Some Results on Subspace Cesaro-Hypercyclic Operators

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Abstract. In this paper we characterize the notion of subspace Cesàro-hypercyclic. At the same time, we also provide a Subspace Cesàro-hypercyclic Criterion and offer an equivalent conditions of this criterion.

1. Introduction

Let $H$ be a separable infinite dimensional Hilbert space over the scalar field $C$. As usual, $N$ is the set of all non-negative integers, $Z$ is the set of all integers, and $B(H)$ is the space of all bounded linear operators on $H$. A bounded linear operator $T : H \to H$ is called hypercyclic if there is some vector $x \in H$ such that $\text{Orb}(T, x) = \{T^n x : n \in N\}$ is dense in $H$, where such a vector $x$ is said hypercyclic for $T$.

The first example of hypercyclic operator was given by Rolewicz in [16]. He proved that if $B$ is a backward shift on the Banach space $l^p$, then $\lambda B$ is hypercyclic if and only if $|\lambda| > 1$.

Let $\{e_n\}_{n \geq 0}$ be the canonical basis of $l^2(N)$. If $\{w_n\}_{n \geq 1}$ is a bounded sequence in $C \setminus \{0\}$, then the unilateral backward weighted shift $T : l^2(N) \to l^2(N)$ is defined by $T_{e_n} = w_n e_{n-1}$, $n \geq 1$, $T_{e_0} = 0$, and let $\{e_n\}_{n \in Z}$ be the canonical basis of $l^2(Z)$. If $\{w_n\}_{n \in Z}$ is a bounded sequence in $C \setminus \{0\}$, then the bilateral weighted shift $T : l^2(Z) \to l^2(Z)$ is defined by $T_{e_n} = w_n e_{n-1}$.

The definition and the properties of supercyclicity operators were introduced by Hilden and Wallen [9]. They proved that all unilateral backward weighted shifts on a Hilbert space are supercyclic.

A bounded linear operator $T \in B(H)$ is called supercyclic if there is some vector $x \in H$ such that the projective orbit $C.\text{Orb}(T, x) = \{\lambda T^n x : \lambda \in C, n \in N\}$ is dense in $X$. Such a vector $x$ is said supercyclic for $T$. Refer to ([1], [8], [4], [19]) for more informations about hypercyclicity and...
A nice criterion namely Hypercyclicity Criterion, was developed independently by Kitai [11] and, Gethner and Shapiro [7]. The Hypercyclicity Criterion has been widely used to show that many different types of operators are hypercyclic. For instance hypercyclic operators arise in the classes of composition operators [3], adjoints of multiplication operators [7], cohyponormal operators [6], and weighted shifts [17].

For the following theorem, see ([1], [8]).

**Theorem 1.1. (Hypercyclicity Criterion).** Suppose that $T \in B(\mathcal{H})$. If there exist two dense subsets $X_0$ and $Y_0$ in $\mathcal{H}$ and an increasing sequence $n_j$ of positive integer such that:

1. $T^{n_j}x \to 0$ for each $x \in X_0$, and
2. there exist mappings $S_{n_j}: Y_0 \to \mathcal{H}$ such that $S_{n_j}y \to 0$, and $T^{n_j}S_{n_j}y \to y$ for each $y \in Y_0$,

then $T$ is hypercyclic.

In [17] and [18], Salas characterized the bilateral weighted shifts that are hypercyclic and those that are supercyclic in terms of their weight sequence. In [5], N. Feldman gave a characterization of the invertible bilateral weighted shifts that are hypercyclic or supercyclic.

Let $M_n(T)$ denote the arithmetic mean of the powers of $T \in B(\mathcal{H})$, that is

$$M_n(T) = \frac{I + T + T^2 + \ldots + T^{n-1}}{n}, n \in \mathbb{N}^*.$$ 

If the arithmetic means of the orbit of $x$ are dense in $\mathcal{H}$ then the operator $T$ is said to be Cesàro-hypercyclic. In [13], Fernando León-Saavedra proved that an operator is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{n^{-1}T^n x\}_{n \geq 1}$ is dense in $\mathcal{H}$ and characterized the bilateral weighted. The following examples give an operator which is Cesàro-hypercyclic but not hypercyclic and vice versa.

**Example 1.1.** [13] Let $T$ the bilateral backward shift with the weight sequence

$$w_n = \begin{cases} 
1 & \text{if } n \leq 0, \\
2 & \text{if } n \geq 1. 
\end{cases}$$

Then $T$ is not hypercyclic, but it is Cesàro-hypercyclic.

