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Recent Developments in General Quasi Variational Inequalities

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Abstract. In this paper, we present a number of new and known numerical techniques for solving general quasi variational inequalities, introduced by Noor [34] in 1988, using various techniques including projection, Wiener-Hopf equations, auxiliary principle, dynamical systems coupled with finite difference approach and sensitivity analysis. Convergence analysis of these methods is investigated under suitable conditions. Sensitivity analysis is also investigated. Some special cases are discussed as applications of the main results. Several open problems are suggested for future research.

1. Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory was introduced and considered in early sixties by Lions and Stampacchia [23], can be viewed as a novel extension and generalization of the variational principles. It is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities. It is well known [20] that the variational inequalities are equivalent to the fixed point problem. This equivalent formulations has been used to study the existence of the solution and to develop numerical methods for variational inequalities. Noor [38, 41] has proposed and suggested three step forward-backward iterative methods for finding the approximate solution of general variational inequalities using the technique of updating the solution and auxiliary principle. These tree-step methods are known as Noor iterations [3, 4, 22]. These forward-backward splitting algorithms are similar to those of the schemes of Glowinski and Le Tallec [16], which they suggested by using the Lagrangian technique. Suantai et. al. [64] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and

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image inpainting. Ashish et. al. [3, 4], Cho et al. [7] and Kwuni et al. [22] explored the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations, which have influenced the research in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. These three-step schemes are a natural generalization of the splitting methods of Ames [2] for solving partial differential equations. Noor (three-step) iterations contain Mann (one-step) iteration and Ishikawa (two-step) iterations as special cases. Inspired and motivated by the usefulness and applications of the splitting three-step methods, several classes of three-step approximation schemes for solving variational inequalities, fixed points and related problems are being investigated. It has been established [38, 60] that Noor iterations, perform better than two-step(Ishikawa iteration) and one step method Mann iteration. If the set involved in the variational inequality depends upon the solution explicitly or implicity, then the variational inequalities are called the quasi-variational inequality, introduced by Bensoussan and Lions [6] in the field of impulse control. Noor [30, 34] proved that the quasi variational inequalities are equivalent to the implicit fixed point problem. This equivalent formulation played an important role in developing numerical methods, sensitivity analysis, dynamical systems and other aspects of quasi-variational inequalities. For the applications, motivations, generalizations, extensions, dynamical systems, sensitivity analysis, numerical methods, error bounds and related optimization programming problems, see [1, 6, 8, 12–19, 21, 22, 25, 26, 28–30, 32–56, 58, 64, 65].

The Wiener-Hopf equations were introduced and studied by Shi [61] and Robinson [59]. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inequalities. Noor [37] have proved that quasi variational inequalities are equivalent to the Wiener-Hopf equations. This equivalence has been used to study the existence and stability of the solution of variational inequalities. Noor et al [40] have been shown that the dynamical system can be used to suggest some implicit iterative method for solving variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [14]. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. This dynamical system is a first order initial value problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It has been shown [19, 26, 41, 50, 54, 66, 67] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems.

We would like to mention that the sensitivity analysis provides useful information for designing or planning various equilibrium systems. Sensitivity analysis can provide new insight and can stimulate new ideas and techniques for problem solving. Dafermos [12] studied the sensitivity analysis of the variational inequalities using the fixed point technique. This approach has strong geometrical flavour and has been investigated for various classes of quasi variational inequalities. For example, see [37,41,48,49,54–56] and the references therein.

Motivated and inspired by ongoing recent research in variational inequalities, we revisit the general quasi variational inequalities involving two operators, which was introduced and studied by Noor [34] in 1988. Noor [34] established the equivalence between the quasi variational inequalities and fixed point problem which was used to consider an iterative method for solving quasi variational inequalities. We prove that the nonlinear programming problems and implicit second order obstacle boundary value problems can be studied via the general quasi variational inequalities. Several special cases are discussed as applications of the quasi variational inequalities, which is discussed in Section 2. In section 3, we discuss the unique existence of the solution as well as to suggest several inertial iterative method along with the convergence analysis. The Wiener-Hopf equation technique is used to suggest some iterative methods, which is considered in Section 4. We also apply the auxiliary principle technique involving an arbitrary operator is used to discuss some iterative schemes for solving the general quasi variational inequalities. Dynamical system approach is applied to study the stability of the solution and to suggest some iterative methods for solving the general quasi variational inequalities exploring the finite difference idea. Parametric quasi variational inequalities are considered in Section 7 to investigate the sensitivity analysis. One of the main purposes of this expository paper is to demonstrate the close connection among various classes of algorithms for the solution of the general variational inequalities and to point out that researchers in different field of variational inequalities and optimization have been considering parallel paths. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovate and potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this fast growing field. The interested reader is advised to explore this field further and discover novel and fascinating applications of general quasi variational inequalities in other areas of sciences such as machine learning, artificial intelligence, data analysis, fuzzy systems, random stochastic, financial analysis and related other optimization problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2. Formulations and basic facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. First of all, we recall some concepts from convex analysis [9,27,44], which are needed in the derivation of the main results.

Definition 2.1. *A set Ω in $\mathcal{H}$ is said to be a convex set, if*

$$\mu + \lambda(\nu - \mu) \in \Omega, \qquad \forall \mu, \nu \in \Omega, \lambda \in [0,1].$$

Definition 2.2. *A function Φ is said to be a convex function, if*

$$\Phi((1-\lambda)\mu + \lambda\nu) \leq (1-\lambda)\Phi(\mu) + \lambda\Phi(\nu), \quad \forall \mu, \nu \in \Omega, \quad \lambda \in [0,1].$$

For $\lambda = \frac{1}{2}$, the convex function reduces to:

$$\Phi\Big(\frac{\mu+\nu}{2}\Big) \leq \frac{1}{2}\{\Phi(\mu)+\Phi(\nu)\}, \quad \forall \mu,\nu \in \Omega,$$

which is known as the mid-convex (Jensen-convex) function. It is known that, if the function is continuous on the interior of the convex set, then convex function and mid-convex are equivalent. Convex functions are closely related to the integral inequalities and variational inequalities. These type of inequalities have played crucial part in developing fields such as: numerical analysis, operations research, transportation, financial mathematics, structural analysis, dynamical systems, sensitivity analysis, etc.

It is well known that a function Φ is a convex functions, if and only if, it satisfies the inequality

$$\Phi\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b \Phi(x)dx \leq \frac{\Phi(a)+\Phi(b)}{2}, \quad \forall a,b \in I=[a,b],$$

which is known as the Hermite-Hadamard type inequality. Such type of the inequalities provide us with the upper and lower bounds for the mean value integral.

If the convex function Φ is differentiable, then $\mu \in \Omega$ is the minimum of the function Φ, if and only if, $\mu \in \Omega$ satisfies the inequality

$$\langle \Phi'(\mu), \nu-\mu\rangle \geq 0, \quad \forall \nu \in \Omega. \tag{2.1}$$

The inequalities of the type (2.1) are called the variational inequalities, which were introduced and studied by Lions and Stampacchia [23]. It is known that the problem (2.1) occurs, which is may not be derivative of the differentiable functions. These facts and observations motivated to consider more general variational inequalities of which (2.1) is a special case. To be more precise, for given nonlinear operator $\mathcal{T}: \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu-\mu\rangle \geq 0, \quad \forall \nu \in \Omega. \tag{2.2}$$

which is called the variational inequality. Note that, for $\Phi'(\mu) = \mathcal{T}\mu$, problem (2.2) is exactly the problem (2.1).

For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [6,7,12–17,20,23,25,26,28–30,32–38,40–46,48–56,58,61,63] and the references therein.

We recall the concept of the general functions involving an arbitrary functions, which are mainly due to Noor [44]. For the sake of completeness and to convey an idea of this result, we include some details.

Definition 2.3. *[44] A set $\Omega_g \subseteq \mathcal{H}$ is said to be a general convex set, if there exists an arbitrary function $g: \mathcal{H} \longrightarrow \mathcal{H}$ such that*

$$g(\mu)+t(\nu-g(\mu)) \in \Omega_g, \quad \forall \mu,\nu \in \Omega_g. \quad t \in [0,1].$$

Note that every convex set is general convex set, but the converse is not true, see Noor [44]. It is worth mentioning that the general convex (g-convex) set is different than the E-convex set of Youness [68] and various general convex sets [10,11]. For the applications of the general convex sets in information technology, railway systems, computer aided design, digital vector optimization and comparison with other concepts, see [9–11]. If $g = I$, then the general convex set Ω_g is exactly the convex set Ω.

Definition 2.4. *The function $\Phi : \Omega_g \longrightarrow \mathcal{H}$ is said to be general convex, if there exists an arbitrary function g, such that*

$$\Phi(g(\mu) + t(\nu - g(\mu))) \leq \Phi(g(\mu)) + t\{\Phi(\nu - \Phi(g(\mu))\}, \quad \forall \mu, \nu \in \Omega_g, \quad t \in [0,1].$$

Clearly every convex function is a general convex, but the converse is not true.
For the differentiable general convex function, we have

$$\Phi(\nu) - \Phi(g(\mu)) \geq \langle \Phi'(g(\mu)), \nu - g(\mu) \rangle, \quad \mu, \nu \in \Omega_g.$$

Theorem 2.1. *[44] Let Φ be a differentiable general convex function on the general convex set Ω_g. Then the minimum $\mu \in \Omega_g$ of the function Φ, if and only if, $\mu \in \Omega_g$ satisfies the inequality*

$$\langle \Phi'(g(\mu), \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega_g, \tag{2.3}$$

where $\Phi'(.)$ is the differential of Φ at $\mu \in \Omega_g$ in the direction $\nu - g(\mu)$.

