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Eigenvalues and Eigenvectors for a Continuous G-Frame Operator

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Abstract. In this paper, we give a simple characterization of the eigenvalues and eigenvectors for the continous G-frame operator for $\{\Lambda_\omega P \in B(H, K_\omega),\ \omega \in \Omega\}$, where H^N is an N-dimensional Hilbert space and P is a rank k orthogonal projection on H^N. Using this, we derive several results.

1. Introduction

Frames introduced by Duffin and Shaeffer [7], reintroduced in 1986 by Daubechies, Grossmann, and Meyer [6], have recently received great attention owing to their wide range of applications in both pure and applied mathematics, specially that it has been extensively used in many fields such as filter bank theory, signal and image processing, coding and communication [9] and other areas. We refer to [3–5, 8] for an introduction to frame theory and its applications.

One of the main virtues of frames is that, given a frame for a Hilbert space H, we can reconstruct each $x \in H$ only from the frame coefficients, (a sequence of complex numbers).

Formally, a frame in a separable Hilbert space H is a sequence $\{x_i\}_{i\in I}$ for which there exist positive constants $A, B > 0$ such that:
$A\|x\|^2 \le \sum_{i\in I} |\langle x, x_i\rangle|^2 \le B\|x\|^2$, for all $x \in H$. The constants A, B are called lower and upper bounds, respectively. If $A = B$, it is called a tight frame and it is said to be a normalized tight or Parseval frame, if $A = B = 1$. The collection $\{x_i\}_{i\in I} \subset H$ is called Bessel if the above second inequality holds. In this case, B is called the Bessel bound.

W. Sun [10] introduced a generalization of frames, called g-frames. Abdollah-pour and Faroughi [1] introduced the concept of continuous g-frames as a generalization of discrete g-frames.

Let us consider the space;

$$l^2\left(\{H_\omega\}_\omega\right) = \{\{f_\omega\}_\omega, \quad f_\omega \in H_\omega, \quad \omega \in \Omega, \int_\Omega \|f_\omega\|^2 d\mu(\omega) < \infty\}$$

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with the inner product given by:

$$\langle \{f_\omega\}_\omega, \{g_\omega\}_\omega \rangle = \int_\Omega \langle f_\omega, g_\omega \rangle d\mu(\omega).$$

and the norm is defined by $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$.
It is clear that $l^2\left(\{H_\omega\}_\omega\right)$ is a Hilbert space.

Definition 1.1. *[1] We say that $\Lambda = \{\Lambda_\omega \in B(H, H\omega), \quad \omega \in \Omega\}$ is a continuous generalized frame or simply a continuous g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if*

(1) *for each $f \in H, \{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable;*

(2) *there exist two constants $0 < A \leq B < \infty$, such that*

$$A\|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \quad f \in H$$

A and B are called the lower and upper continuous g-frame bounds, respectively.

If only the right-hand inequality of (1.1) is satisfied, we call Λ the continuous g-Bessel sequence for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with continuous g-Bessel bound B. If $A = B = \lambda$, we call Λ the λ-tight continuous g-frame. Moreover, if $\lambda = 1$, Λ is called the Parseval continuous g-frame.

We define T_Λ, the synthesis operator for a continuous g-Bessel sequence Λ as follows:

$$\langle T_\Lambda \{f\omega\}_{\omega\in\Omega}, g \rangle = \int_\Omega \langle f_\omega, \Lambda_\omega g \rangle d\mu(\omega) \quad \Big(\{f_\omega\}_{\omega\in\Omega} \in l^2\left(\{H_\omega\}_\omega\right), \quad g \in H\Big).$$

The operator T_Λ is well-defined and bounded.
The adjoint of T_Λ which is called the analysis operator is defined by:

$$T^*_\Lambda : H \longrightarrow l^2\left(\{H_\omega\}_\omega\right) \quad T^*_\Lambda f = \{\Lambda_\omega f\}_{\omega\in\Omega}.$$

The bounded linear operator S_Λ defined by

$$S_\Lambda : H \longrightarrow H \quad \langle S_\Lambda f, g \rangle = \int_\Omega \langle f, \Lambda^*_\omega \Lambda_\omega g \rangle d\mu(\omega).$$

is called the continuous g-frame operator of Λ.

