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# Typical Sequence of Real Numbers From the Unit Interval Has All Distribution Functions

**József Bukor**[1,*], **Kálmán Liptai**[2], **János T. Tóth**[2]

[1]*Department of Informatics, J. Selye University, Komárno, Slovakia*
[2]*Department of Mathematics, J. Selye University, Komárno, Slovakia*

*Corresponding author: bukorj@ujs.sk

**Abstract.** This note is devoted to the study of typical properties (in Baire category sense) of sequences of real numbers in $[0,1]$. We prove that the subset of sequences that have all distribution functions forms a residual set.

## 1. Introduction

The concept of Baire categories is one of the possibilities to compare sets. Let $S$ be a metric space. A subset $A \subseteq S$ is called *meager* (or of first category) if $A$ can be written as a countable union of nowhere dense sets. Any set that is not meager is said to be of second category. The complement of a meager set is called *residual*. We say that a typical element $x$ has property $P$ if the set $A = \{x \in S | x \text{ has property } P\}$ is residual. For more details we refer the reader to Oxtoby [6].

There are analogous results in Baire category sense for the digit sequences of numbers $z \in [0,1]$ and the sequences of real numbers. We mention some results. For a fixed positive integer $s$ the unique, non-terminating, base $s$ expansion of a number $z \in [0,1]$ is

$$z = \frac{d_1(z)}{s} + \frac{d_2(z)}{s^2} + \cdots + \frac{d_n(z)}{s^n} + \cdots \qquad \text{with } d_i(z) \in \{0,1,\ldots,s-1\}.$$

For each digit $i \in \{0,1,\ldots,s-1\}$ let $\Pi_i(z;n)$ denote the frequency of the digit $i$ among the first $n$ digit of $z$. It was proved by Šalát [7] that for a typical $z$, we have $\limsup_{n\to\infty} \Pi_i(z;n) = 1$ and $\liminf_{n\to\infty} \Pi_i(z;n) = 1$. Define the frequency of the digits $i \le x$ among the first $n$ digits of $z$ as

$$F_{z,n}(x) = \sum_{i\le x} \Pi_i(z;n).$$

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Let $\mathcal{F}$ denote the set of all distribution functions of discrete random variables that takes on one of the possible values $0, 1, \dots, s-1$. Using this notation, we mention Olsen's [4] fundamental result. For a typical number $z$ we have that for any $f \in \mathcal{F}$ there exists an increasing sequence $n_1, n_2, \dots$ for that $\lim_{k\to\infty} F_{z,n_k}(x) = f$. Roughly speaking, the digit expansion of a typical number $z$ has all distribution functions from $\mathcal{F}$.

We will consider the metric space $S$ of all sequences of real numbers in $[0,1]$ with the Fréchet metric

$$\rho(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

where $\mathbf{x} = (x_k)$, $\mathbf{y} = (y_k)$. It is known that $(S, \rho)$ is a complete metric space.

In [3] it was proved that the set of all uniformly distributed sequences is a dense subset of the first Baire category in $S$. The same is true for the set of all statistically convergent sequences of real numbers (cf. [8]). The sequence $(x_n)$ is *maldistributed* if for any non-empty interval $I$ the set $\{n \in \mathbb{N} : x_n \in I\}$ has upper asymptotic density 1.

Examples of maldistributed sequences are given in [9] and [2]. In [1] the authors proved that a typical real sequence is maildistributed. The maildistribution property can be characterized by one-jump distribution functions [9], so a typical real sequence has all one-jump distribution functions.

The aim of this not to show that a typical real sequence has all distribution function.

1.1. **Basic notations and properties of distribution functions.** We recall some basic notations and results concerning distribution functions of sequences (e.g., see [11] and [10]).