Proof. Let $\mu \in \Omega_g$ be a minimum of differentiable function Φ on the general convex set Ω_g. Then

$$\Phi(g(\mu)) \leq \Phi(\nu), \quad \forall \nu \in \Omega_g. \tag{2.4}$$

Since Ω_g is a general convex set, so, $\forall \mu, \nu \in\in \Omega_g, t \in [0,1]$,
$g(v_t) = g(\mu) + t(\nu - g(\mu) \in \Omega_g$. Setting $g(\nu) = g(v_t)$ in (2.4), we have

$$\Phi(g(\mu)) \leq \Phi(g(\mu) + t(\nu - g(\mu))),$$

which implies that

$$\Phi(g(\mu)) \quad \leq \quad \Phi(g(\mu)) + t(\nu - g(\mu))) \leq \Phi(g(\mu) + t(\nu - g(\mu))).$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle \Phi'(g(\mu)), \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega_g,$$

which is the required result(2.3).
Conversely, let $\mu \in \Omega_g$ satisfy the inequality (2.3). Since Φ is a general convex function, $\forall \mu, \nu \in \Omega_g,\ t \in [0,1],\ g(\mu) + t(\nu - g(\mu)) \in \Omega_g$. Thus

$$\Phi(g(\mu) + t(\nu - g(\mu)) \leq \Phi(g(\mu)) + t(\varphi(\nu) - \Phi(g(\mu)))$$

which implies that

$$\Phi(\nu) - \Phi(g(\mu)) \geq \langle F'(g(\mu)), \nu - g(\mu)\rangle \geq 0, \quad \text{using (2.3).}$$

Thus

$$\Phi(g(\mu)) - \Phi(\nu) \leq 0,$$

showing that $\mu \in \Omega_g$ is the minimum of Φ on the general convex set $\Phi_g \subseteq H$. □

Theorem 2.1 implies that general convex programming problem can be studied via the general variational inequality(2.3) .

It is known that the inequality of the type (2.3) may not arise as the optimality condition of the differentiable functions. These facts inspired to consider more general variational inequalities involving arbitrary operators.

For given operators $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$, Noor [34] introduced and investigated the problem of finding $\mu \in \Omega \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.5}$$

which is called the general variational inequalities, introduced and studied by Noor [34] in 1988. For the applications, motivations, numerical results, dynamical systems and related optimizations, see [46,56].

In many applications, the convex set Ω depends upon the solution explicitly or implicitly. In such cases, variational inequality is called the quasi variational inequality. Let $\Omega : \mathtt{H} \longrightarrow \mathtt{H}$ be a set-valued mapping which, for any element $\mu \in \mathtt{H}$, associates a convex-valued and closed set $\Omega(\mu) \subseteq \mathtt{H}$. We now consider some new classes of general quasi variational inequalities, which include several new and known classes of variational inequalities as special cases.

For given nonlinear operators $\mathcal{T}, g$, we consider the problem of finding $\mu \in \Omega(\mu)$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.6}$$

Or equivalently find $\mu \in \Omega(\mu)$, such that

$$\langle g(\mu), \nu - \mathcal{T}(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.7}$$

Note the symmetry role played by the mapping $\mathcal{T}$ and g. It is clear all the results, which hold for the problem (2.6), continue to hold for the problem (2.7) and viceversa.

If $\mathcal{T} = I$, the identity operator, then problem (2.6)is called the inverse quasi variational inequalities, see [13], that is, finding $\mu \in \Omega(\mu)$, such that

$$\langle \mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.8}$$

and for $g = I$, the problem (2.7) can be also viewed as the inverse quasi variational inequality.

$$\langle \mu, \nu - \mathcal{T}(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.9}$$

is also called the inverse quasi variational inequality. Consequently, it is evident that all the known results for quasi variational inequalities are also valid for both types of inverse quasi variational inequalities. This is a surprising fascinating fact.

Special Cases. We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2.6).

(I). This problem can be viewed as a problem of finding the minimum of general convex function,. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake.

(II). If $\Omega^*(\mu) = \{\mu \in \mathcal{H} : \langle \mu, \nu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\Omega(\mu)$ in $\mathcal{H}$, then problem (2.6) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \in \Omega(\mu), \quad \mathcal{T}\mu \in \Omega^*(\mu) \quad \text{and} \quad \langle \mathcal{T}\mu, g(\mu) \rangle = 0, \tag{2.10}$$

which is known as the general quasi complementarity problems.
For $\Omega(\mu) = \Omega$, the convex set, the problem (2.10) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad \mathcal{T}\mu \in \Omega^* \quad \text{and} \quad \langle \mathcal{T}\mu, g(\mu) \rangle = 0, \tag{2.11}$$

is called the general nonlinear complementarity problem [33]. Obviously general quasi complementarity problems include the general complementarity problems, nonlinear complementary problems and linear complementarity problems, which were introduced by Cottle et al. [8], Lemake [24], Noor [33, 41] and Noor et al. [47, 57] in game theory, management sciences and quadratic programming as special cases. This inter relations among these problems have played a major role in developing numerical results for these problems and their applications.

(III). For $\mathcal{T} = I$, the identity operator, the problem (2.6)reduces to finding $\mu \in \Omega$ such that

$$\langle \mu, \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.12}$$

is called the inverse variational inequality, which is being investigated extensively in recent years. For example, see Dev et al. [13].

(IV). If $g = I$, then (2.6) collapses to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.13}$$

which is called quasi variational inequality, introduced by Bensoussan and Lions [6] in the impulse control theory. For the numerical analysis, sensitivity analysis, dynamical systems and other aspects of quasi variational inequalities and related optimization programming problems. see [6, 7, 17, 19, 25, 30, 34, 37, 43, 46, 48–52, 56, 57, 62] and the references therein.

(V). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathcal{H}$, then problem (2.13) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.14}$$

is known as the variational inequality, which is mainly due to Lions and Stampacchia [12].

Remark 2.1. *It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}$, g, set-valued convex set $\Omega(\mu)$ and the spaces, one can obtain several classes of variational inequalities, complementarity problems and optimization problems as special cases of the nonlinear quasi-variational inequalities (2.6). This shows that the problem (2.6) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the general quasi-variational inequalities and their variants.*

Example 2.1. *To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding μ such that*

$$\left.\begin{array}{ll} -\mu'' \geq \phi(x) & \text{on } \Omega_1 = [a,b] \\ \mu \geq \mathcal{M}(\mu) & \text{on } \Omega_1 = [a,b] \\ [\mu'' + \phi(x)][\mu - \mathcal{M}(\mu)] = 0 & \text{on } \Omega_1 = [a,b] \\ \mu(a) = 0, \quad \mu(b) = 0. & \end{array}\right\} \tag{2.15}$$

where $\phi(x)$ is a continuous function and $\mathcal{M}(\mu)$ is the cost (obstacle) function. The prototype encountered is

$$\mathcal{M}(\mu) = \eta + \inf_i \{\mu^i\}. \tag{2.16}$$

In (2.16), η represents the switching cost. It is positive, when the unit is turned on and equal to zero when the unit is turned off. The operator $\mathcal{M}$ provides the coupling between the unknowns $\mu = (\mu^1, \mu^2, \ldots, \mu^i)$. We study the problem (2.15) in the framework of quasi variational inequality approach. To do so, we first define the set as

$$\Omega(\mu) = \{\nu : \nu \in \mathcal{H}_0^1(\Omega_1) : \nu \geq \mathcal{M}(\mu), \quad \text{on } \Omega_1\},$$

which is a closed convex-valued set in $\mathcal{H}_0^1(\Omega)$, where $\mathcal{H}_0^1(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (2.15) is

$$\begin{aligned} \mathcal{I}[\nu] &= -\int_a^b \left(\frac{d^2\nu}{dx^2}\right)\nu dx - 2\int_a^b \phi(x)\nu dx, \quad \forall \nu \in \Omega(\mu) \\ &= \int_a^b \left(\frac{d\nu}{dx}\right)^2 dx - 2\int_a^b \phi(x)\nu dx \\ &= \langle \mathcal{T}\nu, \nu \rangle - 2\langle \phi(x), \nu \rangle, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \langle \mathcal{T}\mu, \nu \rangle &= -\int_a^b \left(\frac{d^2\mu}{dx^2}\right)\nu dx = \int_a^b \frac{d\mu}{dx}\frac{d\nu}{dx} dx \qquad (2.18) \\ \phi(\nu) = \langle \phi, \nu \rangle &= \int_a^b \phi(x)\nu dx. \end{aligned}$$

It is clear that the operator $\mathcal{T}$ defined by (2.18) is linear, symmetric and positive. Using the technique of Noor [41] and Noor et al. [49], one can show that the minimum of the functional $\mathcal{I}[\nu]$ defined by (2.17) associated with the problem (2.15) on the closed convex-valued set $\Omega(\mu)$ can be characterized by the inequality of type

$$\langle \mathcal{T}\mu, \nu-\mu\rangle \geq \langle \phi, \nu-\mu\rangle, \quad \forall \nu \in \Omega(\mu), \tag{2.19}$$

which is exactly the quasi variational inequality (2.13).

We also need the following result, known as the projection Lemma(best approximation), which plays a crucial part in establishing the equivalence between the general quasi variational inequalities and the fixed point problems. This result is used in the analyzing the convergence analysis of the implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.1. *[20,49] Let $\Omega(\mu)$ be a closed and convex-valued set in $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega(\mu)$ satisfies the inequality*

$$\langle \mu - z, \nu-\mu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.20}$$

if and only if,

$$\mu = \Pi_{\Omega(\mu)}(z),$$

where $\Pi_{\Omega(\mu)}$ is implicit projection of $\mathcal{H}$ onto the closed convex-valued set $\Omega(\mu)$.

It is well known that the implicit projection operator $\Pi_{\Omega(\mu)}$ is not nonexpansive, but it is required to satisfy the following assumption, which plays an important part in the derivation of the results..

Assumption 2.1.

$$\|\Pi_{\Omega(\mu)}\omega - \Pi_{\Omega(\nu)}\omega\| \leq \eta\|\mu-\nu\|, \forall \mu, \nu, \omega \in \mathcal{H}, \tag{2.21}$$

where $\eta > 0$ is a constant.