**Example 1.2.** [20] Let $T$ the bilateral backward shift with the weight sequence

$$w_n = \begin{cases} 
2 & \text{if } n < 0, \\
\frac{1}{2} & \text{if } n \geq 0. 
\end{cases}$$

Then $T$ is not Cesàro-hypercyclic, but it is hypercyclic and supercyclic.

In 2011, B. F. Madore and R. A. Martinez-Avendano in [14] introduced and studied the concept of subspace-hypercyclicity for an operator. An operator $T$ is subspace-hypercyclic or $M$-hypercyclic for a subspace $M$ of $X$, if there exists $x \in X$ such that $Orb(T, x) \cap M$ is dense in $M$. Such a
vector $x$ is called a $M$-hypercyclic vector for $T$, they showed that there are operators which are $M$-hypercyclic but not hypercyclic. They introduced analogously the concept of subspace-transitivity. Let $T \in \mathcal{B}(X)$ and $M$ be a closed subspace of $X$, we say that $T$ is $M$-transitive, if for any non-empty open sets $U, V$ in $M$, there exists $n \geq 0$ such that $T^{-n}(U) \cap V$ contain a non-empty open subset of $M$. The authors showed that $M$-transitivity implies $M$-hypercyclicity. Note that the converse is not true, this is proven recently by C. M. Le in [12]; for more informations see ([10], [15]).

Similarly, for subspace-supercyclicity, Zhao, Y.L. Sun and Y.H. Zhou in [21] provided a Subspace-Supercyclicity Criterion and offered two necessary and sufficient conditions for a path of bounded linear operators to have a dense $G_δ$ set of common subspace-hypercyclic vectors and common subspace-supercyclic vectors and they also constructed examples to show that subspace-supercyclic is not a strictly infinite dimensional phenomenon and that some subspace-supercyclic operators are not supercyclic.

In this present paper, we will partially characterize the notion of subspace Cesàro-hypercyclic. At the same time, we also provide a Subspace Cesàro-hypercyclic Criterion and offer an equivalent conditions of this criterion.

2. Main results

We will assume that the subspace $M \subset \mathcal{H}$ is topologically closed. We start with our main definitions.

**Definition 2.1.** Let $T \in \mathcal{B}(\mathcal{H})$ and $M$ be a closed subspace of $\mathcal{H}$. We say that $T$ is $M$-cesàro-hypercyclic if there exists a vector $x \in \mathcal{H}$ such that $\text{Orb}(T, x) \cap M = \{n^{-1}T^nx : n \geq 1\} \cap M$ is dense in $M$. We call $x$ a $M$-cesàro-hypercyclic vector.

**Remark 2.1.** The definition above reduces to the classical definition of cesàro-hypercyclic if $M = \mathcal{H}$ and we may assume that the subspace cesàro-hypercyclic vector $x \in M$, if needed.

**Example 2.1.** Let $T$ be a cesàro-hypercyclic operator on $\mathcal{H}$ with cesàro-hypercyclic vector $x$ and let $I$ be the identity operator on $\mathcal{H}$. Then the operator $T \oplus I : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is subspace cesàro-hypercyclic for the subspace $M := \mathcal{H} \oplus \{0\}$ with the subspace cesàro-hypercyclic vector $x \oplus \{0\}$, but $T \oplus I$ is not cesàro-hypercyclic on the space $\mathcal{H} \oplus \mathcal{H}$.

**Theorem 2.1.** Let $T$ be a subspace cesàro-hypercyclic and $M$ be a nonzero subspace of $\mathcal{H}$. Then

$$CH(T, M) = \bigcap_{k \geq 1} \bigcup_{n \geq 1} (n^{-1}T^n)^{-1}(B_k),$$

where $(B_k)_{k \geq 1}$ is a countable open basis for the relative topology of $M$ as a subspace of $\mathcal{H}$.

**Proof.** Let $(B_k)_{k \geq 1}$ is a countable open basis for the relative topology of $M$ as a subspace of $\mathcal{H}$. We have $x \in CH(T, M)$ if and only if $\{n^{-1}T^nx : n \geq 1\} \cap M$ is dense in $M$ if and only if for each $k \geq 1$, there exist $n \geq 1$ such that $n^{-1}T^nx \in B_k$ if and only if $x \in \bigcap_{k \geq 1} \bigcup_{n \geq 1} (n^{-1}T^n)(B_k)$. 

