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Exploring Fixed Points and Common Fixed Points of Contractive Mappings in Complex-Valued Intuitionistic Fuzzy Metric Spaces

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Abstract. The current manuscript aims to introduce complex-valued intuitionisitic fuzzy metric spaces as a fresh perspective on complex-valued fuzzy metric spaces and intuitionistic fuzzy metric spaces. Existence together with distinctiveness of the fixed points within maps with diverse contractive criteria in this novel space are established. Additionally, our work yields a few common fixed-point findings for intuitionistic fuzzy Banach contraction on this newly introduced space. The outcomes presented in this study go beyond the existing literature, adding to the growing body of knowledge in this field. Our research outcomes are exemplified through examples that are included in this paper to help readers better grasp our findings. Our paper concludes with a discussion of how our findings can be applied to the problem of determining the presence of an exclusive solution for Fredholm integral equations.

1. Introduction

Fixed-point theory is a powerful tool in mathematical analysis that has broad applicability. The renowned Banach contraction principle, which originated in [4], is widely employed in solving problems related to the existence of solutions in nonlinear analysis. It has been generalized into various versions of fixed-point theorems and has been developed using different approaches.

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Ambiguous and vague situations in natural phenomena or real-life problems cannot always be expressed by mathematical models using classical set theory. To tackle this issue, Zadeh [27] established fuzzy sets, indicating an element's membership in a set by assigning it a value from the interval $[0,1]$. Later, Atanassov [2] proposed intuitionistic fuzzy sets, which allow for the representation of degree of uncertainty when assigning element's membership and non-membership in a set.

Kramosil and Michalek [18] put forth fuzzy metric spaces, extending probabilistic metric spaces. The investigation into fuzzy metric fixed-point idealogy was pioneered under Grabiec [11]. By introducing G-Cauchy sequences and G-completeness, he laid the foundation for a fuzzy counterpart of Banach contraction principle on fuzzy metric spaces inspired by [18]. George and Veeramani [8] altered fuzzy metric spaces in 1994, which resulted in the emergence of a Hausdorff topology on such spaces. They also proposed modifications to Grabiec's Cauchy sequence concept and demonstrated various fixed-point outcomes on the modified spaces. In 2004, Park [21] put forward the framework of intuitionistic fuzzy metric spaces which broaden fuzzy metric space's scope. Up to the present moment, researchers continue to delve deeper into the exploration of fuzzy metric fixed-point theory. The research diverges primarily into two directions: broadening the category of fuzzy metric spaces to encompass a more general scope (detailed exploration is available in [5, 6, 19, 20, 22, 25]), and investigating the presence of fixed points for mappings subject to numerous contractive conditions (for detailed insights, refer to [1, 9, 10, 23]).

Complex-valued metric spaces were brought into metric fixed-point theory by Azam et al. [3] in 2011. They deviated from the conventional method of employing the set of positive real numbers, instead utilizing ordered complex numbers to attain fixed-point outcomes for mappings subject to rational inequality criteria. Recently, Shukla et al. [24] employed this concept in the realm of fuzzy metric fixed-point theory. By defining complex-valued fuzzy metric spaces, they formulated several fixed-point findings for transformationst fulfilling certain contractive criteria on such spaces. In the present era, significant research interest is directed towards exploring fixed-point findings for mappings with complex-valued fuzzy metric approach. Prominent examples of such research include the works of [7, 28] and the extensive investigations conducted by Humaira et al. [12–16], which derived numerous relevance outcomes, along with practical applications.

This manuscript presents a novel extension to the category of fuzzy metric spaces through introduction of complex-valued intuitionistic fuzzy metric spaces. This new concept generalizes both complex-valued fuzzy metric spaces by [24] alongside intuitionistic fuzzy metric spaces by [21]. Several fixed-point outcomes for transformations subject to contractive constraints in newly defined spaces are presented. Furthermore, we expand fuzzy variant of Banach contraction to intuitionistic fuzzy spaces, establishing common fixed-point outcomes within complex-valued intuitionistic fuzzy metric spaces. Practical examples, including application are provided to demonstrate the usefulness and relevance of our results.

2. Preliminaries

This section offers a concise rundown of essential notions within complex-valued fuzzy metric spaces, as established in prior works by [24]. In the present work, the notations $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{C}$ refer to, in order, the collection of natural numbers, non-negative integers, and complex numbers. For every $z \in \mathbb{C}$, we express $z = a + ib$ by (a, b) where a is the real part and b is the imaginary part. Let $\mathcal{P} = \{(a, b) : 0 \leq a < \infty, 0 \leq b < \infty\} \subset \mathbb{C}$. We denote $(0, 0)$ and $(1, 1)$ in $\mathbb{C}$ as θ and ℓ respectively. We denote closed unit complex interval as $\mathcal{I} = \{(a, b) : 0 \leq a \leq 1, 0 \leq b \leq 1\}$, alongside the open unit complex interval $\mathcal{I}_0 = \{(a, b) : 0 < a < 1, 0 < b < 1\}$. Furthermore, $\mathcal{P}_0$ is designated as $\{(a, b) : 0 < a < \infty, 0 < b < \infty\}$.

A partial order $\preceq$ is imposed on $\mathbb{C}$, where $c_1 \preceq c_2$ if and only if $c_2 - c_1 \in \mathcal{P}$, where $c_1, c_2 \in \mathbb{C}$. We write $c_1 \prec c_2$ to express $\mathrm{Re}(c_2) > \mathrm{Re}(c_1)$ and $\mathrm{Im}(c_2) > \mathrm{Im}(c_1)$. It is evident that $c_1 \prec c_2$ implies and is implied by $c_2 - c_1 \in \mathcal{P}_0$. Let $\{c_n\}$ be a sequence in $\mathbb{C}$. If $c_{n+1} \preceq c_n$ or $c_n \preceq c_{n+1}$ holds for each n belonging to $\mathbb{N}$, then $\{c_n\}$ is termed monotonic sequence in relation to $\preceq$.

In the context of a subset K of $\mathbb{C}$, an element $\inf K \in \mathbb{C}$ is known as the infimum or greatest lower bound of K provided that it acts as lower bound of K, which means $\inf K \preceq k$ for each $k \in K$ along with $l \preceq \inf K$ for any other lower bound l of K. We introduce $\sup K$ in a similar way as the supremum or least upper bound of K.

Remark 2.1 ([24]). *Given that $c_n \in \mathcal{P}$ for every $n \in \mathbb{N}$, all statements below hold:*

1. *If $\{c_n\}$ is a monotonic sequence in relation to $\preceq$ and for some $\alpha, \beta \in \mathcal{P}$ satisfy $\alpha \preceq c_n \preceq \beta$ for each $n \in \mathbb{N}$, it follows that a limit $c \in \mathcal{P}$ exists where $c_n \to c$ as $n \to \infty$.*
2. *While $\preceq$ does not establish a total ordering on $\mathbb{C}$, it does create a lattice structure on $\mathbb{C}$.*
3. *For $K \subset \mathbb{C}$, if every $k \in K$ satisfies $\alpha \preceq k \preceq \beta$ for some $\alpha, \beta \in \mathbb{C}$, then $\inf K$ and $\sup K$ are present.*

Remark 2.2 ([24]). *Given that $c_n, c'_n \in \mathcal{P}_0$ for each $n \in \mathbb{N}$,the following assertions are valid:*

1. *If for every $n \in \mathbb{N}$, we have $c_n \preceq c'_n \preceq \ell$ along with $c_n \to \ell$ as n approaching ∞, it follows that $c'_n = \ell$.*
2. *Whenever $c_n \preceq z$ for each $n \in \mathbb{N}$ plus there is $c \in \mathcal{P}$ such that $\lim_{n\to\infty} c_n = c$, $c \preceq z$ holds.*
3. *Whenever $z \preceq c_n$ for each $n \in \mathbb{N}$ plus there is $c \in \mathcal{P}$ such that $\lim_{n\to\infty} c_n = c$, $z \preceq c$ holds.*

Definition 2.1 ([24]). *Consider Z as nonempty set. Complex fuzzy set Γ is described as the mapping from Z to closed unit complex interval $\mathcal{I}$.*

Definition 2.2 ([24]). *A binary operation $*$ that maps from $\mathcal{I} \times \mathcal{I}$ to $\mathcal{I}$ is referred to as complex-valued t-norm when it satisfies conditions below:*

1. *$\theta * c = \theta, \ell * c = c$ for every $c \in \mathcal{I}$;*
2. *$*$ is associative and commutative;*
3. *$c_3 * c_4 \succeq c_2 * c_1$ given that $c_3 \succeq c_1, c_4 \succeq c_2$ for each c_1, c_2, c_3, c_4 belonging to $\mathcal{I}$.*

Example 2.1 ([24]). *Assuming $c_i = (a_i, b_i) \in \mathcal{I}$ where $i = 1, 2$, binary operations $*_{\mathrm{p}}, *_{\mathrm{m}}, *_{\mathrm{L}} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ are defined as follow:*

(1) $c_1 *_{\mathrm{p}} c_2 = (a_1a_2, b_1b_2)$;
(2) $c_1 *_{\mathrm{m}} c_2 = (\min\{a_1, a_2\}, \min\{b_1, b_2\})$;
(3) $c_1 *_{\mathrm{L}} c_2 = (\max\{a_1 + a_2 - 1, 0\}, \max\{b_1 + b_2 - 1, 0\})$.

Consequently, $*_{\mathrm{p}}, *_{\mathrm{m}}, *_{\mathrm{L}}$ *are complex-valued t-norms.*

Definition 2.3 ([24]). *Suppose* Z *represents nonempty set,* $*$ *is a continuous complex-valued t-norm and* Γ *is a complex fuzzy set defined on* $Z^2 \times \mathcal{P}_0$ *whereby criteria below hold:*

(1) $\Gamma(\varpi, \varkappa, c) \succ \theta$;
(2) $\Gamma(\varpi, \varkappa, c) = \ell$ *for each* $c \in \mathcal{P}_0$ *if and only if* $\varpi = \varkappa$;
(3) $\Gamma(\varpi, \varkappa, c) = \Gamma(\varkappa, \varpi, c)$;
(4) $\Gamma(\varpi, \varsigma, c + c') \succeq \Gamma(\varpi, \varkappa, c) * \Gamma(\varkappa, \varsigma, c')$;
(5) $\Gamma(\varpi, \varkappa, \cdot) : \mathcal{P}_0 \to \mathcal{I}$ *is continuous;*

for every $\varpi, \varkappa, \varsigma \in Z$ *and* $c, c' \in \mathcal{P}_0$.

Then, $(Z, \Gamma, *)$ *is termed complex-valued fuzzy metric space together with* Γ *is referred to as complex-valued fuzzy metric on* Z. Γ *characterizes the closeness degree between a pair of points in the set* Z *relative to a complex factor* $c \in \mathcal{P}_0$.

3. Complex-valued intuitionistic fuzzy metric spaces

The presentation and analysis of properties for complex-valued intuitionistic fuzzy metric spaces are main focus of this section.

Definition 3.1. *A binary operation* $\diamond$ *that maps from* $\mathcal{I} \times \mathcal{I}$ *to* $\mathcal{I}$ *is referred to as complex-valued t-conorm when it satisfies conditions below:*

(1) $c \diamond \theta = c, c \diamond \ell = \ell$ *for every* $c \in \mathcal{I}$;
(2) $\diamond$ *is associative and commutative;*
(3) $c_3 \diamond c_4 \succeq c_1 \diamond c_2$ *given that* $c_3 \succeq c_1, c_4 \succeq c_2$ *for each* c_1, c_2, c_3, c_4 *belonging to* $\mathcal{I}$.

Example 3.1. *Assuming* $c_i = (a_i, b_i) \in \mathcal{I}$ *where* $i = 1, 2$, *binary operations* $\diamond_{\mathrm{n}}, \diamond_{\mathrm{m}}, \diamond_{\mathrm{L}} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ *are defined as follow:*

(1) $c_1 \diamond_{\mathrm{n}} c_2 = (a_1 + a_2, b_1 + b_2) - (a_1a_2, b_1b_2)$;
(2) $c_1 \diamond_{\mathrm{m}} c_2 = (\max\{a_1, a_2\}, \max\{b_1, b_2\})$;
(3) $c_1 \diamond_{\mathrm{L}} c_2 = (\min\{a_1 + a_2, 1\}, \min\{b_1 + b_2, 1\})$.