Assumption 2.1 has been used to prove the existence of a solution of general quasi variational inequalities as well as in analyzing convergence of the iterative methods.

In many important applications, the convex-valued set $\Omega(\mu)$ can be written as

$$\Omega(\mu) = m(\mu) + \Omega,$$

is known as the moving convex set,where $m(\mu)$ is a point-point mapping and Ω is a convex set. In this case, we have

$$\Pi_{\Omega(\mu)}\omega = \Pi_{m(\mu)+\Omega} = m(\mu) + \Pi_{\Omega}[w - m(\mu)], \quad \forall \mu, w \in \Omega.$$

We note that, if $m(\mu)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then

$$\begin{aligned}\|\Pi_{\Omega(\mu)}w - \Pi_{\Omega(\nu)}w\| &= \|m(\mu) - m(\nu) + \Pi_{\Omega}[w - m(\mu)] - \Pi_{\Omega}[w - m(\nu)\| \\ &\leq 2\|m(\mu) - m(\nu)\| \leq 2\gamma\|\mu-\nu\|, \quad \forall \mu, \nu, w \in \Omega.\end{aligned}$$

which shows that Assumption 2.1 holds with $\eta = 2\gamma$.

Definition 2.5. *An operator $\mathcal{T}:\mathcal{H}\rightarrow\mathcal{H}$ is said to be:*

(1) *Strongly monotone, if there exist a constant $\alpha>0$, such that*

$$\langle\mathcal{T}\mu-\mathcal{T}\nu,\mu-\nu\rangle\geq\alpha\|\mu-\nu\|^2,\quad\forall\mu,\nu\in\mathcal{H}.$$

(2) *Lipschitz continuous, if there exist a constant $\beta>0$, such that*

$$\|\mathcal{T}\mu-\mathcal{T}\nu\|\leq\beta\|\mu-\nu\|,\quad\forall\mu,\nu\in\mathcal{H}.$$

(3) *Monotone, if*

$$\langle\mathcal{T}\mu-\mathcal{T}\nu,\mu-\nu\rangle\geq 0,\quad\forall\mu,\nu\in\mathcal{H}.$$

(4) *Pseudo monotone, if*

$$\langle\mathcal{T}\mu,\nu-\mu\rangle\geq 0\quad\Rightarrow\quad\langle\mathcal{T}\nu,\nu-\mu\rangle\geq 0,\quad\forall\mu,\nu\in\mathcal{H}.$$

Remark 2.2. *Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.*

3. Projection method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the general quasi variational inequalities.

Using Lemma 2.1, one can show that the general quasi variational inequalities are equivalent to the fixed point problems.

Lemma 3.1. [46,49] *The function $\mu\in\Omega(\mu)$ is a solution of the general quasi variational inequality (2.6), if and only if, $\mu\in\Omega(\mu)$ satisfies the relation*

$$g(\mu)=\Pi_{\Omega(\mu)}[g(\mu)-\rho T\mu],\tag{3.1}$$

where $\Pi_{\Omega(\mu)}$ is the projection operator and $\rho>0$ is a constant.

Proof. Let $u\in\Omega(\mu)$ be the problem (2.6). Then

$$\langle\rho T\mu+g(\mu)-g(\mu),\nu-g(\mu)\rangle\geq 0,\quad\forall\nu\in\Omega_\mu.$$

Using Lemma 2.1, we have

$$g(\mu)=\Pi_{\Omega(\mu)}[g(\mu)-\rho T\mu],$$

the required result. □

Lemma 3.1 implies that the general quasi variational inequality (2.6) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation (3.1) will play an important role in deriving the main results.

From the equation (3.1), we have

$$u=u-g(\mu)+\Pi_{\Omega(\mu)}[g(\mu)-\rho T\mu].$$

We define the function F associated with (3.1) as

$$F(\mu) = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho T\mu], \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.6), it is enough to show that the map F defined by (3.2) has a fixed point.

Theorem 3.1. *Let the operators $\mathcal{T}, g$ be strongly monotone with constants $\alpha > 0.\sigma > 0$ and Lipschitz continuous with constants $\beta > 0, \zeta > 0$, respectively. If the Assumption 2.1 holds and there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-k}{\beta}, \tag{3.3}$$

where

$$\begin{aligned} \theta &= \rho\beta + k \\ k &= \sqrt{1 - 2\sigma + \zeta^2} + \zeta + \eta. \end{aligned}$$

then there exists a unique solution of the problem (2.6).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.6) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (3.2) has a fixed point.
For all $\nu \neq \mu \in \Omega(\mu)$, we have

$$\begin{aligned} \|F(\mu) - F(\nu)\| &= \|\mu - \nu - (h(\mu) - g(\nu))\| + \Pi_{\Omega(\mu)}\|[g(\mu) - \rho T\mu] - \Pi_{\Omega(\nu)}[g(\nu) - \rho Tv]\| \\ &= \|\nu - \mu - (g(\nu) - g(\mu))\| + \|\Pi_{\Omega(\mu)}[g(\nu) - \rho\mathcal{T}\nu] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}\nu]\| \\ &+ \|\Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}\nu] - \Pi_{\Omega(\mu)}[g(\mu) - \rho(\mathcal{T}\mu]\| \\ &\leq \|\mu - \nu - (g(\mu) - g(\nu))\| + \eta\|\nu - \mu\| + \|g(\nu) - g(\mu) - \rho(T\nu - T\mu)\| \\ &\leq \|\mu - \nu - (g(\mu) - g(\nu))\| + \eta\|\nu - \mu\| + \|g(\nu) - g(\mu)\| + \rho\|T\nu - T\mu\| \\ &\leq \|\mu - \nu - (g(\mu) - g(\nu))\| + \eta\|\mu - \nu\| + \zeta\|\mu - \nu\| + \rho\beta\|\mu - \nu\|. \end{aligned} \tag{3.4}$$

where we have used the fact that the operators $\mathcal{T}, g$ is are Lipschitz continuous operator with constants $\beta > 0, \zeta > 0$, respectively
Since the operator g is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned} \|\mu - \nu - (g(\mu) - g(\nu))\|^2 &\leq \|\mu - \nu\|^2 - 2\langle g(\mu) - g(\nu), \mu - \nu\rangle + \zeta^2\|g(\mu) - g(\nu)\|^2 \\ &\leq (1 - 2\sigma + \zeta^2)\|\mu - \nu\|^2. \end{aligned} \tag{3.5}$$

From (4.5)and (3.5), we have

$$\begin{aligned} \|F(\mu) - F(\nu)\| &\leq \left\{\sqrt{(1 - 2\sigma + \zeta^2)} + \zeta + \eta + \rho\beta\right\}\|\mu - \nu\| \\ &= \theta\|\mu - \nu\|, \end{aligned}$$

where

$$\theta = \rho\beta + k \tag{3.6}$$

$$k = \sqrt{1-2\sigma+\zeta^2} + \zeta + \eta. \tag{3.7}$$

From (3.3), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (3.2) has a fixed point, which is the unique solution of (2.6). □

The fixed point formulation (3.1) is applied to propose and suggest the iterative methods for solving the problem (2.6).

This alternative equivalent formulation (3.1) is used to suggest the following iterative methods for solving the problem (2.6).

Algorithm 3.1. *For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solutions $\{\mu_n\}$, $\{w_n\}$ and $\{y_n\}$ by the iterative schemes*

$$\begin{aligned} g(y_n) &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ g(w_n) &= \Pi_{\Omega(y_n)}[g(y_n) - \rho\mathcal{T}y_n] \\ g(\mu_{n+1}) &= \Pi_{\Omega(w_n)}[g(w_n) - \rho\mathcal{T}w_n]. \end{aligned}$$

Algorithm 3.1 is a three step forward-backward splitting algorithm for solving general quasi variational inequality (2.6). This method is very much similar to that of Glowinski et al. [11] for variational inequalities, which they suggested by using the Lagrangian technique.

We now suggested and analyzed the three step scheme for solving the general quasi variational inequality (2.6).

Algorithm 3.2. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$y_n = (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n]\} \tag{3.8}$$

$$w_n = (1-\beta_n)\mu_n + \beta_n\{y_n - g(y_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho\mathcal{T}y_n]\} \tag{3.9}$$

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \Pi_{\Omega(w_n)}[g(w_n) - \rho\mathcal{T}w_n]\}. \tag{3.10}$$

For $\gamma_n = 0$, Algorithm 3.2 reduces to:

Algorithm 3.3. *For a given $\mu_0 \in \Omega(\mu)$, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} w_n &= (1-\beta_n)\mu_n + \beta_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n]\} \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \Pi_{\Omega(\mu_n)}[g(w_n) - \rho\mathcal{T}w_n\}, \end{aligned}$$

which is known as the Ishikawa iterative scheme for the problem (2.6).
Note that for $\gamma_n = 0$ *and* $\beta_n = 0$, Algorithm 3.1 is called the Mann iterative method, that is.

Algorithm 3.4. *For a given $\mu_0 \in \Omega(\mu)$, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\mu_{n+1} = (1-\beta_n)\mu_n + \beta_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n]\}.$$

We suggest another perturbed iterative scheme for solving the general quasi variational inequality (2.6).

Algorithm 3.5. *For a given $\mu_o \in \mathcal{H}$, compute the approximate solution $\{\mu_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho T\mu_n]\} + \gamma_n h_n \\ w_n &= (1-\beta_n)\mu_n + \beta_n\{y_n - g(y_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho\mathcal{T} y_n]\} + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \Pi_{\Omega(w_n)}[g(w_n) - \rho\mathcal{T} w_n\} + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and $\Pi_{\Omega(\mu_n)}$ is the corresponding perturbed projection operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \le \alpha_n, \beta_n, \gamma_n \le 1; \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving general variational inequality (2.6).
We now study the convergence analysis of Algorithm 3.2, which is the main motivation of our next result.