$\square$
Theorem 2.2. Let \( T \) be a subspace cesàro-hypercyclic and \( M \) be a nonzero subspace of \( \mathcal{H} \). Then the following conditions are equivalent:

1. For every non-empty open \( U \) and \( V \) of \( M \), there exist \( n \geq 1 \) such that \((n^{-1}T^n)^{-1}(U) \cap V\) contains a non-empty open subset of \( M \).

2. For every non-empty open \( U \) and \( V \) of \( M \), there exist \( n \geq 1 \) such that \((n^{-1}T^n)^{-1}(U) \cap V\) is non-empty and \((n^{-1}T^n)(M) \subset M\).

3. For every non-empty open \( U \) and \( V \) of \( M \), there exist \( n \geq 1 \) such that \((n^{-1}T^n)^{-1}(U) \cap V\) is non-empty open subset of \( M \).

Proof. (1) \(\Rightarrow\) (2). Let \( U \) and \( V \) be two nonempty open subsets of \( M \). By (1) there exist \( n \geq 1 \) such that \((n^{-1}T^n)^{-1}(U) \cap V\) contains a non-empty open \( W \) of \( M \), it follows that \( W \subset (n^{-1}T^n)^{-1}(U) \cap V\) and \((n^{-1}T^n)^{-1}(U) \cap V\neq \emptyset\).

Next, we prove that \((n^{-1}T^n)(M) \subset M\).

Let \( x \in M \), we have \( W \subset (n^{-1}T^n)^{-1}(U) \cap V\), this implies that \( n^{-1}T^n(W) \subset U \subset M \). Let \( x_0 \in W \), since \( W \) is open of \( M \) then for all \( r \) enough small we have \( x_0 + rx \in W \), therefore \((n^{-1}T^n)^{-1}(x_0 + rx) = \left((n^{-1}T^n)x_0 + r(n^{-1}T^n)x\right) \in M \). Since \( n^{-1}T^n x_0 \in M \), it follows that \( r(n^{-1}T^n)x \in M \). We then conclude that \((n^{-1}T^n)(M) \subset M\).

(2) \(\Rightarrow\) (3). Let \( U \) and \( V \) be nonempty open subsets of \( M \), by (2) there exist \( n \geq 1 \) such that \((n^{-1}T^n)^{-1}(U) \cap V\) is non-empty and \((n^{-1}T^n)(M) \subset M\).

Since \((n^{-1}T^n)|M : M \to M\) is continuous, then \((n^{-1}T^n)^{-1}(U)\) is open in \( M \), therefore \((n^{-1}T^n)^{-1}(U) \cap V\) is nonempty open of \( M\).

At last, we see that the implication (3) \(\Rightarrow\) (1) is obvious and this completes the whole proof of the theorem. \(\square\)

Corollary 2.1. Let \( T \) be a subspace cesàro-hypercyclic and \( M \) be a nonzero subspace of \( \mathcal{H} \). If any of the conditions in theorem 2.2 is satisfied, then \( CH(T, M) \) is a dense subset of \( M\).

Proof. We may assume the condition (3) in theorem 2.2 is satisfied, then for every non-empty open \( U \) of \( M \) and for all \( k \geq 1 \), there exist \( n \geq 1 \) such that the set \((n^{-1}T^n)^{-1}(U) \cap B_k\) is nonempty and open. Hence the set

\[ A_k = \bigcup_n (n^{-1}T^n)^{-1}(B_k) \]

is nonempty and open. Furthermore, \( U \cap A_k \neq \emptyset \) for all \( k \geq 1 \). Thus each \( A_k \) is dense in \( M \) and so by the Baire category theorem and theorem 2.1 \( CH(T, M) \) is also dense in \( M\). \(\square\)

Next we get the following theorem, which is the Subspace Cesàro-Hypercyclic Criterion, which is similar to the Supercyclicity Criterion that was stated in [2]; see also [8].