Consequently, $\diamond_{\mathrm{n}}, \diamond_{\mathrm{m}}, \diamond_{\mathrm{L}}$ *are complex-valued t-conorms.*

Remark 3.1. *Both binary operations t-norm and t-conorm, are commonly utilized in fuzzy set theory, particularly in* $[0, 1]$ *and lattice cases. The former interprets as the common region between two fuzzy sets, in an alternative expression, conjunction in fuzzy logic. The latter, which serves as a duality of t-norm, is interpret as the combination region of two fuzzy sets, in an alternative expression, disjunction in fuzzy logic. Further insights into both t-norm and t-conorm concepts are encouraged to refer [17] and [26].*

Definition 3.2. *Suppose Z represents nonempty set, $*$ and $\diamond$ are continuous complex-valued t-norm and t-conorm, respectively, and Γ, Λ are complex fuzzy sets defined on $Z^2 \times \mathcal{P}_0$ whereby the following conditions hold:*

(1) $\Gamma(\varpi, \varkappa, c) + \Lambda(\varpi, \varkappa, c) \preceq \ell$;
(2) $\Gamma(\varpi, \varkappa, c) \succ \theta$;
(3) $\Gamma(\varpi, \varkappa, c) = \ell$ *for each* $c \in \mathcal{P}_0$ *if and only if* $\varpi = \varkappa$;
(4) $\Gamma(\varpi, \varkappa, c) = \Gamma(\varkappa, \varpi, c)$;
(5) $\Gamma(\varpi, \varsigma, c + c') \succeq \Gamma(\varpi, \varkappa, c) * \Gamma(\varkappa, \varsigma, c')$;
(6) $\Gamma(\varpi, \varkappa, \cdot) : \mathcal{P}_0 \to \mathcal{I}$ *is continuous;*
(7) $\Lambda(\varpi, \varkappa, c) \prec \ell$;
(8) $\Lambda(\varpi, \varkappa, c) = \theta$ *for every* $c \in \mathcal{P}_0$ *if and only if* $\varpi = \varkappa$;
(9) $\Lambda(\varpi, \varkappa, c) = \Lambda(\varkappa, \varpi, c')$;
(10) $\Lambda(\varpi, \varsigma, c + c') \preceq \Lambda(\varpi, \varkappa, c) \diamond \Lambda(\varkappa, \varsigma, c')$;
(11) $\Lambda(\varpi, \varkappa, \cdot) : \mathcal{P}_0 \to \mathcal{I}$ *is continuous;*

for each $\varpi, \varkappa, \varsigma \in Z$ *and* $c, c' \in \mathcal{P}_0$.

Then, $(Z, \Gamma, \Lambda, *, \diamond)$ *is known as complex-valued intuitionistic fuzzy metric space while the pair* (Γ, Λ) *is referred to as complex-valued intuitionistic fuzzy metric on Z. The pair* (Γ, Λ) *characterizes the closeness degree and the non-closeness degree between a pair of points in the set Z relative to a complex parameter* $c \in \mathcal{P}_0$.

Remark 3.2. *Given a complex-valued fuzzy metric space* $(Z, \Gamma, *)$*, one approach for defining a complex-valued intuitionistic fuzzy metric space is to consider* $(Z, \Gamma, \ell - \Gamma, *, \diamond)$*, where both complex-valued t-norm $*$ and complex-valued t-conorm $\diamond$ have association, for instance,* $c_1 \diamond c_2 = \ell - ((\ell - c_1) * (\ell - c_2))$ *for every* $c_1, c_2 \in \mathcal{I}$.

Example 3.2. *Consider* (Z, d) *as a metric space. For* $c_i = (a_i, b_i) \in \mathcal{I}$ *where* $i = 1, 2$*, two binary operations* $*_m$ *and* $\diamond_m$ *are defined by*

$$c_1 * c_2 = (\min\{a_1, a_2\}, \min\{b_1, b_2\}) \text{ and } c_1 \diamond c_2 = (\max\{a_1, a_2\}, \max\{b_1, b_2\}).$$

Let complex fuzzy sets Γ *and* Λ *be defined as follow:*

$$\Gamma(\varpi, \varkappa, c) = \frac{a + b}{a + b + d(\varpi, \varkappa)} \ell, \quad \Lambda(\varpi, \varkappa, c) = \frac{d(\varpi, \varkappa)}{a + b + d(\varpi, \varkappa)} \ell$$

for all $\varpi, \varkappa \in Z$ *and* $c = (a, b) \in \mathcal{P}_0$*. As a result,* $(Z, \Gamma, \Lambda, *_m, \diamond_m)$ *is a complex-valued intuitionistic fuzzy metric space.*

Example 3.3. *Consider* $Z = (0, \infty)$ *and a mapping* $T : \mathcal{P}_0 \to (0, \infty)$*. Define two binary operations $*$ and $\diamond$ by* $c_1 * c_2 = (a_1 a_2, b_1 b_2)$ *and* $c_1 \diamond c_2 = (a_1 + a_2, b_1 + b_2) - (a_1 a_2, b_1 b_2)$ *where* $c_i = (a_i, b_i) \in \mathcal{I}$ *for* $i = 1, 2$*. Let complex fuzzy sets* Γ *and* Λ *be defined as follow:*

$$\Gamma(\varpi, \varkappa, c) = \left(\exp \frac{-(\varpi - \varkappa)^2}{T(c)}\right) \ell, \quad \Lambda(\varpi, \varkappa, c) = \left(1 - \exp \frac{-(\varpi - \varkappa)^2}{T(c)}\right) \ell$$

for each $\varpi, \varkappa \in Z$ and $c = (a,b) \in \mathcal{P}_0$. As a result, $(Z, \Gamma, \Lambda, *, \diamond)$ is a complex-valued intuitionistic fuzzy metric space.

Lemma 3.1. *Given that $(Z, \Gamma, \Lambda, *, \diamond)$ is a complex-valued intuitionistic fuzzy metric space, $\Gamma(\varpi, \varkappa, \cdot)$ is non-decreasing and $\Lambda(\varpi, \varkappa, \cdot)$ is non-increasing, that is, for any $c, c' \in \mathcal{P}_0$ with $c \prec c'$, it follows that $\Gamma(\varpi, \varkappa, c) \preceq \Gamma(\varpi, \varkappa, c')$ and $\Lambda(\varpi, \varkappa, c) \succeq \Lambda(\varpi, \varkappa, c')$ for every $\varpi, \varkappa \in Z$.*

Proof. Taking into account $c, c' \in \mathcal{P}_0$ where $c \prec c'$, this implies that $c' - c \in \mathcal{P}_0$. Utilizing condition (5) from Definition 3.2, we have

$$\begin{aligned}\Gamma(\varpi, \varkappa, c') &= \Gamma(\varpi, \varkappa, c' - c + c)\\ &\succeq \Gamma(\varpi, \varpi, c' - c) * \Gamma(\varpi, \varkappa, c)\\ &= \ell * \Gamma(\varpi, \varkappa, c)\\ &= \Gamma(\varpi, \varkappa, c).\end{aligned}$$

Hence, $\Gamma(\varpi, \varkappa, c') \succeq \Gamma(\varpi, \varkappa, c)$. On the other hand, utilizing condition (10) from Definition 3.2, we have

$$\begin{aligned}\Lambda(\varpi, \varkappa, c') &= \Lambda(\varpi, \varkappa, c' - c + c)\\ &\preceq \Lambda(\varpi, \varpi, c' - c) \diamond \Lambda(\varpi, \varkappa, c)\\ &= \theta \diamond \Lambda(\varpi, \varkappa, c)\\ &= \Lambda(\varpi, \varkappa, c).\end{aligned}$$

Hence, $\Lambda(\varpi, \varkappa, c') \preceq \Lambda(\varpi, \varkappa, c)$. □

Definition 3.3. *Consider $(Z, \Gamma, \Lambda, *, \diamond)$ as a complex-valued intuitionistic fuzzy metric space. We say that sequence $\{\varpi_n\}$ in Z converges to $\varpi \in Z$ provided that all $r \in \mathcal{I}_0$ as well as $c \in \mathcal{P}_0$, the condition below is satisfied by some $n_0 \in \mathbb{N}$:*

$$\Gamma(\varpi_n, \varpi, c) \succ \ell - r \text{ and } \Lambda(\varpi_n, \varpi, c) \prec r \quad \text{for all } n > n_0.$$

Definition 3.4. *Consider $(Z, \Gamma, \Lambda, *, \diamond)$ as a complex-valued intuitionistic fuzzy metric space. A sequence $\{\varpi_n\}$ in Z shall be referred to as Cauchy sequence provided that*

$$\lim_{n\to\infty} \inf_{m>n} \Gamma(\varpi_n, \varpi_m, c) = \ell,$$
$$\lim_{n\to\infty} \sup_{m>n} \Lambda(\varpi_n, \varpi_m, c) = \theta$$

for all $c \in \mathcal{P}_0$.

A complex-valued intuitionistic fuzzy metric space $(Z, \Gamma, \Lambda, *, \diamond)$ is considered complete provided that each Cauchy sequences in Z converges.

Below are examples to illustrate the concepts outlined in the Definitions 3.3 and 3.4.

Example 3.4. *Examine the complex-valued intuitionistic fuzzy metric space denoted as $(Z, \Gamma, \Lambda, *_m, \diamond_m)$ in Example 3.2. Moreover, set $Z = [2,3]$ and define metric d as $d(\varpi, \varkappa) = |\varpi - \varkappa|$ for all $\varpi, \varkappa \in Z$. Let the sequence $\{\varpi_n\} = \{2 + \frac{1}{n}\}$ and $\varpi = 2$.*

Now we verify that $\Gamma(\varpi_n, \varpi, c) \succ \ell - r$ *for each* $r = (r_1, r_2) \in \mathcal{I}_0$ *and* $c \in \mathcal{P}_0$. *For the real part,*

$$\begin{aligned}\operatorname{Re}(\Gamma(\varpi_n, \varpi, c) - \ell + r) &= \frac{a+b}{a+b+d(\varpi_n, \varpi)} - 1 + r_1 \\ &= \frac{a+b}{a+b+|2+\frac{1}{n}-2|} - 1 + r_1 \\ &= \frac{a+b}{a+b+\frac{1}{n}} - 1 + r_1.\end{aligned}$$

As n approaches infinity, we have $\operatorname{Re}(\Gamma(\varpi_n, \varpi, c) - \ell + r) \to r_1$. *Consequently, for each* $r \in \mathcal{I}_0$ *together with* $c \in \mathcal{P}_0$, *there is always an* $N_1 \in \mathbb{N}$ *in which* $\operatorname{Re}(\Gamma(\varpi_n, \varpi, c) - \ell + r) > 0$ *holds for all* $n > N_1$. *The procedure for the imaginary part follows the same steps, leading to* $\operatorname{Im}(\Gamma(\varpi_n, \varpi, c) - \ell + r) \to r_2$ *as n approaches infinity. Consequently, for each* $r \in \mathcal{I}_0$ *and* $c \in \mathcal{P}_0$, *there is always an* $N_2 \in \mathbb{N}$ *in which* $\operatorname{Im}(\Gamma(\varpi_n, \varpi, c) - \ell + r) > 0$ *holds for all* $n > N_2$. *Therefore, for every* $r \in \mathcal{I}_0$ *and* $c \in \mathcal{P}_0$, *by taking* $n_0 = \max\{N_1, N_2\}$, *we establish* $\Gamma(\varpi_n, \varpi, c) \succ \ell - r$ *for all* $n > n_0$.

Now we verify that $\Lambda(\varpi_n, \varpi, c) \prec r$ *for every* $r = (r_1, r_2) \in \mathcal{I}_0$ *and* $c \in \mathcal{P}_0$. *For the real part,*

$$\begin{aligned}\operatorname{Re}(r - \Lambda(\varpi_n, \varpi, c)) &= r_1 - \frac{d(\varpi_n, \varpi)}{a+b+d(\varpi_n, \varpi)} \\ &= r_1 - \frac{|2+\frac{1}{n}-2|}{a+b+|2+\frac{1}{n}-2|} \\ &= r_1 - \frac{\frac{1}{n}}{a+b+\frac{1}{n}}.\end{aligned}$$

As n approaches infinity, we have $\operatorname{Re}(r - \Lambda(\varpi_n, \varpi, c)) \to r_1$. *Consequently, for each* $r \in \mathcal{I}_0$ *and* $c \in \mathcal{P}_0$, *there is always an* $N_1 \in \mathbb{N}$ *in which* $\operatorname{Re}(\Lambda(\varpi_n, \varpi, c) - \ell + r) > 0$ *holds for all* $n > N_1$. *The procedure for the imaginary part follows the same steps, leading to* $\operatorname{Im}(r - \Lambda(\varpi_n, \varpi, c)) \to r_2$ *as n approaches infinity. Consequently, for each* $r \in \mathcal{I}_0$ *and* $c \in \mathcal{P}_0$, *there is always an* $N_2 \in \mathbb{N}$ *in which* $\operatorname{Im}(r - \Lambda(\varpi_n, \varpi, c)) > 0$ *holds for all* $n > N_2$. *Therefore, for every* $r \in \mathcal{I}_0$ *and* $c \in \mathcal{P}_0$, *by taking* $n_0 = \max\{N_1, N_2\}$, *we establish* $\Lambda(\varpi_n, \varpi, c) \prec r$ *for each* $n > n_0$.