- Let $\mathbf{x} = (x_n)$ be a sequence from unit interval $[0,1]$.
- Let $\chi_A(x)$ denote the characteristic function of the set $A$.
- Denote by

$$F_N(x) = \frac{\#\{n \le N; x_n \in [0,x)\}}{N} = \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,x)}(x_n)$$

  the step distribution function for $x \in [0,1)$, and for $x = 1$ we define $F_N(1) = 1$.
- A non-decreasing function $g : [0,1] \to [0,1]$, $g(0) = 0$, $g(1) = 1$ is called a *distribution function* (abbreviated d.f.). We shall identify any two d.f.s coinciding at common points of continuity. Denote by $\mathcal{G}$ the set of all distribution functions.
- A d.f. $g(x)$ is a d.f. of the sequence $\mathbf{x}$, if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\lim_{k\to\infty} F_{n_k}(x) = g(x)$$

  almost everywhere on $[0,1]$. This is equivalent to the weak convergence, i.e., the preceding limit holds for every point $x \in [0,1]$ of continuity of $g(x)$. Let $G(\mathbf{x})$ denote the set of all d.f.s of $\mathbf{x}$.
- $c_\gamma(x)$ is one-step d.f. for which $c_\gamma(x) = 0$ for $x \in [0,\gamma]$ and $c_\gamma(x) = 1$ for $x \in (\gamma, 1]$.

- For every sequence $\mathbf{x}$ there hold that $G(\mathbf{x})$ is closed and $G(\mathbf{x})$ is connected in the weak topology defined by the metric

$$d(g_1, g_2) = \left( \int_0^1 (g_1(x) - g_2(x))^2 \mathrm{d}x \right)^{\frac{1}{2}}. \tag{1.1}$$

- For given a non–empty set $H$ of d.f.s there exists a sequence $\mathbf{x}$ in $[0,1)$ such that $G(\mathbf{x}) = H$ if and only if $H$ is closed and connected.
- First Helly theorem. Every sequence $g_n(x)$ of d.f.s contains a subsequence $g_{k_n}(x)$ such that $\lim_{n\to\infty} g_{k_n}(x) = g(x)$ for every $x \in [0,1]$. Furthermore, the point limit $g(x)$ is d.f. again.

## 2. Results

First, we show that a typical sequence has distribution function, which in given point has the function value "near" to the prescribed value.

**Lemma 2.1.** *Let $a, b \in (0,1)$. For a positive number $\gamma < \min\{b, \frac{a}{4}, \frac{1-a}{4}\}$ denote by $\mathcal{A}(a,b,\gamma)$ the set of all $\mathbf{x} = (x_k) \in \mathbf{S}$ for which there is an $n_0$ such that for any $n \geq n_0$ we have*

$$\sum_{i=1}^{n} \chi_{[0,a-\gamma)}(x_i) < (b-\gamma)n \quad or \quad \sum_{i=1}^{n} \chi_{[0,a+\gamma)}(x_i) > (b+\gamma)n. \tag{2.1}$$

*Then $\mathcal{A}(a,b,\gamma)$ is a set of the first Baire category in $\mathbf{S}$.*

*Proof.* We define continuous functions $h_{a,\gamma} : [0,1] \to [0,1]$ and $t_{a,\gamma} : [0,1] \to [0,1]$ by

$$h_{a,\gamma}(x) = \begin{cases} 1 & \text{for } x \in \left[0, a-2\gamma\right] \\ \frac{a-\gamma-x}{\gamma} & \text{for } x \in [a-2\gamma, a-\gamma] \\ 0 & \text{for } x \in [a-\gamma, 1] \end{cases}, \quad t_{a,\gamma}(x) = \begin{cases} 1 & \text{for } x \in [0, a+\gamma] \\ \frac{a+2\gamma-x}{\gamma} & \text{for } x \in [a+\gamma, a+2\gamma] \\ 0 & \text{for } x \in [a+2\gamma, 1] \end{cases},$$

see Figure 1.

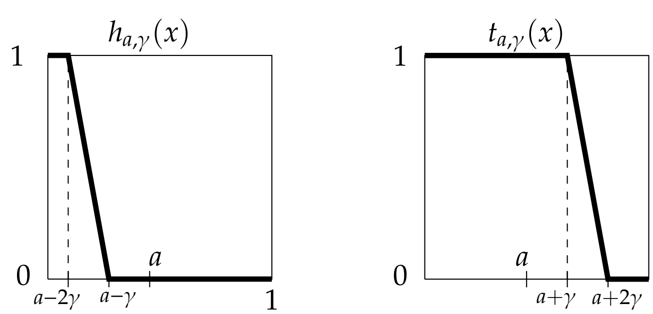


Figure 1. Functions $h_{a,\gamma}(x)$ and $t_{a,\gamma}(x)$

For these functions, we have $h_{a,\gamma}(x) \leq \chi_{[0,a)}(x) \leq t_{a,\gamma}(x)$, where $x \in [0,1]$. Using the functions $h_{a,\gamma}$, $t_{a,\gamma}$ we define for $\mathbf{x} \in \mathbf{S}$ and fixed $n$ the function $f_n : \mathbf{S} \to [0,1]$ in the following way:

$$f_n(\mathbf{x}) = \min\left\{1, \left(\frac{\sum_{i=1}^{n} h_{a,\gamma}(x_i)}{(b-\frac{\gamma}{2})n}\right)^n\right\}.\min\left\{1, \left(\frac{(b+\frac{\gamma}{2})n}{1+\sum_{i=1}^{n} t_{a,\gamma}(x_i)}\right)^n\right\}.$$

Denote $\mathcal{A}^*(a,b,\gamma)$ the set of all $\mathbf{x} \in S$ for which there exists the limit $\lim_{n\to\infty} f_n(\mathbf{x})$.
One can easily check that if (2.1) holds for all sufficiently large $n$, then $f_n(\mathbf{x}) \to 0$ for $n \to \infty$. Therefore $\mathcal{A}(a,b,\gamma) \subset \mathcal{A}^*(a,b,\gamma)$.
Put $f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x})$ for $\mathbf{x} \in \mathcal{A}^*(a,b,\gamma)$. We shall prove that
(a) the function $f_n$ $(n = 1,2,\dots)$ is a continuous function on $S$,
(b) $f$ is discontinuous at each point of $\mathcal{A}^*(a,b,\gamma)$.

(a) the continuity of the functions $f_n$ follows from the facts that the functions $h_{a,\gamma}$, $t_{a,\gamma}$ are continuous and the convergence in the space $S$ is the coordinate convergence.

(b) Let $\mathbf{y} = (y_k) \in \mathcal{A}^*(a,b,\gamma)$. We have the following two possibilities.
(1) $f(\mathbf{y}) < 1$,
(2) $f(\mathbf{y}) = 1$.

In case (1) we choose a positive $\varepsilon$ such that $\varepsilon < 1 - f(\mathbf{y})$. It is suffice to prove that in each ball $K(\mathbf{y},\delta) = \{\mathbf{x} \in \mathcal{A}^*(a,b,\gamma),\ \rho(\mathbf{x},\mathbf{y}) < \delta\}$ $(\delta > 0)$ of the subspace $\mathcal{A}^*(a,b,\gamma)$ of $S$ there exists an element $\mathbf{x} \in S$ with $|f(\mathbf{x}) - f(\mathbf{y})| > \varepsilon$.

Let $\delta > 0$ is given. Choose a positive integer $m$ such that $\sum_{k=m+1}^{\infty} 2^{-k} < \delta$. Choose a d.f. $g(x) \in \mathcal{G}$ which is continuous in $x = a$ and $g(a) = b$. Then there exists a sequence $z \in S$ for that $G(z) = \{g(x)\}$. Define the sequence $\mathbf{x}$ in the following way:

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ \frac{a}{2}, & \text{if } k > m \text{ and } z_k \in [0,a), \\ \frac{a+1}{2}, & \text{if } k > m \text{ and } z_k \in [a,1] \end{cases}$$

Hence $\rho(\mathbf{x},\mathbf{y}) < \delta$. Furthermore, $\frac{1}{n}\sum_{i=1}^{n} \chi_{[0,a)}(x_i)$, $\frac{1}{n}\sum_{i=1}^{n} h_{a,\gamma}(x_i)$ and $\frac{1}{n}\sum_{i=1}^{n} t_{a,\gamma}(x_i)$ tend to $b$ as $n \to \infty$. Then $f_n(\mathbf{x}) = 1$ for all sufficiently large $n$ and therefore $f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x}) = 1$. Then immediately follows

$$f(\mathbf{x}) - f(\mathbf{y}) = 1 - f(\mathbf{y}) > \varepsilon.$$

In case (2) we have $g(\mathbf{y}) = 1$. Let $\delta, m, \mathbf{x}$ have the previous meaning. Put

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ \frac{a+1}{2}, & \text{if } k > m \text{ and } b \ge \frac{1}{2} \\ \frac{a}{2}, & \text{if } k > m \text{ and } b < \frac{1}{2}. \end{cases}$$

Then, clearly $\rho(\mathbf{x},\mathbf{y}) < \delta$, and for sufficiently large $n$ one of the inequalities (2.1) must be true. So, we have $f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x}) = 0$, and therefore $f(\mathbf{y}) - f(\mathbf{x}) = 1 - 0 > 0$. Hence the discontinuity of $f$ at $\mathbf{y} \in \mathcal{A}^*(I,\gamma)$ has been proved.