Theorem 2.3. (Subspace Cesàro-Hypercyclic Criterion) Let \( T \) be a subspace cesàro-hypercyclic and \( M \) be a nonzero subspace of \( \mathcal{H} \). Assume that there exist \( M_0 \) and \( M_1 \), dense subsets of \( M \), an increasing sequence \((n_k)\) of positive integer and a sequence of mappings \( S_{n_k} : M_1 \to M \) such that
(1) For each \( x \in M_0 \), \( n_k^{-1}T^nx \to 0 \),
\( \quad \) (2) For each \( y \in M_1 \), \( S_n y \to 0 \),
\( \quad \) (3) For each \( y \in M_1 \), \( (n_k^{-1}T^nx \circ S_n)y \to y \),
\( \quad \) (4) \( M \) is an invariant subspace for \( n_k^{-1}T^nx \) for all \( n \geq 1 \).

Then \( T \) is subspace cesàro-hypercyclic for \( M \).

**Proof.** Let \( U \) and \( V \) be non-empty open subsets of \( M \). By Theorem 2.2, it is enough to prove that there exist \( n \geq 1 \) such that

\[
(n_k^{-1}T^nx)^{-1}(U) \cap V \text{ is non-empty and } n_k^{-1}T^nx(M) \subset M.
\]

Since \( M_0 \) and \( M_1 \) are dense in \( M \), there exist \( x \in M_0 \cap V \), \( y \in M_1 \cap U \). And since \( U \) and \( V \) are nonempty open subsets, there exists \( \varepsilon > 0 \) such that \( B_M(x, \varepsilon) \subseteq V \) and \( B_M(y, \varepsilon) \subseteq U \). By assumption, there exist \( (n_k) \) such that

\[
||n_k^{-1}T^nx|| \leq \varepsilon, \quad ||S_n y|| \leq \varepsilon \quad \text{and} \quad ||n_k^{-1}T^nx S_n y - y|| \leq \varepsilon.
\]

Define \( u = x + S_n y \). We know that \( u \in M \) and \( u \in V \), since \( ||x - u|| = ||S_n y|| \leq \varepsilon/2 \). Observe that \( n_k^{-1}T^nu = n_k^{-1}T^nx + n_k^{-1}T^nx S_n y \), so \( n_k^{-1}T^nu \in M \). Since

\[
||n_k^{-1}T^nu - y|| = ||n_k^{-1}T^nx|| + ||n_k^{-1}T^nx S_n y - y|| < \varepsilon,
\]

we have that \( n_k^{-1}T^nu \in U \). Then \( (n_k^{-1}T^nx)^{-1}(U) \cap V \neq \emptyset \) and \( T \) is subspace cesàro-hypercyclic for \( M \).

**Theorem 2.4.** Let \( T \in B(\mathcal{H}) \) and \( M \) be a nonzero subspace of \( \mathcal{H} \). Then the following (1) and (2) are equivalent:

(1) \( T \) satisfies Subspace Cesàro-Hypercyclic Criterion.
(2) (Outer Subspace Cesàro-Hypercyclic Criterion) There exist an increasing sequence \((n_k)\) of positive integer, a dense linear subspace \( Y_0 \subseteq M \) and, for each \( y \in Y_0 \), a dense linear subspace \( X_0 \) of \( M \) such that:

(a) There exists a sequence of mappings \( S_n : Y_0 \to M \), \( k \in \mathbb{N}^+ \) such that \( n_k^{-1}T^nx \circ S_n \to y \),
\( \quad \) For each \( y \in Y_0 \) and
\( \quad \) (b) \( ||n_k^{-1}T^nx|| \cdot ||S_n y|| \to 0 \) For each \( y \in Y_0 \) and \( x \in X_0 \),
\( \quad \) (c) \( M \) is an invariant subspace for \( n_k^{-1}T^nx \) for all \( k \geq 1 \).

**Proof.** It is obvious that any operator satisfying the Subspace Cesàro-Hypercyclic Criterion also satisfy the criteria of (2). It suffices to show that (2) implies (1). Let \( U_i, V_i \subseteq M \) non-empty open sets with \( i = 1, 2 \). The same argument as in the proof of Theorem 3.2 in [2] can be used to show that there exist \( (n_k) \) of positive integer such that

\[
(n_k^{-1}T^nx)^{-1}(U_i) \cap V_i \neq \emptyset, \text{ for } i = 1, 2.
\]
Then we can know that \((T \oplus T)\) is subspace cesàro-hypercyclic for \(M \oplus M\) and \((x, y)\) is subspace cesàro-hypercyclic vector for \((T \oplus T)\). In particular, \(x\) be subspace cesàro-hypercyclic vector for \(T\) and \(CH(T, M) \cap M\) is a dense \(G_δ\) subset of \(M\). Let \((U_k)\) be a base of 0-neighborhoods in \(M\). Then there exist \((n_k)\) of positive integer such that

\[
n_{k}^{-1}T^{n_k}x \in U_k \text{ and } n_{k}^{-1}T^{n_k}y \in x + U_k \text{ for all } k \geq 1.
\]