Each conditions specified in Definition 3.3 are met. Therefore, we can conclude that $\{2 + \frac{1}{n}\}$ *converges to* 2.

Example 3.5. *Employing the same settings as in the previous example, we will show that* $\{2 + \frac{1}{n}\}$ *is a Cauchy sequence. For all* $c \in \mathcal{P}_0$ *and any* $n, m \in \mathbb{N}$ *where* $m > n$,

$$\begin{aligned}\Gamma(\varpi_n, \varpi_m, c) &= \frac{a+b}{a+b+d(\varpi_n, \varpi_m)}\ell \\ &= \frac{a+b}{a+b+|2+\frac{1}{n}-(2+\frac{1}{m})|}\ell \\ &= \frac{a+b}{a+b+|\frac{1}{n}-\frac{1}{m}|}\ell\end{aligned}$$

and

$$\begin{aligned}\Lambda(\varpi_n,\varpi_m,c) &= \frac{d(\varpi_n,\varpi_m)}{a+b+d(\varpi_n,\varpi_m)}\ell\\ &= \frac{|2+\frac{1}{n}-(2+\frac{1}{m})|}{a+b+|2+\frac{1}{n}-(2+\frac{1}{m})|}\ell\\ &= \frac{|\frac{1}{n}-\frac{1}{m}|}{a+b+|\frac{1}{n}-\frac{1}{m}|}\ell.\end{aligned}$$

As m,n approaches infinity, we observe that $\Gamma(\varpi_n,\varpi_m,c)\to\ell$ and $\Lambda(\varpi_n,\varpi_m,c)\to\theta$, which leads to $\lim_{n\to\infty}\inf_{m>n}\Gamma(\varpi_n,\varpi_m,c)=\ell$ and $\lim_{n\to\infty}\sup_{m>n}\Lambda(\varpi_n,\varpi_m,c)=\theta$. Hence, we demonstrate that $\{2+\frac{1}{n}\}$ is a Cauchy sequence.

Lemma 3.2. *Given that $(Z,\Gamma,\Lambda,*,\diamond)$ is a complex-valued intuitionistic fuzzy metric space. The sequence $\{\varpi_n\}$ in Z converges to $\varpi\in Z$ if and only if $\lim_{n\to\infty}\Gamma(\varpi_n,\varpi,c)=\ell$ and $\lim_{n\to\infty}\Lambda(\varpi_n,\varpi,c)=\theta$ are satisfied for each $c\in\mathcal{P}_0$.*

Proof. Assume that $\lim_{n\to\infty}\Gamma(\varpi_n,\varpi,c)=\ell$ and $\lim_{n\to\infty}\Lambda(\varpi_n,\varpi,c)=\theta$ for each $c\in\mathcal{P}_0$. Consider a fixed element c in $\mathcal{P}_0$. Given any $r\in\mathcal{I}_0$, it is possible to locate a real number $\varepsilon>0$ where $z\prec r$ hold for all $z\in\mathbb{C}$ with $|z|<\varepsilon$. Taking into account this specific ε, we can identify an $n_0\in\mathbb{N}$ such that

$$|\ell-\Gamma(\varpi_n,\varpi,c)|<\varepsilon \text{ and } |\Lambda(\varpi_n,\varpi,c)|<\varepsilon \quad \text{for every } n>n_0.$$

These two inequalities imply that

$$\begin{aligned}\ell-\Gamma(\varpi_n,\varpi,c)&\prec r\\ -\Gamma(\varpi_n,\varpi,c)&\prec r-\ell\\ \Gamma(\varpi_n,\varpi,c)&\succ \ell-r\end{aligned}$$

as well as

$$\Lambda(\varpi_n,\varpi,c)\prec r$$

for every $n>n_0$ respectively. Therefore, $\{\varpi_n\}$ is converging to $\varpi\in Z$.

Conversely, let $c\in\mathcal{P}_0$ fixed and a real number $\varepsilon>0$ be given. Assume that $\{\varpi_n\}$ converges to $\varpi\in Z$, that is, for every $r\in\mathcal{I}_o$, an $n_0\in\mathbb{N}$ can be chosen such that $\Gamma(\varpi_n,\varpi,c)\succ\ell-r$ and $\Lambda(\varpi_n,\varpi,c)\prec r$ for all $n>n_0$. A complex number $r\in\mathcal{I}_0$ is picked in a way that $|r|<\varepsilon$. It follows that

$$|\ell-\Gamma(\varpi_n,\varpi,c)|<|r|<\varepsilon \text{ and } |\Lambda(\varpi_n,\varpi,c)|<|r|<\varepsilon \quad \text{for any } n>n_0.$$

Therefore, $\lim_{n\to\infty}\Gamma(\varpi_n,\varpi,c)=\ell$ and $\lim_{n\to\infty}\Lambda(\varpi_n,\varpi,c)=\theta$ is satisfied for every $c\in\mathcal{P}_0$. □

Lemma 3.3. *Given that $(Z,\Gamma,\Lambda,*,\diamond)$ is a complex-valued intuitionistic fuzzy metric space. $\{\varpi_n\}$ in Z is considered Cauchy sequence if and only if for each $r\in\mathcal{I}_0$ and $c\in\mathcal{P}_0$, one can find an $n_0\in\mathbb{N}$ satisfying*

$$\Gamma(\varpi_n,\varpi_m,c)\succ\ell-r \text{ and } \Lambda(\varpi_n,\varpi_m,c)\prec r \text{ for all } n,m>n_0.$$

Proof. Suppose that sequence $\{\varpi_n\}$ is Cauchy. Let $c \in P_0$ be fixed, then for each $r \in \mathcal{I}_0$ one can find $n_0 \in \mathbb{N}$ satisfying $\ell - \inf_{m>n} \Gamma(\varpi_n, \varpi_m, c) \prec r$ and $\sup_{m>n} \Lambda(\varpi_n, \varpi_m, c) \prec r$ for all $n > n_0$. Here we consider 3 situations. For the case where $m > n > n_0$, this leads to $\ell - r \prec \inf_{m>n} \Gamma(\varpi_n, \varpi_m, c) \prec \Gamma(\varpi_n, \varpi_m, c)$ and $\Lambda(\varpi_n, \varpi_m, c) \prec \sup_{m>n} \Lambda(\varpi_n, \varpi_m, c) \prec r$. Now if $m = n > n_0$, then $\ell - r \prec \ell = \Gamma(\varpi_n, \varpi_m, c)$ and $\Lambda(\varpi_n, \varpi_m, c) = \theta \prec r$. Last but not least, for the case where $n > m > n_0$, it follows that $\ell - r \prec \inf_{n>m} \Gamma(\varpi_m, \varpi_n, c) \preceq \Gamma(\varpi_m, \varpi_n, c) = \Gamma(\varpi_n, \varpi_m, c)$ and $\Lambda(\varpi_m, \varpi_n, c) = \Lambda(\varpi_n, \varpi_m, c) \preceq sup_{n>m} \Gamma(\varpi_n, \varpi_m, c) \prec r$. Hence, we conclude that $\Gamma(\varpi_n, \varpi_m, c) \succ \ell - r$ as well as $\Lambda(\varpi_n, \varpi_m, c) \prec r$ for any $n, m > n_0$.

Conversely, let $c \in P_o$ fixed and a real number $\varepsilon > 0$ be given. Assume that for all $r \in \mathcal{I}_0$, one can identify an $n_0 \in \mathbb{N}$ in which $\Gamma(\varpi_n, \varpi_m, c) \succ \ell - r$ and $\Lambda(\varpi_n, \varpi_m, c) \prec r$ for any $n, m > n_0$. It follows that

$$\ell - 2r \prec \ell - r \preceq \inf_{m>n} \Gamma(\varpi_n, \varpi_m, c)$$

and

$$\sup_{m>n} \Lambda(\varpi_n, \varpi_m, c) \preceq r \prec 2r$$

for all $n > n_0$. Pick a complex number $r \in \mathcal{I}_0$ which satisfies $|r| < \frac{\varepsilon}{2}$, then we have

$$|\ell - \inf_{m>n} \Gamma(\varpi_n, \varpi_m, c)| < 2|r| < \varepsilon \text{ and } |\sup_{m>n} \Lambda(\varpi_n, \varpi_m, c)| < 2|r| < \varepsilon \text{ for every } n > n_0.$$

Hence, we have $\lim_{n\to\infty} \inf_{m>n} \Gamma(\varpi_n, \varpi_m, c) = \ell$ and $\lim_{n\to\infty} \sup_{m>n} \Lambda(\varpi_n, \varpi_m, c) = \theta$, which means that sequence $\{\varpi_n\}$ is Cauchy. □

4. Fixed-point results

Our focus will now shift to the presence and distinctiveness of fixed points for self-mappings fulfilling specific contractive criteria within complex-valued intuitionistic fuzzy metric space. Consider a sequence $\{c_n\}$ that belongs to $\mathbb{C}$, we say that $\lim_{n\to\infty} c_n = \infty = (\infty, \infty)$ whenever in the case of each $c \in \mathbb{C}$, one can find an $n_0 \in \mathbb{N}$ satisfy $c_n \succeq c$ for any $n > n_0$.

Theorem 4.1. *Let $(Z, \Gamma, \Lambda, *, \diamond)$ be a complete complex-valued intuitionistic fuzzy metric space with the property that any sequence $\{c_n\}$ in $\mathcal{P}_0$ satisfies $\lim_{n\to\infty} c_n = \infty$ implies*

$$\lim_{n\to\infty} \inf_{\varkappa\in Z} \Gamma(\varpi, \varkappa, c_n) = \ell, \quad \lim_{n\to\infty} \sup_{\varkappa\in Z} \Lambda(\varpi, \varkappa, c_n) = \theta$$

for any $\varpi \in Z$. Suppose a self-mapping $f : Z \to Z$ satisfies subsequent condition:

$$\Gamma(f\varpi, f\varkappa, kc) \succeq \Gamma(\varpi, \varkappa, c) \text{ and } \Lambda(f\varpi, f\varkappa, kc) \preceq \Lambda(\varpi, \varkappa, c) \tag{4.1}$$

for all $\varpi, \varkappa \in Z$ and $c \in \mathcal{P}_0$, where $k \in (0, 1)$. Then mapping f possesses a unique fixed point that lies within Z.

Proof. Consider an arbitrary point ϖ_0 in Z. A sequence $\{\varpi_n\}$ is defined in Z as $\varpi_n = f\varpi_{n-1}$ for every $n \in \mathbb{N}$. The existence of an $n_0 \in \mathbb{N}$ where $\varpi_{n_0} = \varpi_{n_0-1}$ secures ϖ_{n_0} as a fixed point of f. Now, we consider $\varpi_n \neq \varpi_{n-1}$ for each $n \in \mathbb{N}$ and establish the Cauchy nature for sequence $\{\varpi_n\}$.