Theorem 3.2. *Let the operators $\mathcal{T}, g$ satisfy all the assumptions of Theorem 3.1. If the condition (3.3) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 3.2 converges to the exact solution $\mu \in \Omega(\mu)$ of the general quasi variational inequality (2.6) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.1, we see that there exists a unique solution $\mu \in \Omega(\mu)$ of the general quasi variational inequalities (2.6). Let $\mu \in \Omega(\mu)$ be the unique solution of (2.6). Then, using Lemma 3.1, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu + \alpha_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu]\} \quad (3.11) \\ &= (1-\beta_n)\mu + \beta_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu]\} \quad (3.12) \\ &= (1-\gamma_n)\mu + \gamma_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu]\}. \quad (3.13) \end{aligned}$$

From (3.10),(3.11) and Assumption (2.1), we have

$$\begin{aligned} \|\mu_{n+1} - \mu\| &= \|(1-\alpha_n)(\mu_n - \mu) + \alpha_n(w_n - \mu - (g(w_n) - g(\mu))) \\ &+ \alpha_n\Pi_{\Omega((w_n)}[g(w_n) - \rho\mathcal{T} w_n] - \Pi_{(\mu)}[g(\mu) - \rho\mathcal{T}\mu\}\| \\ &\le (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|w_n - \mu - (g(w_n) - g(\mu))\| \\ &+ \alpha_n\Pi_{\Omega((w_n)}[g(w_n) - \rho\mathcal{T} w_n] - \Pi_{\Omega(w_n)}[g(\mu_n) - \rho\mathcal{T}\mu\}\| \\ &+ \alpha_n\{\Pi_{(w_n)}[g(\mu_n) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu\}\| \\ &\le (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|w_n - \mu - (g(w_n) - g(\mu))\| \\ &+ \alpha_n\|g(w_n) - g(\mu) - \rho(\mathcal{T} w_n - \mathcal{T}\mu)\| + \alpha_n\eta\|w_n - \mu\| \end{aligned}$$

$$\begin{aligned} &\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n(k+\rho\beta)\|w_n-\mu\| \\ &= (1-\alpha_n)\|u_n-\mu\|+\alpha_n\theta\|w_n-\mu\|, \end{aligned} \tag{3.14}$$

where θ is defined by (3.6).

In a similar way, from (3.8) and (3.12), we have

$$\begin{aligned} \|w_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+2\beta_n\theta\|y_n-\mu-(g(y_n)-g(\mu))\| \\ &+ \beta_n\|y_n-\mu-\rho(Ty_n-T\mu)\|+\beta_n\eta\|y_n-\mu\| \\ &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n(k+\rho\beta)\|y_n-\mu\|, \\ &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|y_n-\mu\|, \end{aligned} \tag{3.15}$$

where θ is defined by (3.6).

From (3.8) and (3.13), we obtain

$$\begin{aligned} \|y_n-\mu\| &\leq (1-\gamma_n)\|\mu_n-\mu\|+\gamma_n\theta\|\mu_n-\mu\| \\ &\leq (1-(1-\theta)\gamma_n)\|\mu_n-\mu\| \\ &\leq \|\mu_n-\mu\|. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned} \|w_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|\mu_n-\mu\| \\ &= (1-(1-\theta)\beta_n)\|\mu_n-\mu\| \\ &\leq \|\mu_n-\mu\|. \end{aligned} \tag{3.17}$$

Form the above we equations, have

$$\begin{aligned} \|\mu_{n+1}-\mu\| &\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\theta\|\mu_n-\mu\| \\ &= [1-(1-\theta)\alpha_n]\|\mu_n-\mu\| \\ &\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0-\mu\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (3.16), and (3.17), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

Algorithm 3.6. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}\mu_n], \quad n=0,1,2,... \tag{3.18}$$

which is known as the projection method and has been studied extensively.

Algorithm 3.7. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_{n+1})}[g(\mu_n)-\rho\mathcal{T}\mu_{n+1}], \quad n=0,1,2,... \tag{3.19}$$

which is known as the implicit projection method and is equivalent to the following two-step method.

Algorithm 3.8. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n]\\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}[g(\mu_n) - \rho T\omega_n], \quad n = 0,1,2,...\end{aligned}$$

We also propose the following iterative method.

Algorithm 3.9. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1}], \quad n = 0,1,2,... \tag{3.20}$$

which is known as the modified projection method and is equivalent to the iterative method.

Algorithm 3.10. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}u_n]\\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}[g(\omega_n) - \rho\mathcal{T}\omega_n], \quad n = 0,1,2,...\end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2.6).
We can rewrite the equation (3.1) as:

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g\Big(\frac{\mu+\mu}{2}\Big) - \rho\mathcal{T}]. \tag{3.21}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 3.11. *[33]. For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g\Big(\frac{\mu_n+\mu_{n+1}}{2}\Big) - \rho\mathcal{T}\mu_{n+1}]. \tag{3.22}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (2.6).

Algorithm 3.12. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n]\\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}[g\Big(\frac{\omega_n+\mu_n}{2}\Big) - \rho\mathcal{T}\omega_n].\end{aligned}$$

From equation (3.1), we have

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{\mu+\mu}{2})]. \tag{3.23}$$

This fixed point formulation (3.23) is used to suggest the implicit method for solving the problem (2.6) as

Algorithm 3.13. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho\mathcal{T}(\frac{\mu_n+\mu_{n+1}}{2})]. \tag{3.24}$$

We can use the predictor-corrector technique to rewrite Algorithm 3.13 as:

Algorithm 3.14. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n], \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{\mu_n + \omega_n}{2})]. \end{aligned}$$

is known as the mid-point implicit method for solving the problem (2.6).

We again use the above fixed formulation to suggest the following implicit iterative method.

Algorithm 3.15. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2})]. \tag{3.25}$$

Using the predictor-corrector technique, Algorithm 3.15 can be written as:

Algorithm 3.16. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n], \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}[g(\omega_n) - \rho\mathcal{T}(\frac{\mu_n + \omega_n}{2})], \end{aligned}$$

which appears to be new one.

It is obvious that Algorithm 3.8 and Algorithm 3.9 have been suggested using different variant of the fixed point formulations (3.1). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the problem (2.6) and related optimization problems.

One can rewrite (3.1) as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\frac{\mu + \mu}{2}) - \rho\mathcal{T}(\frac{\mu + \mu}{2})]. \tag{3.26}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.6).

Algorithm 3.17. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2})]. \tag{3.27}$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 3.9 as the predictor and Algorithm 3.17 as corrector. Thus, we obtain a new two-step method for solving the problem (2.6).

Algorithm 3.18. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\left[\left(\frac{\omega_n + \mu_n}{2}\right) - \rho\mathcal{T}\left(\frac{\omega_n + \mu_n}{2}\right)\right], \end{aligned}$$

which is a new predictor-corrector two-step method.
For a parameter ξ, one can rewrite the (3.1) as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(u)}[g((1-\xi)\mu + \xi\mu) - \rho\mathcal{T}\mu].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (2.6).

Algorithm 3.19. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g((1-\xi)\mu_n + \xi\mu_{n-1}) - \rho\mathcal{T}\mu_n], \quad n = 0,1,2,....$$

It is noted that Algorithm 3.19 is equivalent to the following two-step method.

Algorithm 3.20. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\omega_n) - \rho\mathcal{T}\mu_n]. \end{aligned}$$

Algorithm 3.20 is known as the inertial projection method, which is mainly due to Noor [26] and Noor et al [35, 36].

Using this idea, we can suggest the following iterative methods for solving nonlinear quasi variational inequalities.

Algorithm 3.21. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\omega_n) - \rho\mathcal{T}\omega_n], \quad n = 0,1,2,.... \end{aligned}$$

Algorithm 3.22. *For a given $u_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\alpha)u_n + \alpha u_{n-1} \\ u_{n+1} &= \mu - g(\mu_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho T y_n)], \quad n = 0,1,2,.... \end{aligned}$$

We now suggest multi-step inertial methods for solving the general quasi variational inequalities (2.6).

Algorithm 3.23. *For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ y_n &= (1-\beta_n)\omega_n + \beta_n\Big\{\omega_n - g(\omega_n) + \Pi_{(\omega_n)}\Big[g(\frac{\omega_n + \mu_n}{2}) - \rho\mathcal{T}(\frac{\omega_n + \mu_n}{2})\Big]\Big\}, \\ \mu_{n+1} &= (1-\alpha_n)y_n + \alpha_n\Big\{y_n - g(y_n) + \Pi_{\Omega(\mathrm{y}_n)}\Big[g(\frac{\omega_n + \mathrm{y_n}}{2}) - \rho\mathcal{T}(\frac{\mathrm{y}_n + \omega_n}{2})\Big]\Big\}, \end{aligned}$$

where $\Theta_n \in [0,1], \forall n \geq 1$.

Algorithm 3.23 is a three-step modified inertial method for solving general quasi variational inclusion(2.6).
Similarly a four-step inertial method for solving the general quasi variational inequalities (2.6) is suggested.

Algorithm 3.24. *For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned}
\omega_n &= \mu_n - \Theta_n\left(\mu_n - \mu_{n-1}\right), \\
y_n &= (1-\gamma_n)\omega_n + \gamma_n\Big\{\omega_n - g(\omega_n) + \Pi_{(\omega_n)}\Big[g(\frac{\omega_n + \mu_n}{2}) - \rho\mathcal{T}(\frac{\omega_n + \mu_n}{2})\Big]\Big\}, \\
z_n &= (1-\beta_n)y_n + \beta_n\Big\{y_n - g(y_n) + \Pi_{\Omega(y_n)}\Big[g(\frac{y_n + \omega_n}{2}) - \rho\mathcal{T}(\frac{y_n + \omega_n}{2})\Big]\Big\}, \\
\mu_{n+1} &= (1-\alpha_n)z_n + \alpha_n\Big\{z_n - g(z_n) + \Pi_{\Omega(z_n)}\Big[g(\frac{z_n + y_n}{2}) - \rho\mathcal{T}(\frac{y_n + z_n}{2})\Big]\Big\},
\end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1$.

