Int. J. Anal. Appl. (2024), 22:96

International Journal of Analysis and Applications

Functions in GTSs and GMSs

Yasser Farhat[1], **Vadakasi Subramanian**[2,*]

[1]*Academic Support Department, Abu Dhabi Polytechnic, P. O. Box 111499, Abu Dhabi, UAE*
[2]*Department of Mathematics, A.K.D. Dharma Raja Women's College, Rajapalayam, Tamil Nadu, India*

**Corresponding author:* vadakasivigneswaran@gmail.com

Abstract. In this article, we study the nature of different types of functions, namely, cliquish, lower semi-continuous, and upper semi-continuous functions in generalized G_δ-submaximal, generalized submaximal, and hyperconnected spaces. It also includes a cursory discussion about the properties for generalized G_δ-submaximal, generalized submaximal, and hyperconnected spaces in generalized metric spaces.

1. Introduction

In [7], generalized topological space were introduced by Császár. In topological and generalized topological spaces, different types of continuity were analyzed in [3]- [19]. In topological space, Baire spaces are characterized by using semi-continuous functions [12]. In continuation, cliquish functions have been analyzed in Baire space using sequences by Ewert [11], and the corresponding functions have been introduced by H. P. Thielman [22] whose importance is discussed in [10]- [20].

Using this aspects, Korczak - Kubiak, et. al, [16] redefined the spaces and defined two types of nowhere dense sets along with the lower and upper semi-continuous functions. Finally, they have carried out various properties for cliquish functions in Baire spaces. In [24, 25], we discuss some of the properties for nowhere dense and dense sets in both generalized and bigeneralized topological spaces.

Inspired by these last references, the topic is of impetus to contribute to an improvement of topological theory and authors have been motivated to discuss sections 3 & 4 with new results on generalized topological space. In Section 3, μ, η, ζ will denote generalized topologies.

Received: Apr. 18, 2024.

2020 *Mathematics Subject Classification.* 54A05, 54A10.

Key words and phrases. dense; codense; hyperconnected space; Submaximal space; lower (upper) semi-continuous function.

https://doi.org/10.28924/2291-8639-22-2024-96
ISSN: 2291-8639

© 2024 the author(s).

2. Preliminaries

First, we recall the well-known definitions namely, nowhere dense, dense, and codense sets in GTS.

Let μ be a family of subsets of a non-null set X. Then μ is said to be a *generalized topology* [7] in X if it contains the null set and is closed under arbitrary union so that (X, μ) is called as *generalized topological space* (GTS). The pair (X, μ) is called *strong generalized topological space* (sGTS) [7] if $X \in \mu$.

In [7], if $K \in \mu$, then K is called *μ-open* and if $X - K \in \mu$, then K is called *μ-closed*. And the *interior of L* [17] notated by iL, is the union of all μ-open sets contained in L; the *closure of L* notated by cL, is the intersection of all μ-closed sets containing L.

In [16],

$$\tilde{\mu} = \{Q \in \mu \mid Q \neq \emptyset\};$$

$$\mu(x) = \{Q \in \mu \mid x \in Q\}.$$

For $W \subseteq X$, the *subspace generalized topology* is defined by, $\mu_W = \{H \cap W \mid H \in \mu\}$. Then (W, μ_W) is called the *subspace GTS* [2]. Let $P \subset W$. Then *interior of P* is denoted by $i_W P$ and the *closure of P* is denoted by $c_W P$.

Definition 2.1. In [16], In a GTS (X, μ), a subset Q of X is said to be;
- *μ-nowhere dense* if $ic(Q) = \emptyset$.
- *μ-dense* if $c(Q) = X$.
- *μ-strongly nowhere dense* if for any $E \in \tilde{\mu}$, there is $F \in \tilde{\mu}$ such that $F \subset E$ and $F \cap Q = \emptyset$.
- *μ-codense* [17] if $c(X - Q) = X$.

Moreover,

$$\mathcal{N}(\mu) = \{K \subset X \mid K \text{ is a } \mu\text{-nowhere dense set}\}$$

[17] and we notated

$$\mathfrak{S}(\mu) = \{K \subset X \mid K \text{ is a } \mu\text{-strongly nowhere dense set}\}.$$

Definition 2.2. [16] A subset L of a GTS (X, μ) is said to be;
- *μ-meager* if $L = \bigcup_{n \in \mathbb{N}} L_n$ for each $L_n \in \mathcal{N}(\mu)$.
- *μ-s-meager* if $L = \bigcup_{n \in \mathbb{N}} L_n$ for each $L_n \in \mathfrak{S}(\mu)$ where $\mathbb{N}$ is the set of all natural numbers.

In [17],

$$\mathcal{M}(\eta) = \{D \subset X \mid D \text{ is } \eta\text{-meager}\}$$

and we notated

$$\mathfrak{M}(\eta) = \{K \subset X \mid K \text{ is } \mu\text{-s-meager}\}.$$

Definition 2.3. [16] In a generalized topological space (X, η), a subset H is called as;
- *η-second category* (η-II category) if $H \notin \mathcal{M}(\eta)$.
- *η-s-second category* (η-s-II category) if $H \notin \mathfrak{M}(\eta)$.