2. Main results

These results are generalisations of the results established by Azam Yousefzadeheyni et all in [2] to the countinous case.

Theorem 2.1. *Let H be a Hilbert space and (Ω, μ) be a measure space with finite measure and $(K_\omega)_{\omega\in\Omega}$ a family of Hilbert spaces.*
Let $\{\Lambda_\omega \in B(H, K_\omega); \omega \in \Omega\}$ be a continous g-frame for H, Let $e_1 \in H$, $\|e_1\| = 1$ and let P be the orthogonal projection of H onto span$\{e_1\}$. Then the following statements are equivalent:
(1) *e_1 is an eigenvector for S_Λ with the eigenvalue λ_1.*
(2) *$\int_\Omega \|\Lambda_\omega e_1\|^2 d\mu(\omega) = \lambda_1$ and, $\int_\Omega \langle \Lambda_\omega e_1, \Lambda_\omega f_i \rangle d\mu(\omega) = 0$ for all $f \in (I - P)H$.*

Proof. $(\Rightarrow)$: We have

$$\begin{aligned}\int_{\Omega}\|\Lambda_{\omega}e_1\|^2 d\mu(\omega) &= \int_{\Omega}\langle \Lambda_{\omega}e_1, \Lambda_{\omega}e_1\rangle d\mu(\omega)\\ &= \int_{\Omega}\langle e_1, \Lambda_{\omega}^{*}\Lambda_{\omega}e_1\rangle d\mu(\omega)\\ &= \langle S_{\Lambda}e_1, e_1\rangle = \lambda_1\end{aligned}$$

If $f \in (I-P)H$, then

$$\int_{\Omega}\langle \Lambda_{\omega}e_1, \Lambda_{\omega}f\rangle d\mu(\omega) = \langle S_{\Lambda}e_1, f\rangle = 0$$

$(\Leftarrow)$ Let $\{e_i\}_{i=1}^{N}$ ba an orthonormal basis of H.
Then

$$\begin{aligned}S_{\Lambda}e_1 &= \sum_{i=1}^{N}\langle S_{\Lambda}e_1, e_i\rangle e_i = \sum_{i=1}^{N}\left(\int_{\Omega}\langle \Lambda_{\omega}e_1, \Lambda_{\omega}e_i\rangle d\mu(\omega)\right)e_i\\ &= \int_{\Omega}\|\Lambda_{\omega}e_1\|^2 d\mu(\omega) + \sum_{i=2}^{N}\left(\int_{\Omega}\langle \Lambda_{\omega}e_1, \Lambda_{\omega}e_i\rangle d\mu(\omega)\right)e_i = \lambda_1 e_1\end{aligned}$$

It follows that e_1 is an eigenvector for S_{Λ} with the eigenvalue λ_1 □

Theorem 2.2. *Let $\{\Lambda_{\omega} \in B(H, K_{\omega}); \omega \in \Omega\}$ be a continous g-frame for H and $\{e_i\}_{i=1}^{N}$ ba an orthonormal basis of H,consisting of eigenvectors for S_{Λ} with eigenvalues λ_i respectively.*
Then

$$\int_{\Omega}\|\Lambda_{\omega}f\|^2 d\mu(\omega) = \sum_{i=1}^{N}\lambda_i|\langle f, e_i\rangle|^2, \quad f \in H.$$

and

$$\int_{\Omega}\langle \Lambda_{\omega}f, \Lambda_{\omega}g\rangle d\mu(\omega) = \sum_{i=1}^{N}\lambda_i\langle f, e_i\rangle\langle e_i, g\rangle, \quad f, g \in H.$$

Proof.

$$\begin{aligned}\int_{\Omega}\|\Lambda_{\omega}f\|^2 d\mu(\omega) &= \langle S_{\Lambda}f, f\rangle\\ &= \langle S_{\Lambda}(\sum_{i=1}^{N}\langle f, e_i\rangle e_i), \sum_{j=1}^{N}\langle f, e_j\rangle e_j\rangle\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\langle f, e_i\rangle\langle e_j, f\rangle\langle e_i, e_j\rangle\\ &= \sum_{i=1}^{N}\lambda_i|\langle f, e_i\rangle|^2, \quad \forall f \in H.\end{aligned}$$