Using the technique of Noor et al. [52] and Jabeen et al [17], one can investigate the convergence analysis of these inertial projection methods. We would like to mention that Algorithm 3.23 and Algorithm 3.24 can be viewed as the generalizations of Noor (three-step) iterations [38] for solving the general quasi variational inequalities. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving system of quasi variational inequalities [18], which is an interesting problem.

4. Wiener-Hopf equations technique

In this section, we discuss the Wiener-Hopf equations associated with the quasi variational inequalities. It is worth mentioning that the Wiener-Hopf equations associated with variational inequalities were introduced and studied by Shi [61] and Ronbinson [59] independently using different techniques. Noor [37] proved that the quasi variational inequalities are equivalent to the implicit Wiener-Hopf equations.

We now consider the problem of solving the Wiener-Hopf equations related to the quasi variational inequalities. Let $\mathcal{T}$ be an operator and $\mathcal{R}_{\Omega(\mu)} = \mathcal{I} - \Pi_{\Omega(\mu)}$, where $\mathcal{I}$ is the identity operator and $\Pi_{\Omega(\mu)}$ is the projection operator.
We consider the problem of finding $z \in \mathcal{H}$ such that

$$\mathcal{T}\Pi_{\Omega(\mu)}z + \rho^{-1}\mathcal{R}_{\Omega(\mu)}z = 0. \tag{4.1}$$

The equations of the type (4.1) are called the implicit Wiener-Hopf equations. It have been shown that the implicit Wiener-Hopf equations play an important part in the developments of iterative methods, sensitivity analysis and other aspects of the variational inequalities.

Lemma 4.1. *The element $\mu \in \Omega(\mu)$ is a solution of the quasi variational inequality* (2.6), *if and only if, $z \in \mathcal{H}$ satisfies the resolvent equation* (4.1), *where*

$$g(\mu) = \Pi_{\Omega(\mu)} z, \tag{4.2}$$

$$z = g(\mu) - \rho \mathcal{T} \mu, \tag{4.3}$$

where $\rho > 0$ is a constant.

From Lemma 4.1, it follows that the general quasi variational inequalities (2.6) and the implicit Wiener-Hopf equations (4.1) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the strongly nonlinear quasi variational inequalities and related optimization problems.
We use the Wiener-Hopf equations (4.1) to suggest some new iterative methods for solving the nonlinear quasi variational inequalities. From (4.2) and (4.3),

$$\begin{aligned} z &= \Pi_{\Omega(\mu)} z - \rho \mathcal{T} \Pi_{\Omega(\mu)} z \\ &= \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T} \mu] - \rho \mathcal{T} \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T} \mu]. \end{aligned}$$

Thus, we have

$$g(\mu) = \rho T \mu + g(\mu) - \rho T g^{-1} \Pi_{\Omega(\mu_n)}[g(\mu) - \rho T \mu].$$

implies that

$$\rho T \mu - \rho T g^{-1} \Pi_{\Omega(\mu_n)}[g(\mu) - \rho T \mu] = 0.$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} \mu &= (1 - \alpha_n)\mu + \alpha_n \{\rho T g^{-1} \Pi_{\Omega(\mu_n)}[g(\mu) - \rho T \mu] - \rho \mathcal{T} \mu\} \\ &= (1 - \alpha_n)\mu + \alpha_n \Pi_{\Omega(\mu)}\{\rho \mathcal{T} \omega - \rho \mathcal{T} \mu\}, \end{aligned} \tag{4.4}$$

where

$$\omega = \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T} \mu]. \tag{4.5}$$

Using (4.4) and (4.5), we can suggest the following new predictor-corrector method for solving the quasi variational inequalities.

Algorithm 4.1. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} \mu_n] \\ \mu_{n+1} &= (1 - \alpha_n)\mu_n + \alpha_n \Pi_{(\omega_n)}\Big\{\rho \mathcal{T} \omega_n - \rho \mathcal{T} \mu_n\Big\}. \end{aligned}$$

If $\alpha_n = 1$, then Algorithm 4.1 reduces to

Algorithm 4.2. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho \mathcal{T} \mu_n] \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[\rho \mathcal{T} \omega_n - \rho \mathcal{T} \mu_n], \end{aligned}$$

which appears to be a new one.
In a similar way, we can suggest and analyse the predictor-corrector method for solving the quasi variational inequalities (2.6), which only involve only one projection.

Algorithm 4.3. *For given $u_0, u_1 \in \Omega(\mu)$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - \xi(\mu_n - \mu_{n-1}) \\ \mu_{n+1} &= \Pi_{\Omega(\mu_n)}[\rho\mathcal{T}\omega_n - \rho\mathcal{T}\mu_n]. \end{aligned}$$

One can study the convergence of the Algorithm 4.3 using the technique of Jabeen et al [9].

Remark 4.1. *We have only given some glimpse of the technique of the Wiener-Hopf equations for solving the quasi variational inequalities. One can explore the applications of the Wiener-Hopf equations in developing efficient numerical methods for variational inequalities and related nonlinear optimization problems.*

5. Auxiliary principle technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et al [15] as developed in [36,41,54,55,58], to suggest and analyze some proximal point methods for solving general quasi variational inequalities (2.6).
We apply the auxiliary principle technique involving an arbitrary operator, which is mainly due to Noor [36], for finding the approximate solution of the problem (2.6).

For a given $\mu \in \Omega(\mu)$ satisfying (2.6), find $w \in \Omega(\mu)$ such that

$$\langle \rho T(w + \eta(\mu - w)), \nu - g(w)\rangle + \langle M(w) - M(\mu), \nu - w\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5.1}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (8.4) is called the auxiliary general quasi variational inequality.
If $w = \mu$, then w is a solution of (2.6). This simple observation enables us to suggest the following iterative method for solving (2.6).

Algorithm 5.1. *For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho T(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1})\rangle \\ &\quad + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu). \end{aligned} \tag{5.2}$$

Algorithm 5.1 is called the hybrid proximal point algorithm for solving the general quasi variational inequalities (2.6).
Special Cases: We now discuss some special cases are discussed.
(I). For $\eta = 0$, Algorithm 5.1 reduces to

Algorithm 5.2. *For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho T\mu_{n+1}, \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5.3}$$

is called the implicit iterative methods for solving the problem (2.6).

(II). If $\eta = 1$, then Algorithm 5.1 collapses to

Algorithm 5.3. *For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho T\mu_n, \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

is called the explicit iterative method.

(III). For $\eta = \frac{1}{2}$, Algorithm 5.1 becomes:

Algorithm 5.4. *For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho T(\frac{\mu_{n+1} + \mu_n}{2}), \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

is known as the mid-point proximal method for solving the problem (2.6).

For the convergence analysis of Algorithm 5.2, we need the following concepts.

Definition 5.1. *An operator T is said to be pseudomontone with respect to the operator g if*

$$\langle T\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

implies that

$$-\langle Tv, g(\mu) - \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu).$$

Theorem 5.1. *Let the operator T be a pseudo-monotone with respect to the operators g. Let the approximate solution μ_{n+1} obtained in Algorithm 5.2 converges to the exact solution $\mu \in \Omega(\mu)$ of the problem (2.6). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu - \mu_n\|. \tag{5.4}$$

Proof. Let $\mu\Omega(\mu)$ be a solution of the problem (2.6). Then,

$$-\langle \rho(Tv, g(\mu) - \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5.5}$$

since the operator T is a pseudo-monotone with respect to the operator g.

Takin $v = \mu_{n+1}$ in (5.5), we obtain

$$-\langle \rho T\mu_{n+1}, g(\mu) - \mu_{n+1}\rangle \geq 0. \tag{5.6}$$

Setting $\nu = \mu$ in (8.5), we have

$$\langle \rho T\mu_{n+1}, g(\mu) - \mu_{n+1}\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \geq 0. \tag{5.7}$$

Combining (5.7), (5.6) and (5.5), we have

$$\langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \quad \geq \quad -\langle \rho T\mu_{n+1}, g(\mu) - u_{n+1}\rangle \geq 0. \tag{5.8}$$

From the equation (5.8), we have

$$\begin{aligned} 0 &\leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle \leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strongly monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \zeta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \zeta\|\mu_n - \mu\|,$$

the required result (5.4). □

Theorem 5.2. *Let H be a finite dimensional space and all the assumptions of Theorem 5.1 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 5.2 converges to the exact solution $\mu \in \Omega(\mu)$ of (2.6).*

Proof. Let $\mu \in \Omega(\mu)$ be a solution of (2.6). From (5.4), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi \sum_{n=0}^{\infty} \|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0. \tag{5.9}$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega(\mu)$. Replacing w_n by μ_{n_j} in (7.2), taking the limit $n_j \longrightarrow \infty$ and using (5.9), we have

$$\langle \rho T\hat{\mu}, \nu - g(\hat{\mu})\rangle \geq 0, \qquad \forall \nu \in \Omega(\mu),$$

which implies that $\hat{u}$ solves the problem (2.6) and

$$\|\mu_{n+1} - \mu\| \leq \|\mu_n - \mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (\mu_n) = \hat{\mu}.$$

the required result. □

In recent years inertial type iterative methods have been applied to find the approximate solutions of variational inequalities and related optimizations. We again apply the auxiliary approach to suggest some hybrid inertial proximal point schemes for solving the general Quasi variational inequalities.

For a given $\mu \in \Omega(\mu)$ satisfying (2.6), find $w \in \Omega(\mu)$ such that

$$\langle \rho T(w + \eta(\mu - w)), \nu - g(w) \rangle + \langle M(w) - M(\mu) + \alpha(\mu - \mu), \nu - w \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{5.10}$$

where $\rho > 0, \eta, \alpha \in [0, 1]$ are constants and M is a nonlinear operator.
Clearly $w = \mu$, implies that w is a solution of (2.6). This simple observation enables us to suggest the following iterative method for solving (2.6).