• *η-residual* if $X - H \in \mathcal{M}(\eta)$.
• *η-s-residual* if $X - H \in \mathfrak{M}(\eta)$.

Some other notations are defined by

$$\mathfrak{C}(\zeta) = \{K \subset X \mid K \text{ is of } \zeta\text{-II category}\}$$

and

$$\mathfrak{D}(\zeta) = \{H \subset X \mid H \text{ is of } \zeta\text{-s-II category}\}$$

ζ is a GT on X.

Also, Korczak - Kubiak, et. al, [16] defined two new branch of generalized topologies on (X, μ) defined by

$$\mu^* = \{\textstyle\bigcup_t (H_1^t \cap H_2^t \cap H_3^t \cap ... \cap H_{n_t}^t) \mid H_1^t, H_2^t, ..., H_{n_t}^t \in \mu\}$$

and

$$\mu^{**} = \{Q \subset X \mid Q \in \mathfrak{C}(\mu)\} \cup \{\emptyset\}.$$

Definition 2.4. [16] A space (X, η) is said to be a ;
• *weak Baire space* (wBS) if $\tilde{\eta} \subset \mathfrak{D}(\eta)$.
• *Baire space* (BS) if $\tilde{\eta} \subset \mathfrak{C}(\eta)$.
• *strong Baire Space* (sBS) if $F_1 \cap F_2 \cap ... \cap F_n \in \eta^{**}$ for all $F_1, F_2, ..., F_n \in \eta$ with $F_1 \cap F_2 \cap .... \cap F_n \neq \emptyset$.

In [17], Li et. al gave some other definitions for Baire space using dense sets. That is, (X, ζ) is said to be *Baire* if $c(\bigcap\limits_{n \in \mathbb{N}} H_n) = X$ where each $H_n \in \tilde{\zeta}; c(H_n) = X$.

In generally, [16],
• η^* is closed under finite intersection.
• $\eta \subset \eta^*$.
• $\eta^{**} \supset \eta$ if X is Baire.
• η^* is a topology if η is a sGT.

Definition 2.5. A generalized topological space (X, η) is called as ;
• *hyperconnected* space [10] if $\tilde{\eta} \subset \mathcal{D}(\eta)$.
• *generalized submaximal* [9] if $\mathcal{D}(\eta) \subset \eta$ where $\mathcal{D}(\eta) = \{J \subset X \mid c_\eta(J) = X\}$.

Definition 2.6. [16] A map $h : (X, \eta) \to \mathbb{R}$ is said to be ;
• *η-lower semi-continuous* at $p_0 \in X \Leftrightarrow$ for any $\beta \in \mathbb{R}; \beta < h(p_0)$, there is $K \in \eta(p_0)$ such that $h(K) \subset (\beta, \infty)$.
• *η-upper semi-continuous* at $p_0 \in X \Leftrightarrow$ for any $\beta \in \mathbb{R}; \beta > h(p_0))$, there is $K \in \eta(p_0)$ such that $h(K) \subset (-\infty, \beta)$ where $\mathbb{R}$ is the set of all real numbers.

Equivalently, a function $h : X \to \mathbb{R}$ is μ-lower semi-continuous (resp. μ-upper semi-continuous) $\Leftrightarrow h^{-1}((\beta, \infty)) \in \mu$ (resp. $h^{-1}((-\infty, \beta)) \in \mu$) for any $\beta \in \mathbb{R}$ [16].

Definition 2.7. [16] Let η, ξ be two GT in X. A function $g : X \to \mathbb{R}$ is said to be ;
• (η, ξ)-*lower semi-continuous* at $p_0 \in X$ if for any $\beta \in \mathbb{R}; \beta < g(p_0)$, there is $K \in \xi(p_0)$ being a η-residual set such that $g(K) \subset (\beta, \infty)$.
• (η, ξ)-*upper semi-continuous* at $p_0 \in X$ if for any $\beta \in \mathbb{R}; \beta > g(p_0)$, there is $K \in \xi(p_0)$ being a η-residual set such that $g(K) \subset (-\infty, \beta)$.

We introduce some new notations as follows;
• $\mathfrak{L}(\eta) = \{g \mid g$ is η-lower semi-continuous$\}$;
• $\mathfrak{L}(\eta, \zeta) = \{g \mid g$ is (η, ζ)-lower semi-continuous$\}$;
• $\mathfrak{U}(\eta) = \{g \mid g$ is η-upper semi-continuous$\}$;
• $\mathfrak{U}(\eta, \zeta) = \{g \mid g$ is (η, ζ)-upper semi-continuous$\}$ where g is a map from X to $\mathbb{R}$

In [16], (ζ, η)-l.(u.)s.c. $\Rightarrow$ η-l.(u.)s.c. and also,

$$\begin{array}{ccc} (\zeta^*, \zeta) - l.(u.)s.c. & \longrightarrow & \zeta - l.(u.)s.c. \\ & & \downarrow \\ & & \zeta^* - l.(u.)s.c. \end{array}$$

Now, $C_\eta(h)$ is the family of η-continuity points of $h : X \to \mathbb{R}$ and $\mathcal{D}_\eta(h)$ is collection of η-discontinuity points of h.