For every $n \in \mathbb{N}$ as well as a fixed $c \in \mathcal{P}_0$, let us define

$$\mathcal{A}_n := \{\Gamma(\varpi_n, \varpi_m, c) : m > n\} \subset \mathcal{I},$$
$$\mathcal{B}_n := \{\Lambda(\varpi_n, \varpi_m, c) : m > n\} \subset \mathcal{I}.$$

As $\theta \prec \Gamma(\varpi_n, \varpi_m, c) \preceq \ell$ for each $n \in \mathbb{N}$ where $n < m$, following from Remark 2.1, the infimum of $\mathcal{A}_n$, that is, $\inf \mathcal{A}_n = \alpha_n$ is present in every $n \in \mathbb{N}$. Similarly, since $\theta \preceq \Lambda(\varpi_n, \varpi_m, c) \prec \ell$ for all $n \in \mathbb{N}$ where $m > n$, following from Remark 2.1, the supremum of $\mathcal{B}_n$, that is, $\sup \mathcal{B}_n = \beta_n$ is present in every $n \in \mathbb{N}$. For $c \in \mathcal{P}_0$ and $n, m \in \mathbb{N}$ where $m > n$, using (4.1), we obtain

$$\Gamma(\varpi_{n+1}, \varpi_{m+1}, c) = \Gamma(f\varpi_{n+1}, f\varpi_m, c) \succeq \Gamma\left(\varpi_n, \varpi_m, \frac{c}{k}\right) \tag{4.2}$$

and

$$\Lambda(\varpi_{n+1}, \varpi_{m+1}, c) = \Lambda(f\varpi_n, f\varpi_m, c) \preceq \Lambda\left(\varpi_n, \varpi_m, \frac{c}{k}\right). \tag{4.3}$$

Since $k \in (0,1)$, by Lemma 3.1, it follows that

$$\Gamma\left(\varpi_n, \varpi_m, \frac{c}{k}\right) \succeq \Gamma(\varpi_n, \varpi_m, c) \text{ and } \Lambda\left(\varpi_n, \varpi_m, \frac{c}{k}\right) \preceq \Lambda(\varpi_n, \varpi_m, c).$$

which in turn yields

$$\Gamma(\varpi_{n+1}, \varpi_{m+1}, c) \succeq \Gamma(\varpi_n, \varpi_m, c)$$

and

$$\Lambda(\varpi_{n+1}, \varpi_{m+1}, c) \preceq \Lambda(\varpi_n, \varpi_m, c)$$

for each $n, m \in \mathbb{N}$ where $m > n$. Checking the infimum of Γ and supremum of Λ above, it leads to

$$\theta \preceq \alpha_n \preceq \alpha_{n+1} \preceq \ell, \quad \theta \preceq \beta_{n+1} \preceq \beta_n \preceq \ell$$

for any $n \in \mathbb{N}$. Hence, both $\{\alpha_n\}$ and $\{\beta_n\}$ are monotonic sequences in $\mathcal{P}$. By Remark 2.1, there exist complex numbers $\alpha, \beta \in \mathcal{P}$ satisfying $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} \beta_n = \beta$. By (4.2) and (4.3), we have

$$\alpha_{n+1} = \inf_{m>n} \Gamma(\varpi_{n+1}, \varpi_{m+1}, c) \succeq \inf_{m>n} \Gamma\left(\varpi_n, \varpi_m, \frac{c}{k}\right)$$

and

$$\beta_{n+1} = \sup_{m>n} \Lambda(\varpi_{n+1}, \varpi_{m+1}, c) \preceq \sup_{m>n} \Lambda\left(\varpi_n, \varpi_m, \frac{c}{k}\right)$$

for $c \in \mathcal{P}_0$ and $n \in \mathbb{N}$. Apply (4.1) successively on the inequalities above, we get

$$\begin{aligned}
\alpha_{n+1} &\succeq \inf_{m>n} \Gamma\left(\varpi_n, \varpi_m, \frac{c}{k}\right) \\
&\succeq \inf_{m>n} \Gamma\left(\varpi_{n-1}, \varpi_{m-1}, \frac{c}{k^2}\right) \\
&\succeq \inf_{m>n} \Gamma\left(\varpi_{n-2}, \varpi_{m-2}, \frac{c}{k^3}\right) \\
&\succeq \cdots \\
&\succeq \inf_{m>n} \Gamma\left(\varpi_0, \varpi_{m-n}, \frac{c}{k^{n+1}}\right)
\end{aligned}$$

and

$$\begin{aligned}\beta_{n+1} &\leq \sup_{m>n} \Lambda\left(\varpi_n, \varpi_m, \frac{c}{k}\right)\\ &\leq \sup_{m>n} \Lambda\left(\varpi_{n-1}, \varpi_{m-1}, \frac{c}{k^2}\right)\\ &\leq \sup_{m>n} \Lambda\left(\varpi_{n-2}, \varpi_{m-2}, \frac{c}{k^3}\right)\\ &\leq \cdots\\ &\leq \sup_{m>n} \Lambda\left(\varpi_0, \varpi_{m-n}, \frac{c}{k^{n+1}}\right)\end{aligned}$$

for $c \in \mathcal{P}_0$ and $n \in \mathbb{N}$. Furthermore, we obtain

$$\alpha_{n+1} \geq \inf_{m>n} \Gamma\left(\varpi_0, \varpi_{m-n}, \frac{c}{k^{n+1}}\right) \geq \inf_{\varkappa \in Z} \Gamma\left(\varpi_0, \varkappa, \frac{c}{k^{n+1}}\right)$$

and

$$\beta_{n+1} \leq \sup_{m>n} \Lambda\left(\varpi_0, \varpi_{m-n}, \frac{c}{k^{n+1}}\right) \leq \sup_{\varkappa \in Z} \Lambda\left(\varpi_0, \varkappa, \frac{c}{k^{n+1}}\right)$$

for any $c \in \mathcal{P}_0$ and $n \in \mathbb{N}$. In the case where n approaches infinity on both sides of the inequalities above, as $\lim_{n\to\infty} c/k^{n+1} = \infty$ for any $c \in \mathcal{P}_0$, by the limit of monotonic sequences $\{\alpha_n\}, \{\beta_n\}$ along with the hypothesis, we yield

$$\alpha = \lim_{n\to\infty} \alpha_{n+1} \geq \lim_{n\to\infty} \inf_{\varkappa \in Z} \Gamma\left(\varpi_0, \varkappa, \frac{c}{k^{n+1}}\right) = \ell$$

and

$$\beta = \lim_{n\to\infty} \beta_{n+1} \leq \lim_{n\to\infty} \sup_{\varkappa \in Z} \Lambda\left(\varpi_0, \varkappa, \frac{c}{k^{n+1}}\right) = \theta.$$

which imply $\alpha = \ell$ and $\beta = \theta$. Thus,

$$\begin{aligned}\lim_{n\to\infty} \inf_{m>n} \Gamma(\varpi_{n+1}, \varpi_{m+1}, c) &= \lim_{n\to\infty} \alpha_n = \ell,\\ \lim_{n\to\infty} \sup_{m>n} \Lambda(\varpi_{n+1}, \varpi_{m+1}, c) &= \lim_{n\to\infty} \beta_n = \theta\end{aligned}$$

for all $c \in \mathcal{P}_0$ which show sequence $\{\varpi_n\}$ is Cauchy.

Given that $(Z, \Gamma, \Lambda, *, \diamond)$ is complete, Lemma 3.2 implies the existence of $\varpi \in Z$ satisfying

$$\lim_{n\to\infty} \Gamma(\varpi_n, \varpi, c) = \ell \text{ and } \lim_{n\to\infty} \Lambda(\varpi_n, \varpi, c) = \theta \quad \text{for any } c \in \mathcal{P}_0. \tag{4.4}$$

As a consequence of conditions (5), (10) of Definition 3.2 and (4.1), for any $c \in \mathcal{P}_0$, we can conclude that

$$\begin{aligned}\Gamma(\varpi, f\varpi, c) &\geq \Gamma\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) * \Gamma\left(\varpi_{n+1}, f\varpi, \frac{c}{2}\right)\\ &= \Gamma\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) * \Gamma\left(f\varpi_n, f\varpi, \frac{c}{2}\right)\\ &\geq \Gamma\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) * \Gamma\left(\varpi_n, \varpi, \frac{c}{2k}\right)\end{aligned}$$

and

$$\begin{aligned}\Lambda(\varpi, f\varpi, c) &\leq \Lambda\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) \diamond \Lambda\left(\varpi_{n+1}, f\varpi, \frac{c}{2}\right) \\ &= \Lambda\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) \diamond \Lambda\left(f\varpi_n, f\varpi, \frac{c}{2}\right) \\ &\leq \Lambda\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) \diamond \Lambda\left(\varpi_n, \varpi, \frac{c}{2k}\right).\end{aligned}$$

Now taking the limit as $n \to \infty$ for both inequalities above, using (4.4) along with Remark 2.2, it follows that

$$\Gamma(\varpi, f\varpi, c) = \ell \text{ and } \Lambda(\varpi, f\varpi, c) = \theta$$

for every $c \in \mathcal{P}_0$. By conditions (3) and (8) of Definition 3.2, it can be deduced that $\varpi = f\varpi$, that is, ϖ is a fixed point of f.

To establish the uniqueness, assume z is a different fixed point of f than ϖ. This implies there exist some $c' \in \mathcal{P}_0$ in which $\Gamma(\varpi, z, c') \neq \ell$ and $\Lambda(\varpi, z, c') \neq \theta$. Apply (4.1) successively, we have

$$\begin{aligned}\Gamma(\varpi, z, c') = \Gamma(f\varpi, fz, c') &\geq \Gamma\left(\varpi, z, \frac{c'}{k}\right) \\ &\geq \Gamma\left(\varpi, z, \frac{c'}{k^2}\right) \\ &\geq \cdots \\ &\geq \Gamma\left(\varpi, z, \frac{c'}{k^n}\right)\end{aligned}$$

and

$$\begin{aligned}\Lambda(\varpi, z, c') = \Lambda(f\varpi, fz, c') &\leq \Lambda\left(\varpi, z, \frac{c'}{k}\right) \\ &\leq \Lambda\left(\varpi, z, \frac{c'}{k^2}\right) \\ &\leq \cdots \\ &\leq \Lambda\left(\varpi, z, \frac{c'}{k^n}\right)\end{aligned}$$

for each n in $\mathbb{N}$. Subsequently, we deduce that

$$\Gamma(\varpi, z, c') \geq \Gamma\left(\varpi, z, \frac{c'}{k^n}\right) \geq \inf_{\varkappa \in Z} \Gamma\left(\varpi, z, \frac{c'}{k^n}\right)$$

and

$$\Lambda(\varpi, z, c') \leq \Lambda\left(\varpi, z, \frac{c'}{k^n}\right) \leq \sup_{\varkappa \in Z} \Lambda\left(\varpi, z, \frac{c'}{k^n}\right).$$

Since $k \in (0,1)$, it is clear that $\lim_{n\to\infty} c'/k^n = \infty$. Therefore, taking the limit of $n \to \infty$ on the both inequalities, by hypothesis, it leads to

$$\Gamma(\varpi, z, c') = \ell \text{ and } \Lambda(\varpi, z, c') = \theta$$

which is a contradiction. Hence, $\Gamma(\varpi, z, c) = \ell$ and $\Lambda(\varpi, z, c) = \theta$ for every $c \in \mathcal{P}_0$. From conditions (3) and (8) of Definition 3.2, it can be deduced that $\varpi = z$, which validates the uniqueness fixed point of f. □

Remark 4.1. *In Theorem 4.1, replacing* (4.1) *with the following contractive condition of mapping f and the proof remains similar:*

$$\Gamma(f\varpi, f\varkappa, \mathcal{K}(c)c) \succeq \Gamma(\varpi,\varkappa,c) \text{ and } \Lambda(f\varpi, f\varkappa, \mathcal{K}(c)c) \preceq \Lambda(\varpi,\varkappa,c)$$

for every $\varpi, \varkappa \in Z$ and $c \in \mathcal{P}_0$, with $\mathcal{K}$ represents a mapping from $\mathcal{P}_0$ to $(0,1)$.

Example 4.1. *Assume (Z,d) is a metric space where $Z = [0,1]$ together with $d(\varpi,\varkappa) = |\varpi - \varkappa|$ for all $\varpi,\varkappa \in Z$. Define complex-valued t-norm $*$ and complex-valued t-conorm $\diamond$ by $\omega_1 * \omega_2 = (\mu_1\mu_2, \nu_1\nu_2)$ and $\omega_1 \diamond \omega_2 = (\max\{\mu_1,\mu_2\}, \max\{\nu_1,\nu_2\})$ for all $\omega_1 = (\mu_1,\nu_1), \omega_2 = (\mu_2,\nu_2) \in \mathcal{I}$ respectively. Let complex-valued fuzzy sets Γ and Λ be defined as*

$$\Gamma(\varpi,\varkappa,c) = \frac{ab}{ab + d(\varpi,\varkappa)}\ell, \quad \Lambda(\varpi,\varkappa,c) = \frac{d(\varpi,\varkappa)}{ab + d(\varpi,\varkappa)}\ell$$

for all $\varpi,\varkappa \in Z$ and $c = (a,b) \in \mathcal{P}_0$. The fact that $(Z,\Gamma,\Lambda,*,\diamond)$ is a complete complex-valued intuitionistic fuzzy metric space induced by metric d can be established without much effort. Suppose we have a sequence $\{c_n\}$ in $\mathcal{P}_0$ in which $c_n = (a_n,b_n)$ for each $n \in \mathbb{N}$ satisfying $\lim_{n\to\infty} c_n = \infty$. When $\varpi \in Z$ is fixed and $n \in \mathbb{N}$ is arbitrary, together with the fact that $0 \le d(\varpi,\varkappa) \le 1$ for every $\varkappa \in Z$, this implies that

$$\begin{aligned} \ell &\succeq \inf_{\varkappa\in Z} \Gamma(\varpi,\varkappa,c_n) \\ &= \inf_{\varkappa\in Z} \frac{a_nb_n}{a_nb_n + d(\varpi,\varkappa)}\ell \\ &= \frac{a_nb_n}{a_nb_n + \sup_{\varkappa\in Z} d(\varpi,\varkappa)}\ell \\ &\succeq \frac{a_nb_n}{a_nb_n+1}\ell. \end{aligned}$$

As n becomes infinitely large, we arrive at

$$\ell \succeq \lim_{n\to\infty} \inf_{\varkappa\in Z} \Gamma(\varpi,\varkappa,c_n) \succeq \lim_{n\to\infty} \frac{a_nb_n}{a_nb_n+1}\ell = \ell$$

which leads to the conclusion that $\lim_{n\to\infty} \inf_{\varkappa\in Z} \Gamma(\varpi,\varkappa,c_n) = \ell$. In addition, we have

$$\begin{aligned} \theta &\preceq \sup_{\varkappa\in Z} \Lambda(\varpi,\varkappa,c_n) \\ &= \sup_{\varkappa\in Z} \frac{d(\varpi,\varkappa)}{a_nb_n + d(\varpi,\varkappa)}\ell \\ &= \frac{\sup_{\varkappa\in Z} d(\varpi,\varkappa)}{a_nb_n + \inf_{\varkappa\in Z} d(\varpi,\varkappa)}\ell \\ &\preceq \frac{1}{a_nb_n}\ell. \end{aligned}$$

As n becomes infinitely large, we arrive at

$$\theta \preceq \lim_{n\to\infty} \sup_{\varkappa\in Z} \Lambda(\varpi,\varkappa,c_n) \preceq \lim_{n\to\infty} \frac{1}{a_nb_n}\ell = \theta$$

which leads to the conclusion that $\lim_{n\to\infty} \sup_{\varkappa\in Z} \Lambda(\varpi,\varkappa,c_n) = \theta$.