Algorithm 5.5. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho T(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1}) \rangle + \langle M(\mu_{n+1}) - M(\mu_n) + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1} \rangle \geq 0, \quad \forall \nu \in \Omega(\mu)$$

Algorithm 5.5 is called the hybrid proximal point algorithm for solving the general quasi variational inequalities (2.6). For $\alpha = 0$, Algorithm 5.5 is exactly Algorithm 5.1.
For $M = I$, Algorithm 5.5 reduces to the following method:

Algorithm 5.6. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho T(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1}) \rangle + \langle \mu_{n+1} - \mu_n + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1} \rangle \geq 0, \quad \forall \nu \in \Omega(\mu)$$

Remark 5.1. *For different and suitable choice of the parameters ρ, η, α, operators T, g, M and convex-valued sets, one can recover new and known iterative methods for solving quasi variational inequalities, complementarity problems and related optimization problems. Using the technique and ideas of Theorem 5.1 and Theorem 5.2, one can analyze the convergence of Algorithm 5.5 and its special cases.*

6. Dynamical systems technique

In this section, we consider the dynamical systems technique for solving quasi variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [14]. It is worth mentioning that the dynamical system is a first order initial value problem. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. For more details, see [13,14,41,50,54,66,67]. We consider some iterative methods for solving the quasi variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operators.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = g(\mu) - \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - g(\mu)\}. \tag{6.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(2.6), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{6.2}$$

We now consider a dynamical system associated with the general quasi variational inequalities. Using the equivalent formulation (3.1), we suggest a class of projection dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}u] - g(\mu)\}, \quad \mu(t_0) = \alpha, \tag{6.3}$$

where λ is a parameter. The system of type (6.15) is called the projection dynamical system associated with the problem (2.6). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2.6) can be studied.

Thus it is clear that $\mu \in \Omega(\mu)$ is a solution of the general quasi variational inequality (2.6), if and only if, $\mu \in \Omega(\mu)$ is an equilibrium point.

Definition 6.1. *[14] The dynamical system is said to converge to the solution set S^* of (6.15), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{6.4}$$

where

$$dist(\mu, S^*) = inf_{\nu\in S^*}\|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (6.4) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 6.2. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq u_1\|\mu(t_0) - \mu^*\|exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 6.1. *(Gronwall Lemma) [14] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t) exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now show that the trajectory of the solution of the projection dynamical system (6.15) converges to the unique solution of the general quasi variational inequality (2.6). The analysis is in the spirit of Noor [41] and Xia and Wang [66,67].

Theorem 6.1. *Let the operators $\mathcal{T}, g : H \longrightarrow H$ be Lipschitz continuous with constants $\beta > 0, \zeta > 0$. respectively. If $\rho < \frac{\delta}{(1+\delta)\zeta}$ and Assumption 2.1 then, for each $\mu_0 \in \Omega\mu$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (6.15) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - g(\mu)\}, \quad \forall \mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$, For $\forall \mu, \nu \in H$, we have

$$\begin{aligned}
\|G(\mu) - G(\nu)\| &\leq \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\nu)}[h(\nu) - \rho\mathcal{T}\nu]\| + \|g(\mu) - g(\nu)\|\} \\
&= \lambda\{\|g(\mu) - g(\nu)\| + \|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[g(\nu) - \rho\mathcal{T}\nu]\| \\
&\quad + \|\Pi_{\Omega(\mu)}[g(\nu) - \rho\mathcal{T}\nu] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}\nu]\|\} \\
&\leq \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \|g(\mu) - g(\nu) - \rho(\mathcal{T}\mu - \mathcal{T}\nu)\} \\
&\leq \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \{\|g(\mu) - g(\nu)\| + \rho\|\mathcal{T}(\mu) - \mathcal{T}(\nu)\|\} \\
&\leq \lambda\{(\eta + 2\zeta + \rho\beta)\}\|\mu - \nu\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(\eta + 2\zeta + \beta\rho)\} < 1$ and for each $\mu \in \Omega(\mu)$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (6.15), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider , for any $\mu \in \Omega(\mu)$,

$$\begin{aligned}
\|G(\mu)\| = \|\frac{d\mu}{dt}\| &= \lambda\|[g(u) - \rho\mathcal{T}\mu] - g(\mu)\| \\
&\leq \lambda\{\|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(\mu)\|\} \\
&\leq \lambda\{\delta\|\{g(\mu) - \rho T\mu\| + \|\Pi_{\Omega(\mu)}[g(\mu)] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(u)\|\} \\
&\leq \lambda\{(\rho\beta + 1 + 2\zeta)\|u\| + \|\Pi_{\Omega(\mu)}[0]\|\}.
\end{aligned}$$

Then

$$\begin{aligned} \|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\| ds \\ &\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\| ds, \end{aligned}$$

where $k_1 = \lambda \|\Pi_{\Omega(\mu)}[0]\|$ and $k_2 = \delta\lambda(\rho + 1 + 2\zeta)$. Hence by the Gronwall Lemma 6.1, we have

$$\|\mu(t)\| \leq \{\|u_0\| + k_1(t - t_0)\} e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 6.2. *If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\sigma > 0$ and $\zeta > 0$, then the dynamical system (6.15) converges globally exponentially to the unique solution of the general quasi variational inequality (2.6).*

Proof. Since the operator g is Lipschitz continuous, it follows from Theorem 6.1 that the dynamical system (6.15) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in H$. Let $\mu(t)$ be a solution of the initial value problem (6.15). For a given $\mu^* \in H$ satisfying (2.6), consider the Lyapunov function

$$L(\mu) = \lambda \|\mu(t) - \mu^*\|^2, \quad u(t) \in \Omega(\mu). \tag{6.5}$$

From (6.15) and (6.5), we have

$$\begin{aligned} \frac{dL}{dt} &= 2\lambda \langle \mu(t) - \mu^*, \frac{d\mu}{dt} \rangle \\ &= 2\lambda \langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - g(\mu(t)) \rangle \\ &= 2\lambda \langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - g(\mu^*) + g(\mu^*) - g(\mu(t)) \rangle \\ &= -2\lambda \langle \mu(t) - \mu^*, g(\mu(t)) - g(\mu^*) \rangle \\ &\quad +2\lambda \langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - g(\mu^*) \rangle \\ &\leq -2\lambda \langle \rho(\mathcal{T}\mu(t) - \mathcal{T}\mu^*), g(\mu(t)) - g(\mu^*) \rangle \\ &\quad +2\lambda \langle \mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mathcal{T}\mu^*(t)] \rangle, \\ &\leq -2\lambda\sigma \|\mu(t) - \mu^*\|^2 + \lambda \|\mu(t) - \mu^*\|^2 \\ &\quad +\lambda \|\Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}\mu(t)] - \Pi_{\Omega(\mu)}[h(\mu^*(t)) - \rho\mathcal{T}\mu^*(t)]\|^2 \end{aligned} \tag{6.6}$$

Using the Lipschitz continuity of the operators $\mathcal{T}, g$, we have

$$\begin{aligned} \|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[g(\mu^*) - \rho\mathcal{T}\mu^*]\| &\leq \delta \|g(\mu) - g(\mu^*) - \rho(\mathcal{T}\mu - \mathcal{T}\mu^*)\| \\ &\leq \delta(\zeta + \rho\beta) \|\mu - \mu^*\|. \end{aligned} \tag{6.7}$$

From (6.6) and (6.7), we have

$$\frac{d}{dt} \|\mu(t) - \mu^*\| \leq 2\xi\lambda \|\mu(t) - \mu^*\|,$$

where

$$\xi = (\delta(\zeta + \rho\beta) - 2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t) - \mu^*\| \leq \|\mu(t_0) - \mu^*\| e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (6.15) converges globally exponentially to the unique solution of the general quasi variational inequality (2.6). □

We use the projection dynamical system (6.15) to suggest some iterative for solving the quasi variational inequalities (2.6). These methods can be viewed in the sense of Korpelevich [21] and Noor [41] involving the double projection.

For simplicity, we take $\lambda = 1$. Thus the dynamical system(6.15) becomes

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega(\mu)}\Big[g(\mu) - \rho\mathcal{T}u\Big], \quad \mu(t_0) = \alpha. \tag{6.8}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (6), we have

$$\frac{\mu_{n+1} - \mu_n}{h_1} + g(\mu_n) = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}], \tag{6.9}$$

where h_1 is the step size.
Now, we can suggest the following implicit iterative method for solving the general quasi variational inequality (2.6).

Algorithm 6.1. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}\Big[g(\mu_n) - \rho\mathcal{T}\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h_1}\Big],$$

This is an implicit method. Algorithm 6.1 is equivalent to the following two-step method.

Algorithm 6.2. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\Big[g(\mu_n) - \rho\mathcal{T}\omega_n - \frac{\omega_n - \mu_n}{h_1}\Big], \end{aligned}$$

Discretizing (6), we now suggest an other implicit iterative method for solving (2.6).

$$\frac{\mu_{n+1} - \mu_n}{h_1} + g(\mu_n) = \Pi_{\Omega(g(\mu_{n+1}))}[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1}], \tag{6.10}$$

where h_1 is the step size.
This formulation enables us to suggest the two-step iterative method.

Algorithm 6.3. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\Big[g(\omega_n) - \rho\mathcal{T}\omega_n - \frac{\omega_n - \mu_n}{h_1}\Big]. \end{aligned}$$

Discretizing (6), we have

$$\frac{\mu_{n+1} - \mu_n}{h} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1}], \tag{6.11}$$

where h is the step size.
This helps us to suggest the following implicit iterative method for solving the problem (2.6).

Algorithm 6.4. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\Big[g(\omega_n) - \rho\mathcal{T}\omega_n\Big]. \end{aligned}$$

Discretizing (6), we propose another implicit iterative method.

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_{n+1})}[\mu_n - \rho\mathcal{T}\mu_{n+1}],$$

where h is the step size.

For $h = 1$, we can suggest an implicit iterative method for solving the problem (2.6).

Algorithm 6.5. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}].$$

Algorithm 6.5 is an implicit iterative method in the sense of Korpelevich.