In [16], if $c_\eta(C_\zeta(g)) = X$, then $g : X \to \mathbb{R}$ is (η, ζ)-*cliquish*. Equivalently, g is (ζ, η)-cliquish [16], if (X, ζ) is Baire and $\mathcal{D}_\eta(g)$ is ζ-meager.

Lemma 2.1. [16] Let K and L be two subsets of a GTS (X, η) with $K \subset L$. Then
(a) If $L \in \mathcal{M}(\eta)$ (resp. $L \in \mathcal{N}(\eta)$), then $K \in \mathcal{M}(\eta)$ (resp. $K \in \mathcal{N}(\eta)$).
(b) If $K \in \mathfrak{C}(\eta)$, then $L \in \mathfrak{C}(\eta)$.

Lemma 2.2. [Lemma 2.3, [17]] Let W be a subset of a GTS (X, μ) and $Q \subset W$. Then $c_W Q = cQ \cap W$ where $c_W Q$ denote the closure of Q with respect to the subspace GTS (W, μ_W).

Lemma 2.3. [Lemma 3.2, [17]] Let (X, μ) be a GTS and Q, K be two subsets of X. If $K \in \tilde{\mu}$ and $K \cap Q = \emptyset$, then $K \cap cQ = \emptyset$.

Lemma 2.4. [Proposition 4.7, [17]] In a generalized topological space (X, μ), arbitrary union of a μ-meager set is μ-meager.

Lemma 2.5. [Theorem 3.11, [23]] In a GTS (X, η),
(a) $K \in \mathcal{N}(\eta^*) \Leftrightarrow K \in \mathfrak{S}(\eta^*)$.
(b) $K \in \mathcal{M}(\eta^*) \Leftrightarrow K \in \mathfrak{M}(\eta^*)$.
(c) K is η^*-residual $\Leftrightarrow$ K is η^*-s-residual.
(d) $K \in \mathfrak{C}(\eta^*) \Leftrightarrow K \in \mathfrak{D}(\eta^*)$.

Lemma 2.6. [Theorem 5.1, [23]] In a GTS (X, ζ), the followings are equivalent.
(a) X is a weak Baire Space.
(b) If Q is ζ-s-residual, then $Q \in \mathcal{D}(\zeta)$.
(c) If $P \in \mathfrak{M}(\zeta)$, then P is ζ-codense.

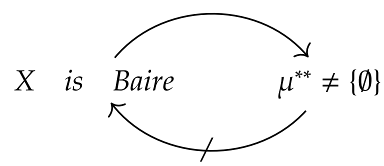
3. Subsets in generalized topological spaces

In this section, it is established that every μ-meager set is not μ-residual in a μ-II category GTS. Also, this section is devoted to the proof of the most important results concerning the generalized topology μ^{**}.

Observation 3.1 follows from the definition of a strongly nowhere dense set and the essay proof of which is omitted.

Observation 3.1. *In a GTS (X, μ), the only μ-open set which is μ-strongly nowhere dense is it the empty set.*

Theorem 3.1 and Example 3.1 are described in the below diagram.



Theorem 3.1. *Let (X, μ) be a GTS. Then*
(a) $\mu^{**} \neq \{\emptyset\} \Leftrightarrow (X, \mu^{**})$ *is a sGTS.*
(b) *If X is Baire, then $\mu^{**} \neq \{\emptyset\}$.*

Proof. (a) Suppose $\mu^{**} \neq \{\emptyset\}$. If $X \notin \mu^{**}$, then we get $X \in \mathcal{M}(\mu)$ so that every subset of X is μ-meager, by Lemma 2.1(a). For that, $\mu^{**} = \{\emptyset\}$ which is impossible. Therefore, $X \in \mathfrak{C}(\mu)$ so that μ^{**} is a sGT. The Converse part is obvious.
(b) Given X is Baire so it result that $\tilde{\mu} \subset \mu^{**}$ thus X is a μ-II category set, by Lemma 2.1(b). Therefore, $X \in \mathfrak{C}(\mu)$ and hence $\mu^{**} \neq \{\emptyset\}$.

□

The reverse implication of Theorem 3.1 (b) is not to be realistic as described in Example 3.1.

Example 3.1. Take $X = \{p, q, r, s\}$ and

$$\mu = \{\emptyset, \{p, s\}, \{q, r\}, \{p, q, r\}, \{q, r, s\}, X\}.$$

Here

$$\mu^{**} = \{\emptyset\} \cup \{P, K \subseteq X \mid \{q\} \subseteq P, \{r\} \subseteq K\}.$$

But $Q = \{p, s\} \in \tilde{\mu}$ is μ-meager which implies (X, μ) is not a BS.

Theorem 3.2. *If X is a μ-II category space, then the below results are true.*
(a) *If $Q \in \mathcal{M}(\mu)$, then Q is not μ-residual.*

(b) *Every μ-residual set is of μ-II category set.*
(c) *If $J \in \mathcal{N}(\mu)$, then $X - J \in \mathfrak{C}(\mu)$.*

Proof. (a) Let $Q \in \mathcal{M}(\mu)$. If Q is μ-residual, then $Q, X - Q \in \mathcal{M}(\mu)$. Also, $Q \cup (X - Q) = X$. By Lemma 2.4, $X \in \mathcal{M}(\mu)$ which is not possible. Therefore, Q is not a μ-residual set in X.