Consider a self-mapping $\mathcal{H} : Z \to Z$ *expressed by* $\mathcal{H}\varpi = \varpi/2$ *for all* $\varpi \in Z$. *If we pick a real number* $k \in [1/2, 1) \subset (0, 1)$, *then* $\mathcal{H}$ *satisfies* (4.1) *for any* $\varpi, \varkappa \in Z$ *and* $c \in \mathcal{P}_0$. *Indeed, since* $2k \geq 1$, *we have*

$$\begin{aligned}\Gamma(\mathcal{H}\varpi, \mathcal{H}\varkappa, kc) &= \frac{kab}{kab + d(\mathcal{H}\varpi, \mathcal{H}\varkappa)}\ell \\ &= \frac{kab}{kab + |\frac{\varpi}{2} - \frac{\varkappa}{2}|}\ell \\ &= \frac{kab}{kab + \frac{1}{2}|\varpi - \varkappa|}\ell \\ &= \frac{2kab}{2kab + |\varpi - \varkappa|}\ell \\ &\succeq \frac{ab}{ab + |\varpi - \varkappa|}\ell \\ &= \Gamma(\varpi, \varkappa, c)\end{aligned}$$

for any $\varpi, \varkappa \in Z$ *and* $c = (a, b) \in \mathcal{P}_0$. *In addition, we have*

$$\begin{aligned}\Lambda(\mathcal{H}\varpi, \mathcal{H}\varkappa, kc) &= \frac{d(\mathcal{H}\varpi, \mathcal{H}\varkappa)}{kab + d(\mathcal{H}\varpi, \mathcal{H}\varkappa)}\ell \\ &= \frac{|\frac{\varpi}{2} - \frac{\varkappa}{2}|}{kab + |\frac{\varpi}{2} - \frac{\varkappa}{2}|}\ell \\ &= \frac{\frac{1}{2}|\varpi - \varkappa|}{kab + \frac{1}{2}|\varpi - \varkappa|}\ell \\ &= \frac{|\varpi - \varkappa|}{2kab + |\varpi - \varkappa|}\ell \\ &\preceq \frac{|\varpi - \varkappa|}{ab + |\varpi - \varkappa|}\ell \\ &= \Lambda(\varpi, \varkappa, c)\end{aligned}$$

for any $\varpi, \varkappa \in Z$ *and* $c = (a, b) \in \mathcal{P}_0$. *As a consequence, each requirements specified in Theorem 4.1 are met. The sole fixed point of* $\mathcal{H}$ *is* 0.

Example below serves to illustrate the assumption of Theorem 4.1 is not redundant.

Example 4.2. *Consider* $Z = \mathbb{N}$. *Define two binary operations* $*$ *and* $\diamond$ *by* $c_1 * c_2 = (a_1a_2, b_1b_2)$ *and* $c_1 \diamond c_2 = (a_1 + a_2, b_1 + b_2) - (a_1a_2, b_1b_2)$ *for any* $c_i = (a_i, b_i) \in \mathcal{I}$ *where* $i = 1, 2$. *Let complex fuzzy sets* Γ *and* Λ *be defined as follow:*

$$\Gamma(\varpi, \varkappa, c) = \frac{\min\{\varpi, \varkappa\}}{\max\{\varpi, \varkappa\}}\ell, \quad \Lambda(\varpi, \varkappa, c) = \left(1 - \frac{\min\{\varpi, \varkappa\}}{\max\{\varpi, \varkappa\}}\right)\ell$$

for every $\varpi, \varkappa \in Z$ *as well as* $c = (a, b) \in \mathcal{P}_0$. *The fact that* $(Z, \Gamma, \Lambda, *, \diamond)$ *is a complete complex-valued intuitionistic fuzzy metric space is established without much effort. Consider a self-mapping* $\mathcal{H} : Z \to Z$ *expressed by* $\varpi^2 + 5$ *for all* $\varpi \in Z$. *Let sequence* $\{c_n\}$ *defined as* $c_n = (n, n)$ *for each* n *in* $\mathbb{N}$. *From the way*

$\{c_n\}$ is constructed, there is no ambiguity in $\lim_{n\to\infty} c_n = \infty$. For each $\varpi \in Z$, a fixed $\varkappa \in Z$ where $\varpi \neq \varkappa$ and any $n \in \mathbb{N}$, it can be concluded that

$$\inf_{\varkappa\in Z} \Gamma(\varpi, \varkappa, c_n) = \inf_{\varkappa\in Z} \frac{\min\{\varpi, \varkappa\}}{\max\{\varpi, \varkappa\}}\ell = \theta$$

and

$$\sup_{\varkappa\in Z} \Lambda(\varpi, \varkappa, c_n) = \sup_{\varkappa\in Z} \left(1 - \frac{\min\{\varpi, \varkappa\}}{\max\{\varpi, \varkappa\}}\right)\ell = \ell.$$

Consequently, we obtain

$$\lim_{n\to\infty} \inf_{\varkappa\in Z} \Gamma(\varpi, \varkappa, c_n) = \theta \neq \ell$$

and

$$\lim_{n\to\infty} \sup_{\varkappa\in Z} \Lambda(\varpi, \varkappa, c_n) = \ell \neq \theta$$

for all $\varpi \in Z$. For any $k \in (0, 1)$, $\varpi, \varkappa \in Z$ and $c \in \mathcal{P}_0$, observe that

$$\begin{aligned}\Gamma(\mathcal{H}\varpi, \mathcal{H}\varkappa, kc) &= \frac{\min\{\varpi^2 + 5, \varkappa^2 + 5\}}{\max\{\varpi^2 + 5, \varkappa^2 + 5\}}\ell \\ &\geq \frac{\min\{\varpi, \varkappa\}}{\max\{\varpi, \varkappa\}}\ell \\ &= \Gamma(\varpi, \varkappa, c)\end{aligned}$$

and

$$\begin{aligned}\Lambda(\mathcal{H}\varpi, \mathcal{H}\varkappa, kc) &= \left(1 - \frac{\min\{\varpi^2 + 5, \varkappa^2 + 5\}}{\max\{\varpi^2 + 5, \varkappa^2 + 5\}}\right)\ell \\ &\leq \left(1 - \frac{\min\{\varpi, \varkappa\}}{\max\{\varpi, \varkappa\}}\right)\ell \\ &= \Lambda(\varpi, \varkappa, c).\end{aligned}$$

Thus mapping $\mathcal{H}$ satisfies (4.1) *but it does not has any fixed point in Z.*

For subsequent result, Ψ is defined as a collection of all mapping $\psi : \mathcal{I} \to \mathcal{I}$ in which ψ is continuous, $\psi(c) \succ c$ for each $c \in \mathcal{I}_0$, $\psi(\ell) = \ell$ and $\lim_{n\to\infty} \psi^n(c) = \ell$ for all $c \in \mathcal{I}_0$. Likewise, Φ is defined as a collection of all mapping $\phi : \mathcal{I} \to \mathcal{I}$ in which ϕ is continuous, $\phi(c) \prec c$ for all $c \in \mathcal{I}$, $\phi(\theta) = \theta$ and $\lim_{n\to\infty} \phi^n(c) = \theta$ for all $c \in \mathcal{I}$.

Theorem 4.2. *Suppose that $(Z, \Gamma, \Lambda, *, \diamond)$ is a complete complex-valued intuitionistic fuzzy metric space. If a self-mapping $f : Z \to Z$ satisfies subsequent condition:*

$$\Gamma(f\varpi, f\varkappa, c) \succeq \psi(\Gamma(\varpi, \varkappa, c)) \text{ and } \Lambda(f\varpi, f\varkappa, c) \preceq \phi(\Lambda(\varpi, \varkappa, c)) \tag{4.5}$$

for all $\varpi, \varkappa \in Z$ and $c \in \mathcal{P}_0$, where $\psi \in \Psi$ and $\phi \in \Phi$. Then mapping f possesses a unique fixed point that lies within Z.

Proof. Consider point $\varpi_0 \in Z$ arbitrary. A sequence $\{\varpi_n\}$ is defined in Z as $\varpi_n = f\varpi_{n-1}$ for every $n \in \mathbb{N}$. The existence of an $n_0 \in \mathbb{N}$ where $\varpi_{n_0} = \varpi_{n_0-1}$ secures ϖ_{n_0} as a fixed point of f. Now, we consider $\varpi_n \neq \varpi_{n-1}$ for each $n \in \mathbb{N}$ and establish the Cauchy nature for sequence $\{\varpi_n\}$.

For every $n \in \mathbb{N}$ as well as a fixed $c \in \mathcal{P}_0$, let us define

$$\mathcal{A}_n := \{\Gamma(\varpi_n, \varpi_m, c) : m > n\} \subset \mathcal{I},$$
$$\mathcal{B}_n := \{\Lambda(\varpi_n, \varpi_m, c) : m > n\} \subset \mathcal{I}.$$

As $\theta \prec \Gamma(\varpi_n, \varpi_m, c) \preceq \ell$ for each $m \in \mathbb{N}$ where $n < m$, following from Remark 2.1, the infimum of $\mathcal{A}_n$, that is, $\inf \mathcal{A}_n = \alpha_n$ is present in every $n \in \mathbb{N}$. Similarly, since $\theta \preceq \Lambda(\varpi_n, \varpi_m, c) \prec \ell$ for all $n \in \mathbb{N}$ where $n < m$, following from Remark 2.1, the supremum of $\mathcal{B}_n$, that is, $\sup \mathcal{B}_n = \beta_n$ is present in every $n \in \mathbb{N}$. By (4.5), for all $n, m \in \mathbb{N}$ where $m > n$, it follows that

$$\Gamma(\varpi_{n+1}, \varpi_{m+1}, c) = \Gamma(f\varpi_n, f\varpi_m, c) \succeq \psi(\Gamma(\varpi_n, \varpi_m, c)) \succ \Gamma(\varpi_n, \varpi_m, c) \tag{4.6}$$

and

$$\Lambda(\varpi_{n+1}, \varpi_{m+1}, c) = \Lambda(f\varpi_n, f\varpi_m, c) \preceq \phi(\Lambda(\varpi_n, \varpi_m, c)) \prec \Lambda(\varpi_n, \varpi_m, c). \tag{4.7}$$