And

$$\begin{aligned}\int_\Omega \langle \Lambda_\omega f, \Lambda_\omega g\rangle d\mu(\omega) &= \langle S_\Lambda f, g\rangle \\ &= \langle S_\Lambda (\sum_{i=1}^N \langle f, e_i\rangle e_i), \sum_{j=1}^N \langle g, e_j\rangle e_j\rangle \\ &= \sum_{i=1}^N \sum_{j=1}^N \lambda_i \langle f, e_i\rangle \langle e_j, g\rangle \langle e_i, e_j\rangle \\ &= \sum_{i=1}^N \lambda_i \langle f, e_i\rangle \langle e_i, g\rangle\end{aligned}$$

□

Proposition 2.1. *Let $\{\Lambda_\omega \in B(H, K_\omega); \omega \in \Omega\}$ be a continous g-frame for H. Let $1 \le k \le N$ fixed and P a rank k orthogonal projection of H onto a subspace F. Let S^1_Λ be the G-frame operator of $\{\Lambda_\omega P \in B(F, K_\omega); \omega \in \Omega\}$ and $\{f_i\}_{i=1}^k$ be an orthonormal basis of F. Then the following are equivalent:*
(1) *$\{f_i\}_{i=1}^k$ is a family of eigenvectors for S^1_Λ with eigenvalues $\eta_1, \eta_2, ..., \eta_k$*
(2) *We have*

$$(i) \int_\Omega \|\Lambda_\omega f_i\|^2 d\mu(\omega) = \eta_i; \quad 1 \le i \le k$$

$$(ii) \int_\Omega \langle \Lambda_\omega f_i, \Lambda_\omega f_j\rangle d\mu(\omega) = 0, \quad i \neq j$$

Proof. ($\Rightarrow$) If $\{f_i\}_{i=1}^k$ is a family of eigenvectors for S^1_Λ with eigenvalues $\{\eta_i\}_{i=1}^k$. then

$$\int_\Omega \|\Lambda_\omega f_i\|^2 d\mu(\omega) = \langle S^1_\Lambda f_i, f_i\rangle = \eta_i \quad \forall 1 \le i \le k$$

and for $i \neq j$

$$\int_\Omega \langle \Lambda_\omega f_i, \Lambda_\omega f_j\rangle d\mu(\omega) = \langle S^1_\Lambda f_i, f_j\rangle = \eta_i \langle f_i, f_j\rangle = 0$$

($\Leftarrow$) Let $\{f_i\}_{i=k+1}^N$ such that $\{f_i\}_{i=1}^N$ is an orthonormal basis for H. Then

$$\begin{aligned}S^1_\Lambda f_i &= \sum_{j=1}^N \langle S^1_\Lambda f_i, f_j\rangle f_j \\ &= \sum_{j=1}^N \left(\int_\Omega \langle f_i, P\Lambda^*_\omega \Lambda_\omega P f_j\rangle d\mu(\omega)\right) f_j \\ &= \sum_{j=1}^k \left(\int_\Omega \langle \Lambda_\omega f_i, \Lambda_\omega f_j\rangle d\mu(\omega)\right) f_j \\ &= \left(\int_\Omega \langle \Lambda_\omega f_i, \Lambda_\omega f_i\rangle d\mu(\omega)\right) f_i + \sum_{j\in\{1,2,...,k\}\setminus\{i\}} \left(\int_\Omega \langle \Lambda_\omega f_i, \Lambda_\omega f_j\rangle d\mu(\omega)\right) f_j \\ &= \left(\int_\Omega \|\Lambda_\omega f_i\|^2 d\mu(\omega)\right) f_i = \mu_i f_i\end{aligned}$$

so, f_i is an eigenvector of S^1_Λ with eigenvalue μ_i for $i = 1, 2, ..., k$. □

Theorem 2.3. *Let $\{\Lambda_\omega \in B(H, K_\omega); \omega \in \Omega\}$ be a continous g-frame for H and $\{e_i\}_{i=1}^N$ ba an orthonormal basis of H,consisting of eigenvectors for S_Λ with eigenvalues λ_i respectively. Let $\{\Lambda_\omega \in B(H, K_\omega); \omega \in \Omega\}$ be a continous g-frame for H. Let $1 \le k \le N$ fixed and P a rank k orthogonal projection of H onto a subspace F. Let S^1_Λ be the G-frame operator of $\{\Lambda_\omega P \in B(F, K_\omega); \omega \in \Omega\}$ and $\{f_i\}_{i=1}^k$ be an orthonormal basis of F. Then the following statements are equivalent:*