This implies that \(n_{k}^{-1}T^{n_k}x \to 0\) and \(n_{k}^{-1}T^{n_k}y \to x\). Let \(M_0 = M_1 = Orb(T, x) \cap M\), which is dense \(G_δ\) in \(M\). Also for all \(k \geq 1\) define

\[
S_{n_k}(n^{-1}T^n x) = n^{-1}T^n x.
\]

Note that

\[
n_{k}^{-1}T^{n_k}S_{n_k}(n_{-1}T^n x) = n_{k}^{-1}T^{n_k}(n_{-1}T^n y) = n^{-1}T^n(n_{k}^{-1}T^{n_k}y) \to n^{-1}T^n x.
\]

Hence (1) holds. We complete the proof. \(\square\)

**Proposition 2.1.** Let \(T \in B(H)\) satisfy the subspace cesàro-hypercyclicity criterion with respect to a sequence \((n_k)\). Then \(T\) is subspace cesàro-mixing.

**Proof.** We show that \(T\) is subspace cesàro-mixing. Let \(M_0\) and \(M_1\) be dense sets in \(M\), that are given in the subspace cesàro-hypercyclicity criterion. Let \(U\) and \(V\) are two nonempty open sets in \(M\), then choose \(x \in M_0 \cap V\) and \(y \in U \cap M_1\) and \(\varepsilon > 0\) such that \(B(x, \varepsilon) \subset V\) and \(B(y, \varepsilon) \subset U\). By Theorem 2.3, there exist \(k_0 \in \mathbb{N}^*\) so that for all \(k \geq k_0\), \(\|T^{n_k}x\| \leq \varepsilon\), \(\|S_{n_k}(y)\| \leq \varepsilon\), and \(\|T^{n_k}_{n_k}S_{n_k}(y) - y\| \leq \varepsilon\). Then for each \(k \geq k_0\) we have \(z_k = x + S_{n_k}y \in B(x, \varepsilon) \subset V\) and \(\|T_{n_k}^{n_k}z_k \in B(y, \varepsilon) \subset U\). That is, \(\frac{T_{n_k}}{n_k}V \cap U \neq \emptyset\), \(\forall k \geq k_0\). Hence \(T\) is subspace cesàro-mixing. \(\square\)

**Theorem 2.5.** Let \(T\) and \(S\) in \(B(H)\) and \(M_1, M_2\) be a nonzero closed subspaces of \(H\) and \((T \oplus S)\) is \((M_1 \oplus M_2)\)-cesàro-mixing operator, then \(T\) and \(S\) are \(M_1\)-cesàro-mixing and \(M_2\)-cesàro-mixing operators, respectively.

**Proof.** Let \(U_1\) and \(U_2\) be open sets in \(M_1\), and \(V_1\) and \(V_2\) be open sets in \(M_2\), then \(U_1 \oplus V_1\) and \(U_2 \oplus V_2\) are open in \(M_1 \oplus M_2\). So there exists an \(n_0 \geq 1\) such that

\[
\left(\left(\left(n^{-1}(T \oplus S)\right)^{-1}\right)(U_1 \oplus V_1) \cap (U_2 \oplus V_2) \neq \emptyset,
\right.
\]

and

\[
\left(\left(\left(n^{-1}(T \oplus S)\right)^{-1}\right)(M_1 \oplus M_2) \subseteq (M_1 \oplus M_2)
\right.
\]

for all \(n \geq n_0\). Then

\[
\left(\left(\left(n^{-1}T^n\right)^{-1}\right)(U_1) \cap U_2 \neq \emptyset, \left(\left(n^{-1}S^n\right)^{-1}\right)(V_1) \cap V_2 \neq \emptyset, \left(\left(n^{-1}T^n\right)\right)(M_1) \subset M_1
\right.
\]

and

\[
\left(\left(\left(n^{-1}S^n\right)^{-1}\right)(M_2) \subset M_2.
\right.
\]

Therefore, \(T\) and \(S\) are \(M_1\)-cesàro-mixing and \(M_2\)-cesàro-mixing operators, respectively. \(\square\)
Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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