(b) If P is a μ-residual set, then $P \notin \mathcal{M}(\mu)$, by (a) so that $P \in \mathfrak{C}(\mu)$.

(c) Suppose $J \in \mathcal{N}(\mu)$ we get $J \in \mathcal{M}(\mu)$ it turns out $X - J$ is μ-residual and hence $X - J \in \mathfrak{C}(\mu)$, by (b).

□

Example 3.2 explains that the condition "X is of μ-II category" is necessary in Theorem 3.2.

Example 3.2. Take $X = [0, 3]$ and

$$\mu = \{\emptyset, [0, 2), (1, 3], [0, 1) \cup (2, 3], [0, 3]\}.$$

Here $X \notin \mathfrak{C}(\mu)$. Let $Q = [0, 2]$. Then $Q \in \mathcal{M}(\mu)$. Also, $X - Q = (2, 3] \in \mathcal{M}(\mu)$.

Example 3.3. *Consider the GTS $(\mathbb{R}, \eta)$. Then $\mathbb{R}$ is of η-II category if η is any one of the following GT.*
(a) *η is the co-singleton GT, that is, $\eta = \{\emptyset\} \cup \{K \subset \mathbb{R} \mid K - \{x\} \subset K$ for some $x \in \mathbb{R}\}$.*
(b) *η is the $\mathbb{Z}$ forbidden GT on $\mathbb{R}$, that is, $\eta = \{K \subset \mathbb{R} \mid K \subset \mathbb{R} - \mathbb{Z}\}$, $\mathbb{Z}$ is the set of all integers.*

Corollary 3.1. *Let E, F be two subsets of a η-II category space X with $F \subseteq E$. Then the following hold and also they are equivalent.*
(a) *If $E \in \mathcal{M}(\eta)$, then F is not η-residual.*
(b) *If F is η-residual, then $E \in \mathfrak{C}(\eta)$.*
(c) *If $E \in \mathcal{M}(\eta)$, then $X - F \in \mathfrak{C}(\eta)$.*

4. Generalized G_δ-submaximal spaces

Here, we discuss the significance of three kinds of functions namely, cliquish, lower semi-continuous and upper semi-continuous functions in generalized G_δ-submaximal and generalized submaximal spaces. Finally, we prove various properties of generalized G_δ-submaximal and generalized submaximal spaces in a generalized metric space.

Lemma 4.1. [1, Proposition 2.12] ζ is a sGT if (X, ζ) is generalized submaximal.

Lemma 4.2. [1, Lemma 3.7] ζ is a sGT if (X, ζ) is generalized G_δ-submaximal.

Observation 4.1. *Let (X, ζ) be a GTS. Then $X \in \zeta^*$ and hence ζ^* is a topology space for the case of anyone as follows as true.*
(a) *X is a generalized submaximal space.*
(b) *X is a generalized G_δ-submaximal space.*

X is generalized submaximal $\longrightarrow$ $X \in \mu^*$

$\uparrow$

X is generalized $G_\delta -$ submaximal

From the above Observation 4.1 we get the above diagram.

In a generalized G_δ-submaximal space, the below relations are true.

$$\begin{array}{ccc} \zeta - dense & \longrightarrow & \zeta - residual \\ & & \uparrow \\ & & \zeta^* - dense \end{array}$$

Diagram (a).

$$\begin{array}{ccc} \zeta^{**} \supset \mathcal{D}(\zeta) & \longleftarrow & X \in \mathfrak{C}(\zeta) \\ & & \downarrow \\ & & \zeta^{**} \supset \mathcal{D}(\zeta^*) \end{array}$$

Diagram (b).

Theorem 4.1 describes the above diagrams (a) & (b). Also, this theorem is an easy way to explore whether the given set is residual or not in a generalized G_δ-submaximal space.

Theorem 4.1. *Let (X, μ) be a generalized G_δ-submaximal space. Then*
(a) *If $K \subset X$ is μ-dense, then it is μ-residual.*
(b) *If $L \subset X$ is μ^*-dense, then it is μ-residual.*
(c) *If $X \in \mathfrak{C}(\mu)$, then $\mu^{**} \supset \mathcal{D}(\mu)$.*
(d) *$\mu^{**} \supset \mathcal{D}(\mu^*)$ if $X \in \mathfrak{C}(\mu)$.*

Proof. (a) If $K \in \mathcal{D}(\mu)$, then K is μ-G_δ-set, by hypothesis. Here $K = \bigcap_{n=1}^{\infty} K_n$ where $K_n \in \tilde{\mu}$ for every $n \in \mathbb{N}$. Since $K \subset K_n$ and $K \in \mathcal{D}(\mu)$ we have each $K_n \in \mathcal{D}(\mu)$ so it result that $i_\mu(X - K_n) = \emptyset$ and $X - K_n$ is μ-closed for each $n \in \mathbb{N}$, since each $K_n \in \tilde{\mu}$. For that, each $X - K_n \in \mathcal{N}(\mu)$. Hence $X - K \in \mathcal{M}(\mu)$ so that K is μ-residual.