(1) *$\{f_i\}_{i=1}^k$ is a family of eigenvectors for S^1_Λ with eigenvalues $\eta_1, \eta_2, ..., \eta_k$*
(2) *We have*

$$(i) \sum_{i=1}^{N} \lambda_i |\langle f, e_i\rangle|^2 = \eta_i, \quad 1 \le i \le k$$

$$(ii) \sum_{i=1}^{N} \lambda_i \langle f, e_i\rangle\langle e_i, g\rangle = 0, \quad i \ne j$$

Proof. It follows immediatly Theorem 2.2 and Proposition 2.3 □

Proposition 2.2. *Let $\{\Lambda_\omega \in B(H, K_\omega); \omega \in \Omega\}$ be a continous g-frame for H and $\{e_i\}_{i=1}^N$ ba an orthonormal basis of H,consisting of eigenvectors for S_Λ with eigenvalues λ_i respectively. Let $\{\delta_j\}_{j=1}^k$ be a partition of $\{1, 2, ..., N\}$ and for every $1 \le j \le k$ set $f_j = \sum_{i\in\delta_j} a_i e_i$ with $\|f_j\| = \sum_{i\in\delta_j} |a_i|^2 = 1$. Let P be the orthogonal projection of H onto $F = span\{f_j\}_{j=1}^k$ and let S^1_Λ be the g-frame operator for $\{\Lambda_\omega P \in B(F, K_\omega); \omega \in \Omega\}$. Then $\{f_j\}_{j=1}^k$ is an orthonormal basis for F and f_j is an eigenvector of S^1_Λ with eigenvalue $\eta_j = \sum_{i\in\delta_j} \lambda_i |a_i|^2$ for $j = 1, 2, ..., k$.*

Proof.

$$\begin{aligned}
\int_\Omega \|\Lambda_\omega f_i\|^2 d\mu(\omega) &= \int_\Omega \langle \Lambda_\omega f_i, \Lambda_\omega f_i\rangle d\mu(\omega) \\
&= \int_\Omega \langle \Lambda_\omega (\sum_{j\in\delta_i} a_j e_j), \Lambda_\omega (\sum_{l\in\delta_i} a_l e_l)\rangle d\mu(\omega) \\
&= \int_\Omega \sum_{j\in\delta_i} \sum_{l\in\delta_i} a_j \overline{a_l} \langle \Lambda_\omega e_j, \Lambda_\omega e_l\rangle d\mu(\omega) \\
&= \int_\Omega \sum_{j\in\delta_i} |a_j|^2 \langle \Lambda_\omega e_j, \Lambda_\omega e_j\rangle d\mu(\omega) + \int_\Omega \sum_{l\in\delta_i\setminus\{j\}} a_j \overline{a_l} \langle \Lambda_\omega e_j, \Lambda_\omega e_l\rangle d\mu(\omega) \\
&= \sum_{j\in\delta_i} |a_j|^2 \int_\Omega \langle e_j, \Lambda^*_\omega \Lambda_\omega e_j\rangle d\mu(\omega) \\
&= \sum_{j\in\delta_i} |a_j|^2 \langle S_\Lambda e_j, e_j\rangle \\
&= \sum_{j\in\delta_i} \lambda_j |a_j|^2 \\
&= \eta_j.
\end{aligned}$$

And it is easy to see that for $j \neq l$ we have:

$$\int_{\Omega} \langle \Lambda_{\omega} f_j, \Lambda_{\omega} f_l \rangle d\mu(\omega) = 0$$