From this, we can conclude that

$$\Gamma(\varpi_{n+1}, \varpi_{m+1}, c) \succ \Gamma(\varpi_n, \varpi_m, c)$$

and

$$\Lambda(\varpi_{n+1}, f\varpi_{m+1}, c) \prec \Lambda(\varpi_n, \varpi_m, c)$$

for any $n, m \in \mathbb{N}$ where $m > n$ along with $c \in \mathcal{P}_0$. Taking the infimum of Γ and supremum of Λ above, it follows that

$$\ell \succeq \alpha_{n+1} \succeq \alpha_n \succeq \theta, \quad \theta \preceq \beta_{n+1} \preceq \beta_n \preceq \ell$$

for each n belongs to $\mathbb{N}$. Thus, both $\{\alpha_n\}$ and $\{\beta_n\}$ are monotonic sequences in $\mathcal{P}$. By Remark 2.1, there exist two elements $\alpha, \beta \in \mathcal{P}$ satisfying $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} \beta_n = \beta$. From (4.6) and (4.7), by applying (4.5) successively, we have

$$\begin{aligned}
\Gamma(\varpi_{n+1}, \varpi_{m+1}, c) &\succeq \psi(\Gamma(\varpi_n, \varpi_m, c)) \\
&\succeq \psi^2(\Gamma(\varpi_{n-1}, \varpi_{m-1}, c)) \\
&\succeq \cdots \\
&\succeq \psi^n(\Gamma(\varpi_0, \varpi_{m-n}, c))
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(\varpi_{n+1}, \varpi_{m+1}, c) &\preceq \phi(\Lambda(\varpi_n, \varpi_m, c)) \\
&\preceq \phi^2(\Lambda(\varpi_{n-1}, \varpi_{m-1}, c)) \\
&\preceq \cdots \\
&\preceq \phi^n(\Lambda(\varpi_0, \varpi_{m-n}, c))
\end{aligned}$$

for each $n \in \mathbb{N}$ where $m > n$ and $c \in \mathcal{P}_0$. It follows that

$$\alpha_{n+1} \succeq \inf_{m>n} \psi^n(\Gamma(\varpi_0, \varpi_{m-n}, c))$$

and

$$\beta_{n+1} \preceq \sup_{m>n} \phi^n(\Lambda(\varpi_0, \varpi_{m-n}, c)).$$

for each $n \in \mathbb{N}$ along with $c \in \mathcal{P}_0$. In the case where n approaches infinity on both sides of the inequalities above, we deduce that

$$\begin{aligned}\alpha &\succeq \lim_{n\to\infty} \inf_{m>n} \psi^n(\Gamma(\varpi_0, \varpi_{m-n}, c)) \\ &= \lim_{n\to\infty} \psi^n(\Gamma(\varpi_0, \varpi_{m-n}, c)) \\ &= \ell\end{aligned}$$

and

$$\begin{aligned}\beta &\preceq \lim_{n\to\infty} \sup_{m>n} \phi^n(\Lambda(\varpi_0, \varpi_{m-n}, c)) \\ &= \lim_{n\to\infty} \phi^n(\Lambda(\varpi_0, \varpi_{m-n}, c)) \\ &= \theta.\end{aligned}$$

Hence, $\alpha = \ell$ and $\beta = \theta$.This means that

$$\begin{aligned}\lim_{n\to\infty} \inf_{m>n} \Gamma(\varpi_{n+1}, \varpi_{m+1}, c) &= \lim_{n\to\infty} \alpha_n = \ell, \\ \lim_{n\to\infty} \sup_{m>n} \Lambda(\varpi_{n+1}, \varpi_{m+1}, c) &= \lim_{n\to\infty} \beta_n = \theta\end{aligned}$$

for all $c \in \mathcal{P}_0$ which indicate that sequence $\{\varpi_n\}$ is Cauchy.

Given that $(Z, \Gamma, \Lambda, *, \diamond)$ is complete, Lemma 3.2 implies the existence of $\varpi \in Z$ satisfying

$$\lim_{n\to\infty} \Gamma(\varpi_n, \varpi, c) = \ell \text{ and } \lim_{n\to\infty} \Lambda(\varpi_n, \varpi, c) = \theta \quad \text{for any } c \in \mathcal{P}_0. \tag{4.8}$$

As a consequence of conditions (5), (10) of Definition 3.2 and (4.5), for any $c \in \mathcal{P}_0$, we can conclude that

$$\begin{aligned}\Gamma(\varpi, f\varpi, c) &\succeq \Gamma\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) * \Gamma\left(\varpi_{n+1}, f\varpi, \frac{c}{2}\right) \\ &= \Gamma\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) * \Gamma\left(f\varpi_n, f\varpi, \frac{c}{2}\right) \\ &\succeq \Gamma\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) * \psi\left(\Gamma\left(\varpi_n, \varpi, \frac{c}{2}\right)\right) \\ &\succ \Gamma\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) * \Gamma\left(\varpi_n, \varpi, \frac{c}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\Lambda(\varpi, f\varpi, c) &\preceq \Lambda\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) \diamond \Lambda\left(\varpi_{n+1}, f\varpi, \frac{c}{2}\right) \\ &= \Lambda\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) \diamond \Lambda\left(f\varpi_n, f\varpi, \frac{c}{2}\right) \\ &\preceq \Lambda\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) \diamond \phi\left(\Lambda\left(\varpi_n, \varpi, \frac{c}{2}\right)\right) \\ &\prec \Lambda\left(\varpi, \varpi_{n+1}, \frac{c}{2}\right) \diamond \Lambda\left(\varpi_n, \varpi, \frac{c}{2}\right).\end{aligned}$$

When we consider both of the above inequalities and take the limits as n approaches infinity, by utilizing (4.8), we can conclude that

$$\Gamma(\varpi, f\varpi, c) = \ell \text{ and } \Lambda(\varpi, f\varpi, c) = \theta$$

for any $c \in \mathcal{P}_0$. By conditions (3) and (8) of Definition 3.2, it can be deduced that $\varpi = f\varpi$, that is, ϖ is a fixed point of f.

To establish uniqueness, assume z is a different fixed point of f than $\varpi \neq z$. For any $c \in \mathcal{P}_0$, by (4.5), it leads to

$$\Gamma(\varpi, z, c) = \Gamma(f\varpi, fz, c) \succeq \psi(\Gamma(\varpi, z, c)) \succ \Gamma(\varpi, z, c)$$

and

$$\Lambda(\varpi, z, c) = \Lambda(f\varpi, fz, c) \preceq \phi(\Lambda(\varpi, z, c)) \prec \Lambda(\varpi, z, c)$$

resulting in a contradiction. As a result, $x = z$ which demonstrates the uniqueness. □

5. Common fixed-point results

This section presents a generalization of fuzzy Banach contraction concept to complex-valued intuitionistic fuzzy metric spaces and provides several common fixed-point findings for two mappings fulfilling the contraction below on these spaces.

Definition 5.1. *Suppose that $(Z, \Gamma, \Lambda, *, \diamond)$ is a complex-valued intuitionistic fuzzy metric space. A pair of self-mappings $\mathcal{F}, \mathcal{G} : Z \to Z$ is referred to as an intuitionistic fuzzy Banach contraction provided that there is real number $k \in (0, 1)$ where*

$$\begin{aligned} \ell - \Gamma(\mathcal{F}\varpi, \mathcal{G}\varkappa, c) &\preceq k(\ell - \Gamma(\varpi, \varkappa, c)), \\ \Lambda(\mathcal{F}\varpi, \mathcal{G}\varkappa, c) &\preceq k\Lambda(\varpi, \varkappa, c) \end{aligned} \tag{5.1}$$

holds for any $\varpi, \varkappa \in Z$ and $c \in \mathcal{P}_0$.

Theorem 5.1. *Let $(Z, \Gamma, \Lambda, *, \diamond)$ be a complete complex-valued intuitionistic fuzzy metric space and a pair of self-mappings $\mathcal{F}, \mathcal{G} : Z \to Z$ be a intuitionistic fuzzy Banach contraction. Then mappings $\mathcal{F}$ and $\mathcal{G}$ possess a unique common fixed point which lies within Z.*

Proof. Consider an arbitrary point $\varpi_0 \in Z$. A sequence $\{\varpi_n\}$ is defined in Z as

$$\begin{aligned} \varpi_{2n+1} &= \mathcal{F}\varpi_{2n}, \\ \varpi_{2n+2} &= \mathcal{G}\varpi_{2n+1} \end{aligned}$$

for any $n \in \mathbb{N}_0$. The existence of an $n_0 \in \mathbb{N}$ where $\varpi_{n_0} = \varpi_{n_0+1}$ guarantees that ϖ_{n_0} is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$. Indeed, if there is $n \in \mathbb{N}_0$ in which $\varpi_{2n} = \varpi_{2n+1}$, it indicates ϖ_{2n} is a fixed point of $\mathcal{F}$. Furthermore, utilizing (5.1), we have

$$\begin{aligned} \ell - \Gamma(\varpi_{2n+1}, \varpi_{2n+2}, c) &= \ell - \Gamma(\mathcal{F}\varpi_{2n}, \mathcal{G}\varpi_{2n+1}, c) \\ &\preceq k(\ell - \Gamma(\varpi_{2n}, \varpi_{2n+1}, c)) \\ &= k(\ell - \ell) \\ &= \theta \end{aligned}$$

and

$$\begin{aligned}\Lambda(\varpi_{2n+1},\varpi_{2n+2},c) &= \Lambda(\mathcal{F}\varpi_{2n},\mathcal{G}\varpi_{2n+1},c)\\ &\preceq k\Lambda(\varpi_{2n},\varpi_{2n+1},c)\\ &= k(\theta)\\ &= \theta.\end{aligned}$$

for every $c \in \mathcal{P}_0$. It follows that $\Gamma(\varpi_{2n+1},\varpi_{2n+2},c) = \ell$ and $\Lambda(\varpi_{2n+1},\varpi_{2n+2},c) = \theta$. By conditions (3) and (8) of Definition 3.2, $\varpi_{2n+1} = \varpi_{2n+2} = \mathcal{G}\varpi_{2n+1}$, which indicate that ϖ_{2n+1} is a fixed point of $\mathcal{G}$. Since $\varpi_{2n} = \varpi_{2n+1}$, we can infer that ϖ_{2n} is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$. In similar fashion, if there is $n \in \mathbb{N}_0$ in which $\varpi_{2n+1} = \varpi_{2n+2}$, using (5.1) we can show that ϖ_{2n+1} is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$.

Assume both ϖ_n, ϖ_{n+1} are always distinct for each $n \in \mathbb{N}_0$. We shall consider two cases. For the first case suppose n is odd. Substitute $\varpi = \varpi_{n-1}$ and $\varkappa = \varpi_n$ in (5.1), for all $c \in \mathcal{P}_0$ we obtain

$$\begin{aligned}\ell - \Gamma(\varpi_n,\varpi_{n+1},c) &= \ell - \Gamma(\mathcal{F}\varpi_{n-1},\mathcal{G}\varpi_n,c)\\ &\preceq k(\ell - \Gamma(\varpi_{n-1},\varpi_n,c))\\ &\prec \ell - \Gamma(\varpi_{n-1},\varpi_n,c)\end{aligned}$$

and

$$\begin{aligned}\Lambda(\varpi_n,\varpi_{n+1},c) &= \Lambda(\mathcal{F}\varpi_{n-1},\mathcal{G}\varpi_n,c)\\ &\preceq k\Lambda(\varpi_{n-1},\varpi_n,c)\\ &\prec \Lambda(\varpi_{n-1},\varpi_n,c).\end{aligned}$$

It follows that

$$\Gamma(\varpi_n,\varpi_{n+1},c) \succ \Gamma(\varpi_{n-1},\varpi_n,c)$$

and

$$\Lambda(\varpi_n,\varpi_{n+1},c) \prec \Lambda(\varpi_{n-1},\varpi_n,c)$$

for any $c \in \mathcal{P}_0$. For second case suppose n is even. Substitute $\varpi = \varpi_n$ and $\varkappa = \varpi_{n-1}$ in (5.1), for all $c \in \mathcal{P}_0$ we obtain

$$\begin{aligned}\ell - \Gamma(\varpi_{n+1},\varpi_n,c) &= \ell - \Gamma(\mathcal{F}\varpi_n,\mathcal{G}\varpi_{n-1},c)\\ &\preceq k(\ell - \Gamma(\varpi_n,\varpi_{n-1},c))\\ &\prec \ell - \Gamma(\varpi_n,\varpi_{n-1},c)\end{aligned}$$

and

$$\begin{aligned}\Lambda(\varpi_{n+1},\varpi_n,c) &= \Lambda(\mathcal{F}\varpi_n,\mathcal{G}\varpi_{n-1},c)\\ &\preceq k\Lambda(\varpi_n,\varpi_{n-1},c)\\ &\prec \Lambda(\varpi_n,\varpi_{n-1},c).\end{aligned}$$