The function $f$ is a limit function (on $\mathcal{A}^*(a,b,\gamma)$ ) of the sequence of continuous functions $(f_n)_{n=1}^{\infty}$ on $\mathcal{A}^*(a,b,\gamma)$. Then the function $f$ is a function in the first Baire class on $\mathcal{A}^*(a,b,\gamma)$. According to the well-known fact that the set of discontinuity points of an arbitrary function of the first Baire class is a set of the first Baire category (cf. [6], p. 32), we see that the set $\mathcal{A}^*(a,b,\gamma)$ is of the first Baire category in $\mathcal{A}^*(a,b,\gamma)$. Thus $\mathcal{A}^*(a,b,\gamma)$ is in $\mathbf{S}$, too. Since $\mathcal{A}(a,b,\gamma) \subset \mathcal{A}^*(a,b,\gamma)$, the assertion follows. □

**Consequence 2.1.** *For any $a,b \in (0,1)$ the set*

$$\mathcal{P} = \{\mathbf{x} \in \mathbf{S} |\ \textit{there is a } g(x) \in G(\mathbf{x}) \textit{ for that } g(a) = b\}$$

*is residual in $\mathbf{S}$.*

*Proof.* By Lemma 2.1 we have that the infinite union $\bigcup_{n=n_0}^{\infty} \mathcal{A}(a,b,\frac{1}{n})$ (where $\frac{1}{n_0} < \min\{b, \frac{a}{4}, \frac{1-a}{4}\}$) is a meager set in $S$. Let $\mathbf{x}$ be a sequence from the complementary set to the mentioned infinite union. For the step d.f. of $\mathbf{x}$ we have

$$b-\gamma \le F_N(a-\gamma) \le F_N(a) \le F_N(a+\gamma) \le b+\gamma$$

for any $\gamma = \frac{1}{n}$ ($n \ge n_0$) and infinitely many $N$. In this case, First Helly theorem implies that there exists a pointwise convergent subsequence with limit $g(x) \in G(\mathbf{x})$ for that $g(a) = b$. So, $\mathbf{x} \in \mathcal{P}$ and it means that the set $\mathcal{P}$ is residual in $\mathbf{S}$. □

Remark. The assertion of Lemma 2.1 holds for the case $b = 1$, too. For the case $b = 0$ we only need to consider right-hand side inequality of (2.1).

In what follows, for simplicity, we will use the notation $\mathbf{a}_l$ for a finite sequence $(a_k)_{k=1}^{l}$ and the notation $\mathbf{b}_l$ for a finite sequence $(b_k)_{k=1}^{l}$. We extend the assertion of Lemma 2.1 for arbitrary number of finite points.

**Lemma 2.2.** *Let a positive integer $l$ and finite sequences $\mathbf{a}_l$ and $\mathbf{b}_l$ are given, where $0 < a_1 < a_2 \cdots < a_l < 1$ and $0 < b_1 \le b_2 \cdots \le b_l \le 1$. For a positive number*

$$\gamma < \min\left\{b_1, \frac{a_1}{4}, \frac{a_2-a_1}{4}, \frac{a_3-a_2}{4}, \dots, \frac{a_l-a_{l-1}}{4}, \frac{1-a_l}{4}\right\}$$

*denote by $\mathcal{A}(\mathbf{a}_l,\mathbf{b}_l,\gamma)$ the set of all $\mathbf{x} = (x_k) \in \mathbf{S}$ for which there is an $n_0$ such that for any $n \ge n_0$ we have that at least one of the inequalities*

$$\sum_{i=1}^{n} \chi_{[0,a_j-\gamma)}(x_i) < (b_j-\gamma)n, \qquad \sum_{i=1}^{n} \chi_{[0,a_j+\gamma)}(x_i) > (b_j+\gamma)n \tag{2.2}$$