(b) It is trivial.

(c) If $G \in \mathcal{D}(\mu)$, then G is μ-residual, by (a) and so $G \in \mu^{**}$, by Theorem 3.2(b).

(d) Follows from (a) and the fact that $\mu \subset \mu^*$.

□

Example 4.1. Take $X = [0, 6]$ and

$$\mu = \{\emptyset, [0, 2), (1, 3], [2, 3], [0, 3]\}.$$

Choose $Q = [1, 2]$ then we get $Q \in \mathcal{D}(\mu)$. But Q is not a G_δ-set it turns out (X, μ) is not generalized G_δ-submaximal.

(a) Choose $K = [0, 2]$ we have $K \in \mathcal{D}(\mu)$ and $X - K = (2, 6]$. Take $Q = [3, 6]$ then $Q \in \tilde{\mu}^{**}$. By Lemma 2.1(b), $X - K \notin \mathcal{M}(\mu)$. Thus, K is not μ-residual.

(b) Here

$$\mu^* = \{\emptyset, [0,2), (1,2), (1,3], [2,3], [0,3]\}.$$

Let $H = (1,2]$. Then $H \in \mathcal{D}(\mu^*)$ and $X - H = [0,1] \cup (2,6]$. Take $B = [3,6]$. Then $B \in \mathfrak{C}(\mu)$. By Lemma 2.1(b), $X - H \notin \mathcal{M}(\mu)$ so that H is not μ-residual.

Here X is of μ-II category. So we take

$$\mu = \{\emptyset, [0,2), (1,3], [0,3], [0,1) \cup (2,4], [0,2) \cup (2,4], [0,1) \cup (1,4], [0,4]\}.$$

Choose $L = [0,4)$ e get $L \in \mathcal{D}(\mu)$. But L is not a G_δ-set for that (X, μ) is not generalized G_δ-submaximal. Now $i_\mu c_\mu([0,1]) = i_\mu([0,1] \cup (3,6]) = \emptyset, i_\mu c_\mu[1,2] = i_\mu([1,2] \cup (4,6]) = \emptyset$ and $i_\mu c_\mu([2,6]) = i_\mu([2,6]) = \emptyset$. Therefore, $[0,1], [1,2], [2,6] \in \mathcal{N}(\mu)$. Hence $X \in \mathcal{M}(\mu)$.

(c) Take $Q = [0,3)$. Then $Q \in \mathcal{D}(\mu)$. Since $Q \subset X$ we have $Q \in \mathcal{M}(\mu)$, by Lemma 2.1(a). Therefore, $Q \notin \mu^{**}$.

(d) Here

$$\mu^* = \{\emptyset, [0,2), (1,2), (1,3], [0,3], [0,1) \cup (2,4], [0,2) \cup (2,4], [0,1) \cup (1,4], \\ [0,1) \cup (1,2) \cup (2,4], [0,1), [0,1) \cup (2,3], (2,3], [0,1) \cup (1,2), (1,2) \cup (2,3], \\ [0,2) \cup (2,3], [0,1) \cup (1,3], [0,1) \cup (1,2) \cup (2,3], [0,4]\}.$$

Let $W = [0,5)$. Then $W \in \mathcal{D}(\mu^*)$. Since $W \subset X$ we have $W \in \mathcal{M}(\mu)$, by Lemma 2.1(a). Therefore, $V \notin \mu^{**}$.

Theorem 4.2. *Let ζ, η be two GT on a non-null set X with $\eta \subset \zeta$. Then*
(a) *Every (ζ, η)-cliquish function is a (η, η)-cliquish function.*
(b) *Every (ζ, ζ)-cliquish function is a (η, ζ)-cliquish function.*

Theorem 4.3. *Let (X, μ) be a GTS, $h : X \to \mathbb{R}$ be a function. If (X, μ^*) is a wBS, $\mathcal{D}_\eta(h) \in \mathcal{M}(\mu^*)$ for $\eta \in \{\mu, \mu^*, \mu^{**}\}$, then h is a (μ^*, η)-cliquish function.*

Proof. By hypothesis and Lemma 2.5 (b), $\mathcal{D}_\eta(h) \in \mathfrak{M}(\mu^*)$ which implies $\mathcal{D}_\eta(h)$ is μ^*-codense, by Lemma 2.6. So that $C_\eta(h) \in \mathcal{D}(\mu^*)$ which implies h is (μ^*, η)-cliquish. □

The below Theorem 4.4 gives a shortcut for finding the significance of a given map, reducing the computational complexity.

Theorem 4.4. *Let (X, μ) be a generalized submaximal space. If (X, μ^*) is a wBS, then the below results are said to be true.*
(a) $h^{-1}((-\infty, \vartheta]) \in \mathcal{M}(\mu^*)$ *for every* $\vartheta \in \mathbb{R} \Rightarrow h \in \mathfrak{L}(\mu)$.
(b) $h^{-1}([\vartheta, \infty)) \in \mathcal{M}(\mu^*)$ *for every* $\vartheta \in \mathbb{R} \Rightarrow h \in \mathfrak{U}(\mu)$.