It follows that

$$\Gamma(\varpi_n,\varpi_{n+1},c) \succ \Gamma(\varpi_{n-1},\varpi_n,c)$$

and

$$\Lambda(\varpi_n,\varpi_{n+1},c) \prec \Lambda(\varpi_n,\varpi_{n-1},c)$$

for any $c \in \mathcal{P}_0$. Therefore, we conclude that

$$\Gamma(\varpi_n, \varpi_{n+1}, c) \succ \Gamma(\varpi_{n-1}, \varpi_n, c), \quad \Lambda(\varpi_n, \varpi_{n+1}, c) \prec \Lambda(\varpi_n, \varpi_{n-1}, c)$$

for every $n \in \mathbb{N}_0$ and $c \in \mathcal{P}_0$. Denote $\Gamma(\varpi_n, \varpi_{n+1}, c) = \mathcal{A}_n$ and $\Lambda(\varpi_n, \varpi_{n+1}, c) = \mathcal{B}_n$ for all $n \in \mathbb{N}_0$. Since

$$\ell \succeq \mathcal{A}_n \succ \mathcal{A}_{n-1} \succ \theta$$

and

$$\theta \preceq \mathcal{B}_n \prec \mathcal{B}_{n-1} \prec \ell$$

for each $n \in \mathbb{N}_0$, it leads to the conclusion that both sequences $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ are monotonic in $\mathcal{P}$. By Remark 2.1, one is possible to locate $\alpha, \beta \in \mathcal{P}$ satisfying

$$\lim_{n\to\infty} \mathcal{A}_n = \alpha, \quad \lim_{n\to\infty} \mathcal{B}_n = \beta.$$

Utilizing (5.1), for $n \in \mathbb{N}_0$ and $c \in \mathcal{P}_0$ we obtain

$$\ell - \Gamma(\varpi_n, \varpi_{n+1}, c) \preceq k(\ell - \Gamma(\varpi_{n-1}, \varpi_n, c))$$
$$\ell - \mathcal{A}_n \preceq k(\ell - \mathcal{A}_{n-1})$$

and

$$\Lambda(\varpi_n, \varpi_{n+1}, c) \preceq k\Lambda(\varpi_{n-1}, \varpi_n, c)$$
$$\mathcal{B}_n \preceq k\mathcal{B}_{n-1}.$$

As n becomes infinitely large for both inequalities, we arrive at

$$\ell - \alpha \preceq k(\ell - \alpha)$$

and

$$\beta \preceq k\beta.$$

As $k \in (0,1)$, if $\alpha \prec \ell$ and $\beta \succ \theta$, it will lead to a contradiction. Therefore, $\alpha = \ell$ and $\beta = \theta$ which means that

$$\lim_{n\to\infty} \Gamma(\varpi_n, \varpi_{n+1}, c) = \ell,$$
$$\lim_{n\to\infty} \Lambda(\varpi_n, \varpi_{n+1}, c) = \theta$$

for every $n \in \mathbb{N}_0$ and $c \in \mathcal{P}_0$.

We will now establish the Cauchy nature for sequence $\{\varpi_n\}$. For every $n \in \mathbb{N}_0$ as well as fixed $c \in \mathcal{P}_0$, consider

$$\mathcal{C}_n = \{\Gamma(\varpi_n, \varpi_m, c) : m > n\} \subseteq \mathcal{I},$$
$$\mathcal{D}_n = \{\Lambda(\varpi_n, \varpi_m, c) : m > n\} \subseteq \mathcal{I}.$$

Since $\theta \prec \Gamma(\varpi_n, \varpi_m, c) \preceq \ell$ and $\theta \preceq \Lambda(\varpi_m, \varpi_n, c) \prec \ell$, by Remark 2.1, the infimum of complex fuzzy set $\Gamma(\varpi_n, \varpi_m, c)$ and the supremum of complex fuzzy set $\Lambda(\varpi_n, \varpi_m, c)$ exist. For any positive integer $m > n$, by applying condition (5) of Definition 3.2 successively, we have

$$\Gamma(\varpi_n, \varpi_m, c) \succeq \Gamma\left(\varpi_n, \varpi_{n+1}, \frac{c}{m-n}\right) * \Gamma\left(\varpi_{n+1}, \varpi_{n+2}, \frac{c}{m-n}\right) * \cdots * \Gamma\left(\varpi_{m-1}, \varpi_m, \frac{c}{m-n}\right).$$

It follows that

$$\lim_{n\to\infty}\inf_{m>n}\Gamma(\varpi_n,\varpi_m,c) \geq \ell * \ell * \cdots * \ell$$
$$= \ell$$

which leads to

$$\lim_{n\to\infty}\inf_{m>n}\Gamma(\varpi_n,\varpi_m,c) = \ell$$

for every $c \in \mathcal{P}_0$. On top of that, For any positive integer $m > n$, by applying condition (10) of Definition 3.2 successively, we have

$$\Lambda(\varpi_n,\varpi_m,c) \preceq \Lambda\left(\varpi_n,\varpi_{n+1},\frac{c}{m-n}\right) \diamond \Lambda\left(\varpi_{n+1},\varpi_{n+2},\frac{c}{m-n}\right) \diamond \cdots \diamond \Lambda\left(\varpi_{m-1},\varpi_m,\frac{c}{m-n}\right).$$

This implies

$$\lim_{n\to\infty}\sup_{m>n}\Lambda(\varpi_n,\varpi_m,c) \preceq \theta \diamond \theta \diamond \cdots \diamond \theta$$
$$= \theta$$

which leads to

$$\lim_{n\to\infty}\sup_{m>n}\Lambda(\varpi_n,\varpi_m,c) = \theta$$

for all $c \in \mathcal{P}_0$. Hence, sequence $\{\varpi_n\}$ is Cauchy.

Given that $(Z,\Gamma,\Lambda,*,\diamond)$ is complete, Lemma 3.2 indicates the presence of $u \in Z$ satisfying

$$\lim_{n\to\infty}\Gamma(\varpi_n,u,c) = \ell \text{ and } \lim_{n\to\infty}\Lambda(\varpi_n,u,c) = \theta$$

for all $c \in \mathcal{P}_0$. For any $n \in \mathbb{N}_0$ and $c \in \mathcal{P}_0$, by (5.1) we yield

$$\ell - \Gamma(\mathcal{F}u,\mathcal{G}\varpi_{2n+1},c) \preceq k(\ell - \Gamma(u,\varpi_{2n+1},c))$$
$$\prec \ell - \Gamma(u,\varpi_{2n+1},c)$$

and

$$\Lambda(\mathcal{F}u,\mathcal{G}\varpi_{2n+1},c) \preceq k\Lambda(u,\varpi_{2n+1},c)$$
$$\prec \Lambda(u,\varpi_{2n+1},c).$$

These imply that

$$\Gamma(\mathcal{F}u,\mathcal{G}\varpi_{2n+1},c) \succ \Gamma(u,\varpi_{2n+1},c) \tag{5.2}$$

and

$$\Lambda(\mathcal{F}u,\mathcal{G}\varpi_{2n+1},c) \prec \Lambda(u,\varpi_{2n+1},c) \tag{5.3}$$

for each $n \in \mathbb{N}_0$ and $c \in \mathcal{P}_0$. As a consequence of conditions (5), (10) of Definition 3.2, (5.2) and (5.3), for any $n \in \mathbb{N}_0$ and $c \in \mathcal{P}_0$, we can conclude that

$$\Gamma(u,\mathcal{F}u,c) \succeq \Gamma\left(u,\varpi_{2n+2},\frac{c}{2}\right) * \Gamma\left(\varpi_{2n+2},\mathcal{F}u,\frac{c}{2}\right)$$
$$= \Gamma\left(u,\varpi_{2n+2},\frac{c}{2}\right) * \Gamma\left(\mathcal{G}\varpi_{2n+1},\mathcal{F}u,\frac{c}{2}\right)$$
$$= \Gamma\left(u,\varpi_{2n+2},\frac{c}{2}\right) * \Gamma\left(\mathcal{F}u,\mathcal{G}\varpi_{2n+1},\frac{c}{2}\right)$$
$$\succeq \Gamma\left(u,\varpi_{2n+2},\frac{c}{2}\right) * \Gamma\left(u,\varpi_{2n+1},\frac{c}{2}\right)$$

and

$$\begin{aligned}\Lambda(u,\mathcal{F}u,c) &\preceq \Lambda\left(u,\varpi_{2n+2},\frac{c}{2}\right)\diamond\Lambda\left(\varpi_{2n+2},\mathcal{F}u,\frac{c}{2}\right)\\ &= \Lambda\left(u,\varpi_{2n+2},\frac{c}{2}\right)\diamond\Lambda\left(G\varpi_{2n+1},\mathcal{F}u,\frac{c}{2}\right)\\ &= \Lambda\left(u,\varpi_{2n+2},\frac{c}{2}\right)\diamond\Lambda\left(\mathcal{F}u,\mathcal{G}\varpi_{2n+1},\frac{c}{2}\right)\\ &\preceq \Lambda\left(u,\varpi_{2n+2},\frac{c}{2}\right)\diamond\Lambda\left(u,\varpi_{2n+1},\frac{c}{2}\right)\end{aligned}$$

As n becomes infinitely large for both inequalities, we arrive at

$$\Gamma(u,\mathcal{F}u,c)=\ell \text{ and } \Lambda(u,\mathcal{F}u,c)=\theta$$

for all $c\in\mathcal{P}_0$. By conditions (3) and (8) of Definition 3.2, it means that $u=\mathcal{F}u$. Using similar steps as above one can deduce that

$$\Gamma(u,\mathcal{G}u,c)=\ell \text{ and } \Lambda(u,\mathcal{G}u,c)=\theta$$

for all $c\in\mathcal{P}_0$. By conditions (3) and (8) of Definition 3.2, it means that $u=\mathcal{G}u$. As a result, $u=\mathcal{F}u=\mathcal{G}u$ which indicates u is common fixed point of both both $\mathcal{F}$ and $\mathcal{G}$.

To establish the uniqueness, assume v is a different fixed point of f in which $v\neq u$. It is possible to locate $c\in\mathcal{P}_0$ satisfying $\Gamma(u,v,c)\neq\ell$ and $\Lambda(u,v,c)\neq\theta$. By (5.1),

$$\begin{aligned}\ell-\Gamma(u,v,c) &= \ell-\Gamma(\mathcal{F}u,\mathcal{G}v,c)\\ &\preceq k(\ell-\Gamma(u,v,c))\\ &\prec \ell-\Gamma(u,v,c)\end{aligned}$$

and

$$\begin{aligned}\Lambda(u,v,c) &= \Lambda(\mathcal{F}u,\mathcal{G}v,c)\\ &\preceq k\Lambda(u,v,c)\\ &\prec \Lambda(u,v,c)\end{aligned}$$

which contradicts with our assumption. Thus $\Gamma(u,v,c)=\ell$ and $\Lambda(u,v,c)=\theta$ for all $c\in\mathcal{P}_0$. By conditions (3) and (8) of Definition 3.2, we conclude that $u=v$ which demonstrates the uniqueness. □

Corollary 5.1. *Let $(Z,\Gamma,\Lambda,*,\diamond)$ be a complete complex-valued intuitionistic fuzzy metric space. If a self-mapping $\mathcal{F}:Z\to Z$ satisfying*

$$\ell-\Gamma(\mathcal{F}\varpi,\mathcal{F}\varkappa,c)\preceq k(\ell-\Gamma(\varpi,\varkappa,c)),$$
$$\Lambda(\mathcal{F}\varpi,\mathcal{F}\varkappa,c)\preceq k\Lambda(\varpi,\varkappa,c)$$

for every $\varpi,\varkappa\in Z$ and $c\in\mathcal{P}_0$, in which $k\in(0,1)$. Then mapping $\mathcal{F}$ possesses a unique fixed point that lies within Z.

Proof. The conclusion can be derived by substituting $\mathcal{F}=\mathcal{G}$ into Theorem 5.1. □

Below, an instance effectively highlights the concept expounded in Corollary 5.1.

Example 5.1. *Let $Z = [0, 1]$. Two binary operations $*$ and $\diamond$ are defined as $c_1 * c_2 = (a_1a_2, b_1b_2)$ and $c_1 \diamond c_2 = (\max a_1, a_2, \max b_1, b_2)$ for each $c_i = (a_i, b_i) \in \mathcal{I}$ where $i = 1, 2$. Let complex fuzzy sets Γ and Λ be defined as follow:*

$$\Gamma(\varpi, \varkappa, c) = \left(\frac{ab + \min\{\varpi, \varkappa\}}{ab + \max\{\varpi, \varkappa\}}\right)\ell, \quad \Lambda(\varpi, \varkappa, c) = \left(\frac{\max\{\varpi, \varkappa\} - \min\{\varpi, \varkappa\}}{ab + \max\{\varpi, \varkappa\}}\right)\ell$$

for all $\varpi, \varkappa \in Z$ and $c = (a, b) \in \mathcal{P}_0$. The fact that $(Z, \Gamma, \Lambda, *, \diamond)$ is a complete complex-valued intuitionistic fuzzy metric space is established without much effort.