And $\{f_j\}_{j=1}^{k}$ is an orthonormal basis for F since it is an orthogonal system and $\|f_j\| = 1$ for every $1 \leq j \leq k$.
By Proposition (2.3) f_j is an eigenvector of S_{Λ}^{1} with eigenvalue $\eta_j = \sum_{i \in \delta_j} \lambda_j |a_j|^2$ for $j = 1, 2, ..., k$. □

Example 2.1. *Let* $H = \mathbb{C}^3$, $\Omega = [0,3]$ *and* $K_{\omega} = \mathbb{C} \quad \forall \omega \in \Omega$, *then*
$l^2(\{H_{\omega}\}_{\omega}) = L^2(\Omega)$.
Let us define

$$\Lambda_{\omega}(x, y, z) = x - y \quad if \quad \omega \in [0, 1[$$
$$\Lambda_{\omega}(x, y, z) = x + y \quad if \quad \omega \in [1, 2[$$
$$\Lambda_{\omega}(x, y, z) = \quad 2z \quad if \quad \omega \in [2, 3].$$

Then

$$\begin{aligned} \int_{\Omega} \|\Lambda_{\omega}(x, y, z)\|^2 d\mu(\omega) &= \|x - y\|^2 + \|x + y\|^2 + \|2z\|^2 \\ &= 2\|x\|^2 + 2\|y\|^2 + 4\|z\|^2. \end{aligned}$$

So

$$2\|(x, y, z)\|^2 \leq \int_{\Omega} \|\Lambda_{\omega}(x, y, z)\|^2 d\mu(\omega) \leq 4\|(x, y, z)\|^2$$

This means that $\{\Lambda_{\omega} \in B(H, K_{\omega} : w \in \Omega\}$ *is a countinous g-frame for* H *with respect to* $\{K_{\omega}\}_{\omega \in \Omega}$ *with bounds 2 and 4.*
The g-frame operator of $\{\Lambda_{\omega} \in B(H, K_{\omega} : w \in \Omega\}$ *is:*

$$S_{\Lambda} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3; \quad S_{\Lambda}(x, y, z) = (2x, 2y, 4z).$$

Let $\{e_i\}_{i=1}^{3}$ *be the standard orthonormal basis for* $\mathbb{C}^3$.
One can see that 2 is an eigenvalue of S_{Λ} *with eigenvectors* e_1 *and* e_2 *and 4 is an eigenvalue of* S_{Λ} *with eigenvector* e_3.
We consider the orthogonal projection:

$$P : \mathbb{C}^3 \longrightarrow span\{e_3\}.$$

Then

$$\int_{\Omega} \|\Lambda_{\omega}(e_3)\|^2 d\mu(\omega) = 4$$

and $\forall f \in (I - P)\mathbb{C}^3$:

$$\int_{\Omega} \langle \Lambda_{\omega}(e_3), \Lambda_{\omega} f \rangle d\mu(\omega) = \langle S_{\Lambda}(e_3), f \rangle = \langle 4e_3, \alpha e_1 + \beta e_2 \rangle = 0.$$

On the other hand

$$\int_{\Omega} \|\Lambda_{\omega}(x,y,z)\|^2 d\mu(\omega) = 2\|x\|^2 + 2\|y\|^2 + 4\|z\|^2.$$
$$= 2\langle (x,y,z), e_1\rangle + 2\langle (x,y,z), e_2\rangle + 2\langle (x,y,z), e_3\rangle$$
$$= \sum_{i=1}^{3} \lambda_i |\langle (x,y,z), e_i\rangle|^2$$

We define now the projection P as follows;

$$P : \mathbb{C}^3 \longrightarrow F = span\{e_1, e_2\};$$

then the g-frame operator of $\{\Lambda_\omega P\}$ is the operator $S : F \longrightarrow F$ defined by:

$$S(x,y,z) = PS_{\Lambda}P(x,y,z) = (2x, 2y, 0)$$

e_1 and e_2 are eigenvectors of S with the eigenvalue 2. We have

$$\int_{\Omega} \|\Lambda_{\omega}(e_1)\|^2 d\mu(\omega) = 2$$

and

$$\int_{\Omega} \|\Lambda_{\omega}(e_2)\|^2 d\mu(\omega) = 2.$$

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