Proof. (a) Given $h^{-1}((-\infty, \vartheta]) \in \mathcal{M}(\mu^*)$ for all $\vartheta \in \mathbb{R}$. By Lemma 2.5 (b) and Lemma 2.6, $h^{-1}((-\infty, \vartheta])$ is μ^*-codense for all $\vartheta \in \mathbb{R}$. Then $X - h^{-1}((-\infty, \vartheta]) = h^{-1}(\mathbb{R} - (-\infty, \vartheta]) = h^{-1}((\vartheta, \infty))$ is a μ^*-dense set so that $h^{-1}((\vartheta, \infty)) \in \mathcal{D}(\mu)$ for all $\vartheta \in \mathbb{R}$, since $\mu \subset \mu^*$. By hypothesis, $h^{-1}((\vartheta, \infty)) \in \mu$ for all $\vartheta \in \mathbb{R}$. Therefore, $h \in \mathfrak{L}(\mu)$.

(b) By similar arguments in (a), we get this result.

□

Now let us examine the significance of cliquish functions in generalized submaximal and generalized G_δ-submaximal spaces.

$$\begin{array}{ccc} (\mu,\eta)-cliquish & \longrightarrow & (\mu^{**},\eta)-cliquish \\ & & \uparrow \\ & & (\mu^{*},\eta)-cliquish \end{array}$$

$$\text{where, } \eta \in \{\mu, \mu^*, \mu^{**}\}$$

Theorem 4.5 describes the above diagram also, which gives an easy way to check whether a function is (μ^{**}, η)-cliquish or not.

Theorem 4.5. *Let (X, μ) be a generalized submaximal space, $\eta \in \{\mu, \mu^*, \mu^{**}\}$ and $h : X \to \mathbb{R}$ be a map. Then the below results are true.*
(a) $\mathcal{D}(\mu) \subset \mathcal{D}(\mu^{**})$.
(b) *h is (μ^{**}, η)-cliquish if h is (μ, η)-cliquish.*
(c) *h is (μ^{**}, η)-cliquish if h is (μ^{*}, η)-cliquish.*

Proof. We will present the elaborate proof for (a) only. Choose

$$Q \in \mathcal{D}(\mu). \tag{4.1}$$

Then by hypothesis, $Q \in \mu$. Take

$$H \in \tilde{\mu}^{**} \tag{4.2}$$

From equation (4.2), $H \notin \mathcal{M}(\mu)$. Thus, $H \notin \mathcal{N}(\mu)$ and hence $i_\mu c_\mu(H) \neq \emptyset$. Thus, $i_\mu c_\mu(H) \in \tilde{\mu}$. By equation (4.1), we have $Q \cap i_\mu c_\mu(H) \neq \emptyset$ and hence $Q \cap c_\mu H \neq \emptyset$. By Lemma 2.3, $Q \cap H \neq \emptyset$. Therefore, $Q \in \mathcal{D}(\mu^{**})$. □

Theorem 4.6. *Let (X, μ) is a generalized G_δ-submaximal space. If $h : X \to \mathbb{R}$ is a (μ, η)-cliquish function for η is a GT on X, then $\mathcal{D}_\eta(h) \in \mathcal{M}(\mu)$.*

Proof. Suppose h is (μ, η)-cliquish. Then $C_\eta(h) \in \mathcal{D}(\mu)$ so that $C_\eta(h)$ is μ-residual, by Theorem 4.1(a). Therefore, $\mathcal{D}_\eta(h) \in \mathcal{M}(\mu)$. □

The below Theorem 4.7 describes the below diagram.

$$\begin{array}{ccc} h \quad is \quad (\mu,\eta)-cliquish & \longrightarrow & C_\eta(h) \in \mu^{**} \\ & & \uparrow \\ & & h \quad is \quad (\mu^{*},\eta)-cliquish \end{array}$$

$$\text{where, } \eta \in \{\mu, \mu^*, \mu^{**}\}$$

Theorem 4.7 provides some tricks to find the character of the collection of all continuity points using the cliquish function.

Theorem 4.7. *In a generalized G_δ-submaximal space (X, μ), if $\eta \in \{\mu, \mu^*, \mu^{**}\}$ and $X \in \mu^{**}$, then the below results are true.*
(a) $C_\eta(h) \in \mu^{**}$ *if h is (μ, η)-cliquish;*
(b) $C_\eta(h) \in \mu^{**}$ *if h is (μ^*, η)-cliquish for $h : X \to \mathbb{R}$.*

Theorem 4.8. *Let (X, μ) be a generalized G_δ-submaximal space. If $\eta \in \{\mu, \mu^*, \mu^{**}\}$ and (X, μ) is a BS, then*
(a) $\mathcal{D}(\mu^{**}) \subset \mu^{**}$.
(b) $C_\eta(h) \in \mu^{**}$ *if h is (μ^{**}, η)-cliquish where $h : X \to \mathbb{R}$.*

Proof. (a) By Lemma 4.2, X in μ. Also,

$$\mu \subset \mu^{**} \tag{4.3}$$

so that $X \in \mu^{**}$. Let $Q \in \mathcal{D}(\mu^{**})$. Then $Q \in \mathcal{D}(\mu)$, by equation (4.3). Therefore, $Q \in \mu^{**}$, by Theorem 4.1 (c).

