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Generation of Anti-Magic Graphs

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Abstract. An anti-magic labeling of a graph G is a one-to-one correspondence between $E(G)$ and $\{1, 2, \cdots, |E|\}$ such that the vertex-sum for distinct vertices are different. Vertex-sum of a vertex $u \in V(G)$ is the sum of labels assigned to edges incident to the vertex u. In this paper, we prove that the splittance of an anti-magic graph admits anti-magic labeling. It was conjectured by Hartsfield and Ringel that every tree other than K_2 has an anti-magic labeling. In this paper, we prove that there exists infinitely many trees that are anti-magic.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. Terms that are not defined in this paper can be refered from book [10]. Let $G = (V, E)$ be a graph and $f : E \to \{1, 2, \cdots, |E|\}$ is a bijective function. For each vertex $u \in V(G)$, the vertex-sum $\varphi_f(u)$ at u is defined as $\varphi_f(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. If $\varphi_f(u) \neq \varphi_f(u)$ for any two distinct vertices u, v of G, then f is called an anti-magic labeling of G. A graph G is called anti-magic if G has an anti-magic labeling. The problem of anti-magic labeling of graphs was introduced by Hartsfield and Ringel [4]. They posed the following conjectures on anti-magic labeling of graphs.

Conjecture 1.1. *[4] Every connected graph other than K_2 is anti-magic.*

Conjecture 1.2. *[4] Every tree other than K_2 is anti-magic.*

In spite of much attention given by many researchers, both conjectures remain open. Alon et al. [1] proved that there is an absolute constant C such that graphs with minimum degree $\delta(G) \geq C\ log|V(G)|$ are anti-magic. Also they proved that all complete partite graphs except K_2

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are anti-magic. Liang and Zhu [6] proved that cubic graphs are anti-magic. Cranston, Liang and Zhu [2] proved that odd degree regular graphs are anti-magic.
For Conjecture 2, J. Shang [9] proved that spiders are antimagic. Kaplan et al. [5] showed that trees without vertices of degree 2 are anti-magic. Liang, Wong and Zhu [7] studied trees with many degree 2 vertices, with restriction on the subgraph induced by degree 2 vertices and its complement. They proved that such trees are anti-magic. For an exhaustive survey on anti-magic graphs, we refer the dynamic survey by Gallian [3].

2. Splittance of an anti-magic graph is anti-magic

In this section, we prove that the splittance of an anti-magic graph is anti-magic. Splittance of a graph was introduced by Sampathkumar and Walikar [8] in the year 1980. Let G be a graph. Add a new vertex u' for every vertex u of G. Add edges between u' and all the vertices of G that are adjacent to vertex u. The graph thus obtained is called splitting graph of G and is denoted as $S(G)$. One can easily observe that if G is a (p,q) graph, then $S(G)$ is a $(2p,3q)$ graph. In [8], Sampathkumar and Walikar proved the following characterization result on splittance of a graph.

Theorem 2.1. *A graph G is a splitting graph if and only if $V(G)$ can be partitioned into two sets $V_1 \cup V_2$ such that (i) there exists a bijective mapping $V_1 \to V_2$ and (ii) $N(v_2) = N(v_1) \cap V_1$, where $N(v) = \{u : uv \in E(G)\}$.*

Now, let us prove one of our main results.

Theorem 2.2. *Let G be an anti-magic graph such that $\delta(G) \geq 2$. Then the splittance graph of G is anti-magic.*