From (6), we have

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega((1-\alpha)\mu+\alpha\mu)}[g((1-\alpha)\mu + \alpha\mu) - \rho\mathcal{T}((1-\alpha)\mu + \alpha\mu)], \tag{6.12}$$

where $\alpha \in [0, 1]$ is a constant.

Discretization (6) and taking $h = 1$, we have

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega((1-\alpha)\mu_n+\alpha\mu_{n-1})}\Big[g\Big((1-\alpha)\mu_n + \alpha\mu_{n-1}\Big) - \rho\mathcal{T}((1-\alpha)\mu_n + \alpha\mu_{n-1})\Big],$$

which is an inertial type iterative method for solving the general quasi variational inequality (2.6).
Using the predictor-corrector techniques, we have

Algorithm 6.6. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative schemes*

$$\begin{aligned}\omega_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1}\\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\omega_n}\Big[g(\omega_n) - \rho\mathcal{T}(\omega_n)\Big],\end{aligned}$$

which is known as the inertial two-step iterative method.

We now introduce the second order dynamical system associated with the variational inequality (2.6). To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\ddot{\mu} + \dot{\mu} = \lambda\{\Pi_{\Omega(mu)}[g(\mu) - \rho\mathcal{T}\mu] - g(\mu)\}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \tag{6.13}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (6.13) is indeed a second order boundary vale problem.

The equilibrium point of the dynamical system (6.13) is naturally defined as follows.

Definition 6.3. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (6.13), if,*

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the variational inequality (2.6), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
From (6.13), we have

$$g(\mu) = \Pi_{\Omega(\mu)}[\mu - \rho\mathcal{T}\mu].$$

Thus, we can rewrite (6.13) as follows:

$$g(\mu) = |Pi_{\Omega(\mu)}\Big[g(\mu) - \rho\mathcal{T}\mu + \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\Big]. \tag{6.14}$$

For $\lambda = 1$, the problem (6.13) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + g(\mu) = P_{\Omega(\mu)}\Big[g(\mu) - \rho\mathcal{T}\mu\Big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{6.15}$$

The problem (6.15) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various areas is fruitful from numerical analysis in developing implementable numerical methods for finding the approximate solutions of the variational inequalities. Consequently, we can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the general quasi variational inequalities and related optimization problems.
We discretize the second-order dynamical systems (6.15) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_n) = P_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}], \tag{6.16}$$

where h is the step size.
If $\gamma = 1, h = 1$, then, from equation(6.16) we have

Algorithm 6.7. *For a given $\mu_0 \in \mathtt{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n + g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}],$$

which is the the extragradient method for solving the general quasi variational inequalities. Algorithm 6.7 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 6.8. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[\mu_n - \rho\mathcal{T}y_n], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n \in [0,1]$ is a constant.
In a similar way, we have the following two-step method.

Algorithm 6.9. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(y_n)}[g(y_n - \rho\mathcal{T}y_n], \end{aligned}$$

which is also called the double projection method for solving the generale quasi variational inequalities (2.6).

We discretize the second-order dynamical systems (6.15) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_{n+1}) = \Phi_{\Omega(\mu_{n+1})}\Big[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}\Big],$$

where h is the step size.
Using this discrete form, we can suggest the following an iterative method for solving the variational inequalities (2.1).

Algorithm 6.10. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) + \Pi_{\Omega(\mu_n)}\Big[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1} - \gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h}\Big].$$

Algorithm 6.10 is called the hybrid inertial proximal method for solving the general quasi variational inequalities and related optimization problems. This is a new proposed method.
Note that, for $\gamma = 0$, Algorithm 6.10 reduces to the following iterative method.

Algorithm 6.11. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) + \Pi_{\Omega(\mu_{n+1})}\Big[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1} + \frac{\mu_n - \mu_{n-1}}{h}\Big],$$

which is called the inertial double projection method.

Remark 6.1. *For appropriate and suitable choice of the operators $\mathcal{T}$, g, convex-valued set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving general quasi variational inequalities and related optimization problems. Using the techniques and ideas of Noor et al [49,50], one can discuss the convergence analysis of the proposed methods.*

7. Sensitivity analysis

In this section, we study the sensitivity analysis of the general quasi variational inequalities, that is, examining how the solutions of such problems change when the data of the problems are changed. This is an important problems for several reasons.

We now consider the parametric versions of the problem (2.6). To formulate the problem, let M be an open subset of $\mathcal{H}$ in which the parameter λ takes values. Let $\mathcal{T}(\mu, \lambda)$ be given operator defined on $\mathcal{H} \times \mathcal{H} \times M$ and take value in $\mathcal{H} \times \mathcal{H}$. From now onward, we denote $\mathcal{T}_\lambda(.) \equiv \mathcal{T}(., \lambda)$ unless otherwise specified.

The parametric general variational inequality problem is to find $(\mu, \lambda) \in \mathcal{H} \times M$ such that

$$\langle \rho T_\lambda \mu + g(\mu) - g(\mu), \nu - g(\mu) \rangle \geq 0, \forall \nu \in \Omega(\mu). \tag{7.1}$$

We also assume that, for some $\overline{\lambda} \in M$, problem (7.1) has a unique solution $\overline{\mu}$. From Lemma 3.1, we see that the parametric general quasi variational inequalities are equivalent to the fixed point problem:

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T}_\lambda(\mu)],$$

or equivalently

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T}_\lambda(\mu)].$$

We now define the mapping F_λ associated with the problem (7.1) as

$$F_\lambda(\mu) = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T}_\lambda \mu], \quad \forall (\mu, \lambda) \in X \times M. \tag{7.2}$$

We use this equivalence to study the sensitivity analysis of the general quasi variational inequalities. We assume that for some $\overline{\lambda} \in M$, problem (7.1) has a solution $\overline{\mu}$ and X is a closure of a ball in $\mathcal{H}$ centered at $\overline{\mu}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (7.1) has a unique solution $z(\lambda)$ near $\overline{u}$ and the function $u(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 7.1. *Let $\mathcal{T}_\lambda(.)$ be an operator on $X \times M$. Then, the operator $\mathcal{T}_\lambda(.)$ is said to:*

(a) Locally strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu), \mu - \nu \rangle \geq \alpha \|\mu - \nu\|^2, \quad \forall \lambda \in M, \mu, \nu \in X.$$

(b) Locally Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|\mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu)\| \leq \beta \|\mu - \nu\|, \quad \forall \lambda \in M, \mu, \nu \in X.$$

We consider the case, when the solutions of the parametric general quasi variational inequality (7.1) lie in the interior of X. Following the ideas of Dafermos [12] and Noor [37,41], we consider the map $F_\lambda(\mu)$ as defined by (7.2). We have to show that the map $F_\lambda(\mu)$ has a fixed point, which is a solution of the parametric general quasi variational inequality (7.1). First of all, we prove that the map $F_\lambda(\mu)$, defined by (7.2), is a contraction map with respect to μ uniformly in $\lambda \in M$.

Lemma 7.1. *Let $\mathcal{T}_\lambda(.)$ be a locally strongly monotone with constant $\alpha > 0$ and locally Lipschitz continuous with constant $\beta > 0$. Let the operator g be strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constants $\zeta > 0$ respectively. If Assumption 2.1 holds and for all $\mu_1, \mu_2 \in X$ and $\lambda \in M$, we have*

$$\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| \leq \theta\|\mu_1 - \mu_2\|,$$

where

$$\theta = \left\{\sqrt{1-2\sigma+\zeta^2} + \eta + \zeta + \rho\beta\right\} = \{k + \rho\beta\} \tag{7.3}$$

for

$$\rho < \frac{1-k}{\beta}, \quad k < 1, \tag{7.4}$$

where

$$k = \sqrt{1-2\sigma+\zeta^2} + \zeta + \eta. \tag{7.5}$$

Proof. In order to prove the existence of a solution of (7.1), it is enough to show that the mapping $F_\lambda(\mu)$, defined by (7.2), is a contraction mapping.
For $\mu_1 \neq \mu_2 \in \mathcal{H}$, and using Assumption 2.1, we have

$$\begin{aligned}
\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \|\mu_1 - \mu_2 - (g(\mu_1) - g(\mu_2))\| \\
&\quad + \|\Pi_{\Omega(\mu_1)}[g(\mu_1) - \rho\mathcal{T}_\lambda\mu_1] - \Pi_{\Omega(u_2)}[g(\mu_2) - \rho\mathcal{T}_\lambda\mu_2]\| \\
&\quad + \|\Pi_{\Omega(u_1)}[g(\mu_1) - \rho\mathcal{T}_\lambda\mu_1] - \Pi_{\Omega(u_2)}[g(\mu_1) - \rho\mathcal{T}_\lambda\mu_1]\| \\
&\leq \|\mu_1 - \mu_2 - (g(\mu_1) - g(\mu_2))\| \\
&\quad + \eta\|\mu_1 - \mu_2\| + \|g(\mu_1) - g(\mu_2) - \rho(T_\lambda\mu_1 - T_\lambda\mu_2)\| \\
&\leq \|\mu_1 - \mu_2 - (g(\mu_1) - g(\mu_2))\| + \eta\|\mu_1 - \mu_2\| \\
&\quad + \|g(\mu_1) - g(\mu_2)\| + \rho\|(\mathcal{T}_\lambda\mu_1 - \mathcal{T}_\lambda\mu_2)\|.
\end{aligned} \tag{7.6}$$

Since the operator g is a strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu_1 - \mu_2 - (g(\mu_1) - g(\mu_2)\|^2 &\leq \|u_1 - u_2\|^2 - 2\langle g(\mu_1) - g(\mu_2), \mu_1 - \mu_2\rangle + \|g(\mu_1) - g(\mu_2)\|^2 \\
&\leq (1 - 2\sigma + \zeta^2)\|\mu_1 - \mu_2\|^2.
\end{aligned} \tag{7.7}$$

In a similar way, we have

$$\|(\mathcal{T}_\lambda\mu_1 - \mathcal{T}_\lambda\mu_2)\| \leq (1 - 2\sigma + \zeta^2)\|\mu_1 - \mu_2\|, \tag{7.8}$$

where we have used the fact that g is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$.