*hold ($j = 1,2,\dots,n$). Then $\mathcal{A}(\mathbf{a}_l,\mathbf{b}_l,\gamma)$ is a set of the first Baire category in $\mathbf{S}$.*

*Proof.* The proof is analogous to the proof of Lemma 2.1. We mention only the differences. The crucial role is played by the function $f_n : \boldsymbol{S} \to [0,1]$ given by

$$f_n(\mathbf{x}) = \prod_{j=1}^{l} \left( \min\left\{1, \left( \frac{\sum_{i=1}^{n} h_{a_j,\gamma}(x_i)}{(b_j - \frac{\gamma}{2})n} \right)^n \right\} . \min\left\{1, \left( \frac{(b_j + \frac{\gamma}{2})n}{1 + \sum_{i=1}^{n} t_{a_j,\gamma}(x_i)} \right)^n \right\} \right).$$

Denote $\mathcal{A}^*(\mathbf{a}_l, \mathbf{b}_l, \gamma)$ the set of all $\mathbf{x} \in \boldsymbol{S}$ for which there exists the limit $\lim_{n\to\infty} f_n(\mathbf{x})$. Similarly as before, $\mathcal{A}(\mathbf{a}_l, \mathbf{b}_l, \gamma) \subset \mathcal{A}^*(\mathbf{a}_l, \mathbf{b}_l, \gamma)$ and put $f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x})$ for $\mathbf{x} \in \mathcal{A}^*(\mathbf{a}_l, \mathbf{b}_l, \gamma)$.
In case (b) (1) we prove that $f$ is discontinuous in any $\mathbf{y} \in \mathcal{A}^*(\mathbf{a}_l, \mathbf{b}_l, \gamma)$, where $f(\mathbf{y}) < 1$. Let a positive $\varepsilon < 1 - f(\mathbf{y})$ be given. For given $\delta > 0$ we choose $m$ by the same way. Choose a d.f. $g(x) \in \mathcal{G}$ which is continuous in points $x = a_j$ and $g(a_j) = b_j$ (for $j = 1, 2, \dots, l$. Then there exists a sequence $\boldsymbol{z} \in \boldsymbol{S}$ for that $G(\boldsymbol{z}) = \{g(x)\}$. For simplicity, denote by $a_0 = 0$ and $a_{l+1} = 1$. Define the sequence $\mathbf{x}$ in the following way:

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ \frac{a_{j-1}+a_j}{2} & \text{if } k > m \text{ and } z_k \in [a_{j-1}, a_j), \quad j = 1, 2, \dots, l+1 \\ 1, & \text{if } k > m, \text{ and } z_k = 1. \end{cases}$$

Then $\frac{1}{n}\sum_{i=1}^{n} \chi_{[0,a_j)}(x_i)$, $\frac{1}{n}\sum_{i=1}^{n} h_{a_j,\gamma}(x_i)$ and $\frac{1}{n}\sum_{i=1}^{n} t_{a_j,\gamma}(x_i)$ tend to $b_j$ as $n \to \infty$ ($j = 1, 2, \dots, l$). So, $f_n(\mathbf{x}) = 1$ for all sufficiently large $n$. Therefore $f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x}) = 1$.

In case (b) (2) we have $g(\mathbf{y}) = 1$. Let $\delta, m, \mathbf{x}$ have the previous meaning. Put

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ \frac{a_l+1}{2}, & \text{if } k > m \text{ and } b_1 \ge \frac{1}{2} \\ \frac{a_1}{2}, & \text{if } k > m \text{ and } b_1 < \frac{1}{2}. \end{cases}$$

Then, clearly $\rho(\mathbf{x}, \mathbf{y}) < \delta$, and for sufficiently large $n$ at least one of the inequalities (2.2) have to be true. So, we have $f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x}) = 0$, and therefore $f(\mathbf{y}) - f(\mathbf{x}) = 1 - 0 > 0$. The rest of the proof follows by the same way as the proof of Lemma 2.1. □