Proof. Let G be an anti-magic graph with n vertices and m edges. Consider $f : E \to \{1, 2, \cdots, |E|\}$ be the anti-magic labeling of G. Also, for each vertex u of G, the vertex-sum $\varphi_f(u) = \sum\limits_{e \in E(u)} f(e)$ at u is distinct, where $E(u)$ is the set of edges incident to u. For convenience, let us arrange and label the vertices of G as $u_1, u_2, \cdots, u_n$ such that $\varphi_f(u_1) < \varphi_f(u_2) < \varphi_f(u_3) \cdots < \varphi_f(u_n)$. Let us arrange and label the edges of G as $e_1, e_2, \cdots, e_m$ such that $f(e_i) = i$ for $1 \leq i \leq m$. Let $u'_1, u'_2, \cdots, u'_n$ are the set of new vertices with respect to the set of vertices $u_1, u_2, u_3, \cdots, u_n$ respectively. For any vertex u'_i, $1 \leq i \leq n$, let us introduce the new edges and label them as $e_1^{(i)}, e_2^{(i)}, \cdots, e_{k_i}^{(i)}$, where $k_i = deg(u_i)$, such that their counterpart edges in the graph G has increasing edge labels as defined by the bijective function f. More precisely, the newly added edges incident to vertex u'_i are arranged in such a way that their arrangement $f(e_1) < f(e_2) < \cdots < f(e_{k_i})$ as defined by the function f.
In the above set up, the vertex set of splittance of graph $S(G)$ can be partitioned as $V(S(G)) = V_1 \cup V_2$, where $V_1 = V(G)$ and $V_2 = \{u'_1, u'_2, \cdots, u'_n\}$. The edges of splittance of graph $S(G)$ can be partitioned as $E(S(G)) = E_1 \cup E_2$, where $E_1 = E(G)$ and $E_2 = \{e_1^{(i)}, e_2^{(i)}, \cdots, e_{k_i}^{(i)}\}$, for $1 \leq i \leq n$. It is clear that $S(G)$ has $2n$ vertices and $3m$ edges. Now, let us define the bijective function $s : E(S(G)) \to \{1, 2, \cdots, 3m\}$ as follows:

For any edge $e_r^{(i)} \in E_2$, for $1 \leq i \leq n$ and $1 \leq r \leq deg(u_i)$:

$$s(e_r^{(i)}) = r + \sum_{j=1}^{i-1} deg(u'_j) \tag{2.1}$$

For any edge $e_i \in E_1$, $1 \leq m$:

$$s(e_i) = 2m + f(e_1) \tag{2.2}$$

It is clear that from equation (2.1), the edge labels of $2m$ edges in E_2 are distinct and are from the set $\{1, 2, 3, \cdots, 2m\}$ as defined by function s. Also from equation (2.2), it is clear that the edge labels of m edges in E_1 are distinct and are from the set $\{2m+1, 2m+2, \cdots, 3m\}$ as defined by function s. Therefore, the function s is bijective.

Claim: Vertex-sum $\varphi_s(u)$ for any vertex $u \in V(S(G))$ is distinct.

Proof. Recall that for each vertex u of $S(G)$, the vertex-sum $\varphi_s(u)$ at u is defined by $\varphi_s(u) = \sum_{e \in E(u)} s(e)$, where $E(u)$ is the set of edges incident to u. By the construction and arrangement of vertices and edges of $S(G)$ and since $\delta(G) \geq 2$ and hence $\delta(S(G)) \geq 2$, we can form the monotonically increasing sequence of vertex-sums of vertices of $S(G)$ as follows:
$\varphi_s(u'_1), \varphi_s(u'_2), \varphi_s(u'_3), \cdots, \varphi_s(u'_n)$ followed by $\varphi_s(u_1), \varphi_s(u_2), \varphi_s(u_3), \cdots, \varphi_s(u_n)$
Therefore, Vertex-sum $\varphi_s(u)$ for any vertex $u \in V(S(G))$ is distinct. □

By the construction and arrangement of vertices and edges of $S(G)$, we defined a bijective function $s : E(S(G)) \to \{1, 2, \cdots, 3m\}$ and hence vertex-sum for any vertex in $S(G)$ is distinct. Therefore, splittance graph $S(G)$ is anti-magic. Hence the proof. □

3. Construction of anti-magic trees

In this section, we construct infinitely many anti-magic trees given an anti-magic tree. To prove our result, we introduce some basic definitions.