From (7.5), (7.6), (7.7) and (7.8), we have

$$\begin{aligned}\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \left\{\nu + \sqrt{(1-2\sigma+\delta^2)} + \sqrt{(1-2\rho\alpha+\rho^2\beta^2)}\right\}\|\mu_1-\mu_2\| \\ &= \left\{k + \sqrt{(1-2\rho\alpha+\rho^2\beta^2)}\right\}\|\mu_1-\mu_2\| \\ &= \theta\|\mu_1-\mu_2\|,\end{aligned}$$

where

$$\theta = k + \rho\beta.$$

From (7.4), it follows that $\theta < 1$. Thus it follows that the mapping $F_\lambda(\mu)$, defined by (7.2), is a contraction mapping and consequently it has a fixed point, which belongs to $\Omega(\mu)$ satisfying the general quasi variational inequality (7.1), the required result. □

Remark 7.1. *From Lemma 3.1, we see that the map $F_\lambda(\mu)$ defined by (7.2) has a unique fixed point $\mu(\lambda)$, that is, $\mu(\lambda) = F_\lambda(\mu)$. Also, by assumption, the function $\overline{\mu}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric general quasi variational inequality (7.1). Again using Lemma 3.1, we see that $\overline{\mu}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(\mu)$ and it is also a fixed point of $F_{\overline{\lambda}}(\mu)$. Consequently, we conclude that*

$$\mu(\overline{\lambda}) = \overline{\mu} = F_{\overline{\lambda}}(\mu(\overline{\lambda})).$$

Using Lemma 3.1, we can prove the continuity of the solution $\mu(\lambda)$ of the parametric general quasi variational inequality (7.1) using the technique of Noor [37, 41]. However, for the sake of completeness and to convey an idea of the techniques involved, we give its proof.

Lemma 7.2. *Assume that the operator $T_\lambda(.)$ is locally Lipschitz continuous with respect to the parameter λ. If the operator $T_\lambda(.)$ is Locally Lipschitz continuous and the map $\lambda \to P_{K_\lambda}u$ is continuous (or Lipschitz continuous), then the function $u(\lambda)$ satisfying (7.2) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. For all $\lambda \in M$, invoking Lemma 3.1 and the triangle inequality, we have

$$\begin{aligned}\|\mu(\lambda) - \mu(\bar{\lambda})\| &\leq \|F_\lambda(\mu(\lambda)) - F_{\bar{\lambda}}(\mu(\bar{\lambda})\| + \|F_\lambda(\mu(\bar{\lambda})) - F_{\bar{\lambda}}(\mu(\bar{\lambda}))\| \\ &\leq \theta\|\mu(\lambda) - \mu(\bar{\lambda})\| + \|F_\lambda(\mu(\bar{\lambda})) - F_{\bar{\lambda}}(\mu(\bar{\lambda}))\|. \qquad (7.9)\end{aligned}$$

From (7.2) and the fact that the operator T_λ is a Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned}\|F_\lambda(\mu(\bar{\lambda})) - F_{\bar{\lambda}}(\nu(\bar{\lambda}))\| &= \|\mu(\bar{\lambda}) - \mu(\bar{\lambda}) - (g(\mu(\bar{\lambda})) - g(\nu(\bar{\lambda})))\| \\ &\quad +\rho\|(T_\lambda(\mu(\bar{\lambda}), \mu(\bar{\lambda})) - T_{\bar{\lambda}}(\mu(\bar{\lambda}), \mu(\bar{\lambda})))\| \\ &\leq \rho\theta\|\lambda - \bar{\lambda}\| + \zeta\|\mu(\bar{\lambda}) - \mu(\bar{\lambda})\|. \qquad (7.10)\end{aligned}$$

, using the Iipschitz continuity of the operator g. Combining (7.9) and (7.10), we obtain

$$\|u(\lambda) - u(\bar{\lambda})\| \leq \frac{\rho\mu}{(1-\theta-\zeta)}\|\lambda - \bar{\lambda}\|, \quad \text{for all } \lambda, \bar{\lambda} \in M,$$

from which the required result follows. □

We now state and prove the main result of this paper and is the motivation our next result.

Theorem 7.1. *Let $\overline{\mu}$ be the solution of the parametric general quasi variational inequality (7.1) for $\lambda = \overline{\lambda}$. Let $T_\lambda(\mu)$ be the locally strongly monotone Lipschitz continuous operator for all $\mu, \nu \in X$. If the map $\lambda \to \Pi_{\Omega_\mu}$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric general quasi variational inequality (7.2) has a unique solution $\mu(\lambda)$ in the interior of $X, u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof.

Proof. Its proof follows from Lemma7.1, Lemma 7.2 and Remark 7.1. □

8. Generalizations and applications

In this section, we show that the quasi variational inequalities are equivalent to the strongly nonlinear general variational inequalities, see Noor [24].
In many applications, the convex-valued set $\Omega(\mu)$ is of the form:

$$\Omega(\mu) = m(\mu) + \Omega, \tag{8.1}$$

where Ω is a convex set and m is a point-to-point mapping.
Let $\mu \in \Omega(\mu)$ be a solution of the problem (2.6). Then from Lemma 3.1, it follows that $\mu \in \Omega(\mu)$ such that

$$g(\mu) = \Pi_{\Omega(\mu)}\Big[g(\mu) - \rho(\mathcal{T}\mu\Big]. \tag{8.2}$$

Combining (8.1) and (8.2), we obtain

$$\begin{aligned} g(\mu) &= \Pi_{\Omega(\eta(\mu)+\Omega)}\Big[g(\mu) - \rho\mathcal{T}\mu\Big] \\ &= m(\mu) + \Pi_{\Omega}\Big[g(\mu) - m(\mu) - \rho\mathcal{T}u\Big]. \end{aligned}$$

This implies that

$$G(\mu) = \Pi_{\Omega}\Big[G(\mu)\Big].$$

with $G(\mu) = g(\mu) - m(\mu)$, which is equivalent to finding $\mu \in g(\mu) \in \Omega$ such that

$$\langle(\mathcal{T}\mu, G(\nu) - G(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega. \tag{8.3}$$

The inequality of the type (8.3) is called the general variational inequality, investigated by Noor [33] in 1988. It have been shown that odd-order and nonsymmetric obstacle boundary value problems can be studied in the general variational inequalities. For more details, see [33,41,54,55]. Thus all the results proved for general quasi variational inequalities continue to hold for general variational inequalities (8.3) with suitable modifications and adjustment. Despite the research activates, very few results are available.

We would like to mention that some of the results obtained and presented in this paper can be extended for more multivalued general quasi variational inequalities. To be more precise, let $C(H)$ be a family of nonempty compact subsets of H. Let $T, V : H \longrightarrow C(H)$ be the multivalued operators. For a given nonlinear bifunction $N(.,.) : H \times H \longrightarrow H$ and operators $g, h : H \longrightarrow H$,consider the problem of finding $u \in \Omega(u), w \in T(u), y \in V(u)$ such that

$$\langle N(w,y), h(v) - g(u)\rangle \geq 0, \quad \forall \quad v \in \Omega(u), \tag{8.4}$$

which is called the multivalued general quasi variational inequality. We would like to mention that one can obtain various classes of general quasi variational inequalities for appropriate and suitable choices of the bifunction $N(.,.)$, the operators g, h, and convex-valued set $\Omega(u)$.
Note that, if $N(w, y) = Tu, \quad h = I$, then the problem (8.4) is equivalent to find $u \in \Omega(u)$, such that

$$\langle Tu, v - g(u) \geq 0 \quad \forall v \in \Omega(u),$$

which is exactly the general quasi variational inequality (2.6).
Using Lemma 3.1, one can prove that the problem (8.4) is equivalent to finding $u \in \Omega(u)$ such that

$$g(u) = \Pi_{\Omega(\mu)}[h(u) - \rho N(w, y)] \tag{8.5}$$

which can be written as

$$u = u - g(u) + \Pi_{\Omega(\mu)}[h(u) - \rho N(w, y)].$$

Thus one can consider the mapping F associated with the problem (8.4) as

$$F(u) = u - g(u) + \Pi_{\Omega(u)}[h(u) - \rho N(w, y)],$$

which can be used to discuss the uniqueness of the solution of the problem (8.4).
From (8.4) and (8.5, it follows that the multivalued general quasi variational inequalities are equivalent to the fixed problems. Consequently, all results obtained for the problem (2.6) continue to hold for the problem (8.4) with suitable modifications and adjustments. Applying the technique and idea of this paper, similar results can be established for solving system of quasi variational inequalities considered in [18] with appropriate modifications. The development of efficient implementable numerical methods for solving the multivalued general quasi variational inequalities and non optimization problems requires further efforts.

Conclusion. In this paper, we have used the equivalence between the general quasi variational inequalities and fixed point problems to suggest some new multi step multi-step iterative methods for solving the quasi variational inequalities. These new methods include extragradient methods, modified double projection methods and inertial type are suggested using the techniques of projection method, Wiener-Hopf equations and dynamical systems. Convergence analysis of the proposed method is discussed for monotone operators. It is an open problem to compare these proposed methods with other methods. Sensitivity analysis is also investigated for general quasi variational inequalities using the equivalent fixed point approach. Applying the technique and ideas of Ashish et. al. [3, 4], Cho et al. [7] and Kwuni et al. [22], can one explore the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. This is an open problem, which deserves further research efforts. We have shown that the general quasi variational inequalities are equivalent to the strongly general variational inequalities under suitable conditions of the convex-valued set. Applications of the fuzzy set theory [39], stochastic [5], quantum calculus, fractal, fractional and random traffic equilibrium [5] can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research and variational inequalities. One may explore these aspects of the general quasi variational inequality and its variant forms.

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Authors contributions: All authors contributed equally to the conception, design of the work, analysis, interpretation of data, reviewing it critically and final approval of the version for publication.

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