V_1)$ are $(\sigma_1, \sigma_2)r$-closed in Y and by (6), we have

$$\tau_1\tau_2\text{-Cl}(F^-(V_1) \cup F^+(V_2)) \subseteq \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(V_2)).$$

It follows from Theorem 3.1 that F is weakly (τ_1, τ_2)-continuous. □

Corollary 3.2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly (τ_1, τ_2)-continuous;*
(2) $\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Int((\sigma_1,\sigma_2)\theta$-$Cl(B)))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta$-$Cl(B))$ *for every subset B of Y;*
(3) $\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(B)))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta$-$Cl(B))$ *for every subset B of Y;*
(4) $\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-$Cl(V))$ *for every $\sigma_1\sigma_2$-open set V of Y;*
(5) $\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-$Cl(V))$ *for every $(\sigma_1,\sigma_2)p$-open set V of Y;*
(6) $\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Int(K))) \subseteq f^{-1}(K)$ *for every $(\sigma_1,\sigma_2)r$-closed set K of Y.*

Theorem 3.3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is weakly (τ_1, τ_2)-continuous;*
(2)
$$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))) \cup F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$$
for every $(\sigma_1,\sigma_2)\beta$-open sets V_1, V_2 of Y;
(3)
$$\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_1))) \cup F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V_2))))$$
$$\subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V_1)) \cup F^+(\sigma_1\sigma_2\text{-}Cl(V_2))$$
for every $(\sigma_1,\sigma_2)s$-open sets V_1, V_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $(\sigma_1,\sigma_2)\beta$-open sets of Y. Then,
$V_i \subseteq \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V_i)))$ and $\sigma_1\sigma_2$-Cl$(V_i) = \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V_i)))$ for $i = 1, 2$.
Since $\sigma_1\sigma_2$-Cl(V_1) and $\sigma_1\sigma_2$-Cl(V_2) are $(\sigma_1,\sigma_2)r$-closed in Y and by Theorem 3.2,
$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(V_2)).$$

(2) $\Rightarrow$ (3): This is obvious since every $(\sigma_1,\sigma_2)s$-open set is $(\sigma_1,\sigma_2)\beta$-open.

(3) $\Rightarrow$ (1): Let V_1, V_2 be any $(\sigma_1,\sigma_2)p$-open sets of Y. Then $\sigma_1\sigma_2$-Cl(V_1) and $\sigma_1\sigma_2$-Cl(V_2) are $(\sigma_1,\sigma_2)r$-closed sets of Y and hence $\sigma_1\sigma_2$-Cl(V_1) and $\sigma_1\sigma_2$-Cl(V_2) are $(\sigma_1,\sigma_2)s$-open in Y. By (3), we have
$$\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_1))) \cup F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V_2)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V_1)) \cup F^+(\sigma_1\sigma_2\text{-Cl}(V_2))$$
and by Theorem 3.2, F is weakly (τ_1,τ_2)-continuous. □

Corollary 3.3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly (τ_1, τ_2)-continuous;*
(2) $\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-$Cl(V))$ *for every $(\sigma_1,\sigma_2)\beta$-open set V of Y;*
(3) $\tau_1\tau_2$-$Cl(f^{-1}(\sigma_1\sigma_2$-$Int(\sigma_1\sigma_2$-$Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-$Cl(V))$ *for every $(\sigma_1,\sigma_2)s$-open set V of Y.*

Acknowledgements

This research project was financially supported by Mahasarakham University.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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