Definition 3.1. *Let G be an anti-magic graph whose anti-magic labeling is given by bijective function $f : E(G) \to \{1, 2, \cdots, |E|\}$. Let $k = \max_{u \in V(G)} \{\varphi_f(u)\}$. A vertex $u \in V(G)$ is said to be anti-magic maximum vertex if $\varphi_f(u) = k$ and we denote such vertex as $\hat{u}$.*

Definition 3.2. *Let T be an anti-magic tree. Construct a tree by considering two copies namely $T^{(1)}$ and $T^{(2)}$ of T. Add an edge between the anti-magic maximum vertex of $T^{(1)}$ and $T^{(2)}$. We denote the tree thus obtained as $\hat{T}$.*

Remark 3.1. *If T has m edges, then $\hat{T}$ has $2m+1$ edges. Further, it is clear that anti-magic maximum vertex in any anti-magic graph is unique with respect to the anti-magic labeling.*

Theorem 3.1. *Let T be an anti-magic tree. Then $\hat{T}$ admits anti-magic labeling.*

Proof. Since T is an anti-magic tree with m edges, there exists a bijective function $f : E(T) \rightarrow \{1,2,\cdots,m\}$ and its vertex-sum of vertices $\varphi_f(u)$ for any vertex $u \in V(T)$ form a monotonically increasing sequence. For convenience, let us arrange the vertices of T as $u_1, u_2, \cdots, u_{m+1}$ such that $\varphi_f(u_1) < \varphi_f(u_1) < \varphi_f(u_3) < \cdots < \varphi_f(u_{m+1})$. In view of definition , $u_{m+1} = \hat{u}$ with respect to the bijective function f. Similarly, let us arrange the edges of T as $e_1, e_2, \cdots, e_m$ such that $f(e_1) < f(e_2) < \cdots < f(e_m)$. Denote $u_1^{(1)}, u_2^{(1)}, \cdots, u_{m+1}^{(1)}$ and $u_1^{(2)}, u_2^{(2)}, \cdots, u_{m+1}^{(2)}$ be the arrangement of vertices in the first copy and second copy of T respectively. Similarly, denote $e_1^{(1)}, e_2^{(1)}, \cdots, e_m^{(1)}$ and $e_1^{(2)}, e_2^{(2)}, \cdots, e_m^{(2)}$ be the arrangement of edges in the first copy and second copy of T respectively. Denote $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ be the anti-magic maximum vertices of first copy and second copy of T respectively. Observe that $\hat{T} = T^{(1)} \cup T^{(2)} + \hat{e}$, where $\hat{e} = (\hat{u}^{(1)}, \hat{u}^{(2)})$.
Now, let us define a bijective function $s : E(\hat{T}) \rightarrow \{1,2,\cdots,2m+1\}$ as follows:
For any edge $e_i^{(1)}$, for $1 \le i \le m$,

$$s(e_i^{(1)}) = 2f(e_i) - 1 \tag{3.1}$$

For any edge $e_i^{(2)}$, for $1 \le i \le m$,

$$s(e_i^{(2)}) = 2f(e_i) \tag{3.2}$$

$$s(\hat{e}) = 2m + 1 \tag{3.3}$$

By the definition of function s, it is clear that it is a bijective function defined on the edge set of $\hat{T}$.

Claim: Vertex-sum $\varphi_s(u)$ for any vertex $u \in V(\hat{T})$ is distinct.

Proof. By the construction and arrangement of vertices and edges of $\hat{T}$, we can form the monotonically increasing sequence of vertex-sum of vertices of $\hat{T}$ as follows:
$\varphi_s(u_1^{(1)}), \varphi_s(u_1^{(2)}), \varphi_s(u_2^{(1)}), \varphi_s(u_2^{(2)}), \cdots, \varphi_s(u_i^{(1)}), \varphi_s(u_i^{(2)}), \cdots, \varphi_s(\hat{u}^{(1)}), \varphi_s(\hat{u}^{(2)})$
Therefore, Vertex-sum $\varphi_s(u)$ for any vertex $u \in V(\hat{T})$ is distinct. □

By the construction and arrangement of vertices and edges of $\hat{T}$, we defined a bijective function $s : E(\hat{T}) \rightarrow \{1,2,\cdots,2m+1\}$ and hence vertex-sum for any vertex in $\hat{T}$ is distinct. Therefore, $\hat{T}$ is anti-magic. Hence the proof.

□

Remark 3.2. *We can construct infinitely many anti-magic trees by recursively applying Theorem 3.*

4. Conclusion

In this paper, we proved that splittance of an anti-magic graph is anti-magic. Further, we proved that there exists infinitely many anti-magic trees. Our results in this paper strongly supports the conjectures that every connected graph other than K_2 is anti-magic and every tree other than K_2 is anti-magic, posed by Hartsfield and Ringel [4].

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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