**Theorem 2.2.** *Let $\mathcal{H}$ be a subset of $\boldsymbol{S}$ with the property: if $\mathbf{x} \in \mathcal{H}$ than for any positive integer $l$ and arbitrary finite rational sequences $\mathbf{a}_l$ and $\mathbf{b}_l$ with the properties $0 < a_1 < a_2 \cdots < a_l < 1$ and $0 < b_1 \le b_2 \cdots \le b_l \le 1$, there is a d.f. $g(x) \in G(\mathbf{x})$ for that $g(a_j) = b_j$ ($j = 1, 2, \dots, l$). Then $\mathcal{H}$ is residual in $\boldsymbol{S}$.*

*Proof.* If we take unions of the sets $\mathcal{A}(\mathbf{a}_l, \mathbf{b}_l, \gamma)$ for all positive integers $l = 1, 2, \dots$, for all rational numbers $a_1, \dots, a_l$, $b_1, \dots, b_l$, $\gamma = \frac{1}{n}$ ($n = 1, 2 \dots$) satisfying the necessary conditions, we get countable union of meager sets, which is still set of first Baire category in $\boldsymbol{S}$. The complement of this set is residual in $\boldsymbol{S}$. □

**Theorem 2.3.** *Let $\mathcal{M} = \{\mathbf{x} \in S \,|\, G(\mathbf{x}) = \mathcal{G}\}$. Then the set of sequences $\mathcal{M}$ is residual in $\boldsymbol{S}$.*

*Proof.* Let us consider a sequence $\mathbf{x} \in \mathcal{H}$ and a d.f. $g(x) \in \mathcal{G}$. As $g(x)$ is monotone on $[0, 1]$, then it is Riemann integrable. Thus, in the sense of (1.1) we can approximate $g(x)$ with arbitrary precision with d.f.s from $G(\mathbf{x})$ which have positive rational function values in points of equidistant partition of the unit interval. It means, that $\mathbf{x} \in \mathcal{M}$ and the assertion follows. □

**Problem 2.1.** *In [5] it was proved that a typical (in the sense of Baire) point x has the following property: the all higher order Cesàro averages of digits of x have all distribution functions for discrete random variables that takes possible values of the digits. Let us denote by $C(\mathbf{x})$ the Cesàro average of the sequence $\mathbf{x}$. It seems to be interesting to ask whether the Cesàro average of a typical sequence has all distribution function? More precisely, is the set of sequences*

$$\{\mathbf{x} \in S \,|\, G(C(\mathbf{x})) = \mathcal{G}\}$$

*also residual in* $\mathbf{S}$*?*

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

[1] J. Bukor, J. T. Tóth, On Topological Properties of the Set of Maldistributed Sequences, Acta Univ. Sapient. Math. 12 (2020), 272–279. https://doi.org/10.2478/ausm-2020-0018.
[2] P. Kostyrko, M. Mačaj, T. Šalát, O. Strauch, On Statistical Limit Points, Proc. Amer. Math. Soc. 129 (2000), 2647–2654. https://doi.org/10.1090/s0002-9939-00-05891-3.
[3] V. László, T. Šalát, The Structure of Some Sequence Spaces, and Uniform Distribution (Mod 1), Period Math. Hung. 10 (1979), 89–98. https://doi.org/10.1007/bf02018376.
[4] L. Olsen, Extremely Non-Normal Numbers, Math. Proc. Camb. Phil. Soc. 137 (2004), 43–53. https://doi.org/10.1017/s0305004104007601.
[5] J. Hyde, V. Laschos, L. Olsen, T. Petrykiewicz, A. Shaw, Iterated Cesàro Averages, Frequencies of Digits, and Baire Category, Acta Arith. 144 (2010), 287–293. https://doi.org/10.4064/aa144-3-6.
[6] J.C. Oxtoby, Measure and Category, Springer, New York, 1996.
[7] T. Šalát, A Remark on Normal Numbers, Rev. Roumaine Math. Pures Appl. 11 (1966), 53–56.
[8] T. Šalát, On Statistically Convergent Sequences of Real Numbers, Math. Slovaca, 30 (1980), 139–150. http://dml.cz/dmlcz/136236.
[9] O. Strauch, Uniformly Maldistributed Sequences in a Strict Sense, Monatsh. Math. 120 (1995), 153–164. https://doi.org/10.1007/bf01585915.
[10] O. Strauch, Distribution of Sequences: A Theory, VEDA, Bratislava; Academia, Prague, 2019.
[11] O. Strauch, Š. Porubský, Distribution of Sequences: A Sampler, Peter Lang, Frankfurt am Main, 2005.