(b) Given that h is (μ^{**}, η)-cliquish. Then $C_\eta(h) \in \mathcal{D}(\mu^{**})$ and hence $C_\eta(h) \in \mu^{**}$, by (a).

□

In the rest of this section, we study the significance of generalized G_δ-submaximal and generalized submaximal spaces in generalized metric spaces.

The pair (X, Ω) is called as a *generalized metric space* [16] (GMS) where X is a non-null set and Ω ia a family of all metric σ defined on a subset X.

Moreover, Ω_X is the family of all the metrics defined on X [16]. Also, if $\sigma \in \Omega_X$ and $Q \subset X$ is a non-null set, then $\sigma|_Q$ [16] for the restriction of the metric σ to $Q \times Q$.

Put $\Omega|_Q = \{\sigma|_Q \mid \sigma \in \Omega\}$ for any $\Omega \subset \Omega_X$.

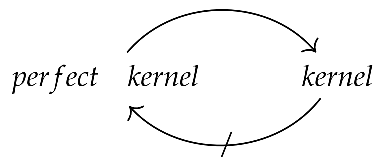
In a GMS (X, Ω), the collection of all Ω-*open sets* [16] in (X, Ω) is notated by μ_Ω, more precisely, $L \in \mu_\Omega \Leftrightarrow$ for any $p \in L$, there is $\sigma \in \Omega$ and $\delta > 0$ such that $B_\sigma(p, \delta) \subset L$ for $B_\sigma(p, \delta) = \{t \in dom(\sigma) \mid \sigma(p, t) < \delta\}$.

Remark 4.1. [16] (a) If (X, Ω_X) is a GMS, $\Omega \subset \Omega_X$ and $Q \neq \emptyset$, then $(Q, \Omega|_Q)$ is a GMS.
(b) (X, μ_Ω) is a generalized topological space.

Lemma 4.3. [16] In a GMS (X, Ω), $B_{\sigma|Q}(p, \varepsilon) = B_\sigma(p, \varepsilon) \cap Q$ where $p \in Q \subset X, \varepsilon > 0$.

Definition 4.1. [16] In a GMS (X, Ω), a finite collection $\Omega_0 \subset \Omega$ is said to be a ;
• *kernel* if for every $D \in \tilde{\mu}_\Omega = \mu_\Omega - \{\emptyset\}$, there is $\sigma \in \Omega_0$ such that $i_\sigma(D) \neq \emptyset$.
• *perfect kernel* if for any finite number of elements $K_1, K_2, \ldots., K_n$ in μ_Ω with $K_1 \cap K_2 \cap \ldots. \cap K_n \neq \emptyset$, there is $\sigma \in \Omega_0$ such that $i_\sigma(\cap_{i=1}^n K_i) \neq \emptyset$.

In [16], generally, in a generalized metric space,



Lemma 4.4. [Lemma 4.3, [16]] If the GMS (X, Ω) has a perfect kernel Ω_0 and $W \in \tilde{\mu}_\Omega$, then $\Omega_0|_W$ is a perfect kernel of $(W, \Omega|_W)$.

Definition 4.2. Let (X, Ω) be a generalized metric space. Then Ω is said to satisfy *$\mathcal{V}$-property* if $\sigma_1, \sigma_2 \in \Omega$ and $r, s \in X$, then $\sigma(r,s) = max\{\sigma_1(r,s), \sigma_2(r,s)\}$ is a metric and hence $\sigma \in \Omega$.

Theorem 4.9. *Let (X, Ω_X) be a GMS, Ω_X satisfy the $\mathcal{V}$-property. If $(W, \Omega_X|_W)$ is an open subspace of X, then the below results are true.*
(a) *If $Q \subset W$ is $\mu_{\Omega_X|_W}$-open in W, then $Q \in \mu_{\Omega_X}$.*
(b) *If $(W, \mu_{\Omega_X|_W})$ is generalized submaximal, then (X, μ_{Ω_X}) is generalized submaximal.*
(c) *If $(W, \mu_{\Omega_X|_W})$ is generalized G_δ-submaximal, then (X, μ_{Ω_X}) is generalized G_δ-submaximal.*
(d) $(X, \mu_{\Omega_X|_W})$ is generalized G_δ-submaximal if $(W, \mu_{\Omega_X|_W})$ is generalized submaximal.

Proof. (a) Given that Q is $\mu_{\Omega_X|_W}$-open in W. Let $r \in Q$. Then there is $\sigma_1|_W \in \Omega_X|_W$ and $\varepsilon_1 > 0$ such that $B_{\sigma_1|_W}(r, \varepsilon_1) \subset Q$ and so $B_{\sigma_1}(r, \varepsilon_1) \cap W \subset Q$, by Lemma 4.3. Since $r \in W$ and W is a μ_{Ω_X}-open subset of X, there exists $\sigma_2 \in \Omega_X$ and $\varepsilon_2 > 0$ such that $B_{\sigma_2}(r, \varepsilon_2) \subset W$. For $s \in X$, take $\sigma_3(r,s) = max\{\sigma_1(r,s), \sigma_2(r,s)\}$. Then $\sigma_3 \in \Omega_X$, by hypothesis. Also,

$$B_{\sigma_3}(r, \varepsilon_1) \subset B_{\sigma_1}(r, \varepsilon_1) \tag{4.4}$$

and

$$B_{\sigma_3}(r, \varepsilon_2) \subset B_{\sigma_2}(r, \varepsilon_2). \tag{4.5}$$