Define $\mathcal{F} : Z \to Z$ by $\mathcal{F} = \varpi/2$ where $\varpi \in Z$. For any $\varpi, \varkappa \in Z$ satisfying $\varpi \le \varkappa$, it is clear that $\mathcal{F}\varpi \le \mathcal{F}\varkappa$. It follows that

$$\begin{aligned}\Gamma(\mathcal{F}\varpi, \mathcal{F}\varkappa, c) &= \left(\frac{ab + \min\{\mathcal{F}\varpi, \mathcal{F}\varkappa\}}{ab + \max\{\mathcal{F}\varpi, \mathcal{F}\varkappa\}}\right)\ell \\ &= \left(\frac{ab + \mathcal{F}\varpi}{ab + \mathcal{F}\varkappa}\right)\ell \\ &\ge \left(\frac{ab + \varpi}{ab + \varkappa}\right)\ell \\ &= \Gamma(\varpi, \varkappa, c).\end{aligned}$$

If we pick any $k \in (\frac{1}{2}, 1)$, we have

$$\ell - \Gamma(\mathcal{F}\varpi, \mathcal{F}\varkappa, c) \le k(\ell - \Gamma(\varpi, \varkappa, c))$$

for every $\varpi, \varkappa \in Z$ and $c = (a, b) \in \mathcal{P}_0$. Similarly, we able to deduce that

$$\Lambda(\mathcal{F}\varpi, \mathcal{F}\varkappa, c) \le k\Lambda(\varpi, \varkappa, c)$$

for every $\varpi, \varkappa \in Z$ and $c = (a, b) \in \mathcal{P}_0$. The graphical view of these two inequalities are shown in Figure 1 and Figure 2 respectively. Consequently, each conditions specified in Corollary 5.1 are met. Particularly, 0 is the unique fixed point of $\mathcal{F}$.

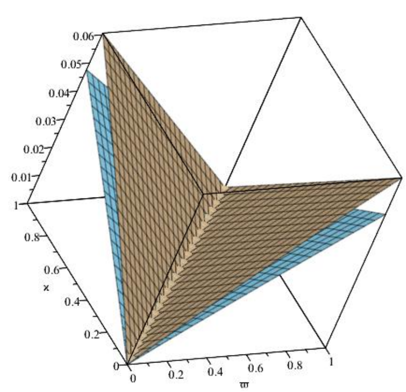

FIGURE 1. Graphical view of inequality $\ell - \Gamma(\mathcal{F}\varpi, \mathcal{F}\varkappa, c) \le k(\ell - \Gamma(\varpi, \varkappa, c))$, where the blue color plane represents the left-hand side and the brown color plane represents the right-hand side, when $k = 2/3$ and $c = (2, 5)$.

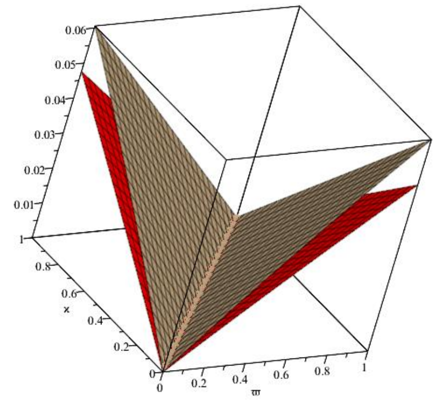

FIGURE 2. Graphical view of inequality $\Lambda(\mathcal{F}\varpi, \mathcal{F}\varkappa, c) \preceq k\Lambda(\varpi, \varkappa, c)$, where the red color plane represents the left-hand side and the brown color plane represents the right-hand side, when $k = 2/3$ and $c = (2, 5)$.

Theorem 5.2. *Suppose that $(Z, \Gamma, \Lambda, *, \diamond)$ is a complete complex-valued intuitionistic fuzzy metric space. If commuting pair of self-mappings $\mathcal{F}, \mathcal{G} : Z \to Z$ satisfying*

$$\ell - \Gamma(\mathcal{F}^n\varpi, \mathcal{G}^n\varkappa, c) \preceq k(\ell - \Gamma(\varpi, \varkappa, c)),$$

$$\Lambda(\mathcal{F}^n\varpi, \mathcal{G}^n\varkappa, c) \preceq k\Lambda(\varpi, \varkappa, c)$$

for any $\varpi, \varkappa \in Z$, $c \in \mathcal{P}_0$ and $n \in \mathbb{N}$, in which $k \in (0, 1)$. Then mappings $\mathcal{F}$ and $\mathcal{G}$ possess a unique common fixed point that lies within Z.

Proof. Each conditions in Theorem 5.1 are fulfilled by both $\mathcal{F}^n$ and $\mathcal{G}^n$. Consequently, they possess a unique common fixed point u in Z, for instance, $\mathcal{F}^n u = \mathcal{G}^n u = u$. From the fact that

$$\mathcal{F}^n\mathcal{F}u = \mathcal{F}\mathcal{F}^n u = \mathcal{F}u,$$

it can be inferred that $\mathcal{F}u$ is a point fixed by $\mathcal{F}^n$. Since mappings $\mathcal{F}$ and $\mathcal{G}$ commute, we can write

$$\mathcal{G}^n\mathcal{F}u = \mathcal{F}\mathcal{G}^n u = \mathcal{F}u$$

which shows that Fu is a point fixed by $\mathcal{G}^n$. Consequently, Fu serves as common fixed point of $\mathcal{F}^n$ and $\mathcal{G}^n$.

Similarly, from the fact that

$$\mathcal{G}^n\mathcal{G}u = \mathcal{G}\mathcal{G}^n u = \mathcal{G}u,$$

it can be inferred that $\mathcal{G}u$ is a point fixed by $\mathcal{G}^n$. Since mappings $\mathcal{F}$ and $\mathcal{G}$ commute, we can write

$$\mathcal{F}^n\mathcal{G}u = \mathcal{G}\mathcal{F}^n u = \mathcal{G}u$$

which shows that $\mathcal{G}u$ is a point fixed by $\mathcal{F}^n$. Consequently, $\mathcal{G}u$ serves as common fixed point of $\mathcal{F}^n$ and $\mathcal{G}^n$.

In light of the common fixed point of $\mathcal{F}^n$ and $\mathcal{G}^n$ being unique, this indicates that $u = \mathcal{G}u = \mathcal{F}u$. As a result, u serve as the point shared and fixed by $\mathcal{F}$ and $\mathcal{G}$. Obviously, any common fixed point of $\mathcal{F}$ and $\mathcal{G}$ remains a common fixed point of $\mathcal{F}^n$ and $\mathcal{G}^n$. For this purpose, the common fixed point of $\mathcal{F}$ and $\mathcal{G}$ is uniquely determined. □

Corollary 5.2. *Suppose that $(Z, \Gamma, \Lambda, *, \diamond)$ is a complete complex-valued intuitionistic fuzzy metric space. If a self-mapping $\mathcal{F} : Z \to Z$ satisfying*

$$\ell - \Gamma(\mathcal{F}^n\varpi, \mathcal{F}^n\varkappa, c) \preceq k(\ell - \Gamma(\varpi, \varkappa, c)),$$
$$\Lambda(\mathcal{F}^n\varpi, \mathcal{F}^n\varkappa, c) \preceq k\Lambda(\varpi, \varkappa, c)$$

for any $\varpi, \varkappa \in Z$, $c \in \mathcal{P}_0$ and $n \in \mathbb{N}$, in which $k \in (0,1)$. Then, $\mathcal{F}$ possess a unique fixed point that lies within Z.

Proof. The conclusion can be derived by substituting $\mathcal{F} = \mathcal{G}$ into Theorem 5.2. □

6. Application to Fredholm integral equations of second kind

This section explore how Theorem 4.1 is employed to demonstrate the presence of a unique solution for Fredholm integral equations. The set $C([0,1],\mathbb{R})$ denotes the collection of every continuous functions that map the interval $[0,1]$ to $\mathbb{R}$. Below is an example of a second-kind nonlinear Fredholm integral equation:

$$\psi(t) = Q(t) + \gamma \int_0^1 \omega(t,s)\chi(s,\psi(s))ds \tag{6.1}$$

where Q represents real valued function that is continuous on the interval $[0,1]$, $\omega(t,s)$ represents the kernel of the integral function, $\chi(s,\psi(s))$ represents continuous and nonlinear function defined on $[0,1]\times\mathbb{R}$ and $\psi(t)$ represents function that we wish to be determined.

Theorem 6.1. *Consider $Z = C([0,1],\mathbb{R})$. Suppose that the conditions outlined below are met:*

(1) *an element $\alpha \in (0,1)$ can be located in which*

$$|\chi(s,\psi(s)) - \chi(s,\phi(s))| \leq \alpha|\psi(s) - \phi(s)|$$

for any $\psi, \phi \in Z$ and $s \in [0,1]$;

(2) $\int_0^1 \omega(t,s)ds \leq \beta$;

(3) $\gamma^2\beta^2\alpha^2 \leq k < 1$.

Consequently, integral equation (6.1) *admits a unique solution in Z.*

Proof. Given a mapping $\mathcal{F} : Z \to Z$ defined by

$$\mathcal{F}\psi(t) = Q(t) + \gamma \int_0^1 \omega(t,s)\chi(s,\psi(s))ds$$

for every $\psi(t) \in Z$ and $t \in [0,1]$. Let complex-valued t-norm and complex valued t-conom be defined by $*_p$ and $\diamond_n$ respectively. Furthermore, $\Gamma(\varpi,\varkappa,c)$ and $\Lambda(\varpi,\varkappa,c)$ defined by

$$\Gamma(\psi(t),\phi(t),c) = \frac{a+b}{a+b+|\psi(t)-\phi(t)|^2}\ell, \quad \Lambda(\psi(t),\phi(t),c) = \frac{|\psi(t)-\phi(t)|^2}{a+b+|\psi(t)-\phi(t)|^2}\ell$$

for all $\psi, \phi \in Z$, $c = (a, b) > 0$ and $t \in [0, 1]$. The fact that $(Z, \Gamma, \Lambda, *, \diamond)$ is a complete complex-valued intuitionistic fuzzy metric space is established without much effort.

For all $\psi, \phi \in Z$ and $t \in [0, 1]$, it follows that

$$\begin{aligned}
|\mathcal{F}\psi(t) - \mathcal{F}\phi(t)|^2 &= \left|Q(t) + \gamma \int_0^1 \omega(t,s)\chi(s,\psi(s))ds - Q(t) - \gamma \int_0^1 \omega(t,s)\chi(s,\phi(s))ds)\right|^2 \\
&= \gamma^2 \left|\int_0^1 \omega(t,s)\chi(s,\psi(s))ds - \int_0^1 \omega(t,s)\chi(s,\phi(s))ds\right|^2 \\
&\le \gamma^2 \left(\int_0^1 \omega(t,s)ds\right)^2 |\chi(s,\psi(s)) - \chi(s,\phi(s))|^2 \\
&\le \gamma^2\beta^2\alpha^2 |\psi(s) - \phi(s)|^2 \\
&\le k|\psi(s) - \phi(s)|^2
\end{aligned}$$

Now, for all $\psi, \phi \in Z$ and $c \in \mathcal{P}_0$, it leads to

$$\begin{aligned}
\Gamma(\mathcal{F}\psi(t), \mathcal{F}\phi(t), kc) &= \frac{k(a+b)}{k(a+b) + |\mathcal{F}\psi(t) - \mathcal{F}\phi(t)|^2}\ell \\
&\ge \frac{k(a+b)}{k(a+b) + k|\psi(t) - \phi(t)|^2}\ell \\
&= \frac{a+b}{a+b+|\psi(t) - \phi(t)|^2}\ell \\
&= \Gamma(\psi(t), \phi(t), c)
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(\mathcal{F}\psi(t), \mathcal{F}\phi(t), kc) &= \frac{|\mathcal{F}\psi(t) - \mathcal{F}\phi(t)|^2}{k(a+b) + |\mathcal{F}\psi(t) - \mathcal{F}\phi(t)|^2}\ell \\
&= \left(1 - \frac{k(a+b)}{k(a+b) + |\mathcal{F}\psi(t) - \mathcal{F}\phi(t)|^2}\right)\ell \\
&\le \left(1 - \frac{k(a+b)}{k(a+b) + k|\psi(t) - \phi(t)|^2}\right)\ell \\
&= \left(1 - \frac{k(a+b)}{k(a+b) + k|\psi(t) - \phi(t)|^2}\right)\ell \\
&= \frac{|\psi(t) - \phi(t)|^2}{a+b+|\psi(t) - \phi(t)|^2}\ell \\
&= \Lambda(\psi(t), \phi(t), c)
\end{aligned}$$

As a consequence, all the requirements specified in Theorem 4.1 are met, indicating $\mathcal{F}$ possesses a unique fixed point in Z. Stated differently, a unique solution to (6.1) exists in $C([0, 1], \mathbb{R})$. □

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