Take $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$. Then from equations (4.4) & (4.5), $B_{\sigma_3}(r, \varepsilon) \subset B_{\sigma_1}(r, \varepsilon_1) \cap B_{\sigma_2}(r, \varepsilon_2)$ and so $B_{\sigma_3}(r, \varepsilon) \subset Q$. Thus, there exists $\sigma_3 \in \Omega_X$ and $\varepsilon > 0$ such that $B_{\sigma_3}(r, \varepsilon) \subset Q$. Therefore, $Q \in \tilde{\mu}_{\Omega_X}$.

(b) Let $(W, \mu_{\Omega_X|_W})$ be a generalized submaximal space and $L \in \mathcal{D}(\mu_{\Omega_X})$. By Lemma 2.2, $c_W L = cL \cap W = X \cap W = W$. Thus, L is $\mu_{\Omega_X|_W}$-dense in W. By assumption, L is $\mu_{\Omega_X|_W}$-open in W and so $L \in \mu_{\Omega_X}$, by (a). Therefore, (X, μ_{Ω_X}) is generalized submaximal.

(c) Given that $(W, \mu_{\Omega_X|_W})$ is a generalized G_δ-submaximal space. Let $K \in \mathcal{D}(\mu_{\Omega_X})$. Then $c_W K = cK \cap W = X \cap W = W$, by Lemma 2.2. Thus, K is a $\mu_{\Omega_X|_W}$-dense set in W and hence K is $\mu_{\Omega_X|_W}$- G_δ-set which implies that K is μ_{Ω_X}-G_δ in X, by (a). Therefore, (X, μ_{Ω_X}) is generalized G_δ-submaximal.

(d) Trivial proof is omitted.

□

Theorem 4.10. *Let (X, Ω_X) be a generalized metric space, Ω_X satisfy the $\mathcal{V}$-property. Then the below results are true.*
(a) $\mu_{\Omega_X} = \mu^*_{\Omega_X}$.
(b) μ_{Ω_X} *is closed under finite intersection.*

Proof. (a) By Remark 4.1, (X, μ_{Ω_X}) is a GTS and hence $\mu_{\Omega_X} \subset \mu^*_{\Omega_X}$. Let $L \in \mu^{\tilde{*}}_{\Omega_X}$ and $x \in L$. Then $L = \bigcup_t (L^t_1 \cap L^t_2 \cap \ldots . \cap L^t_{n_t})$ where $L^t_i \in \mu_{\Omega_X}$ and $i = 1$ to n_t. Take $P_k = L^k_1 \cap L^k_2 \cap \ldots . \cap L^k_{n_k}$ for some k with $P_k \neq \emptyset$ and $x \in P_k$. Then $x \in L^k_i$ for all $i = 1$ to n_k and so there exist $\sigma_i \in \Omega_X$ and $\varepsilon_i > 0$ such that $B_{\sigma_i}(x, \varepsilon_i) \subset L^k_i$ for all $i = 1$ to n_k. Consider $B_{\sigma_1}(x, \varepsilon_1)$ and $B_{\sigma_2}(x, \varepsilon_2)$. For $y \in X$, take $d_1(r,s) = max\{\sigma_1(r,s), \sigma_2(r,s)\}$. Then $d_1 \in \Omega_X$, by hypothesis. Also,

$$B_{d_1}(x, \varepsilon_1) \subset B_{\sigma_1}(x, \varepsilon_1) \tag{4.6}$$

and

$$B_{d_1}(x, \varepsilon_2) \subset B_{\sigma_2}(x, \varepsilon_2) \tag{4.7}$$

Let $\delta_1 = min\{\varepsilon_1, \varepsilon_2\}$. From equations (4.6) & (4.7), $B_{d_1}(x, \delta_1) \subset B_{\sigma_1}(x, \varepsilon_1) \cap B_{\sigma_2}(x, \varepsilon_2)$ and so $B_{d_1}(x, \delta_1) \subset L^k_1 \cap L^k_2$. Consider $B_{d_1}(x, \delta_1)$ and $B_{\sigma_3}(x, \varepsilon_3)$. Take $d_2(r,s) = max\{d_1(r,s), \sigma_3(r,s)\}$. Then $d_2 \in \Omega_X$, by hypothesis. Also,

$$B_{d_2}(x, \delta_1) \subset B_{d_1}(x, \delta_1) \tag{4.8}$$

and

$$B_{d_2}(x, \varepsilon_3) \subset B_{\sigma_3}(x, \varepsilon_3) \tag{4.9}$$

Take $\delta_2 = min\{\delta_1, \varepsilon_3\}$. By equations (4.8) & (4.9), $B_{d_2}(x, \delta_2) \subset B_{d_1}(x, \delta_1) \cap B_{\sigma_3}(x, \varepsilon_3)$ and so $B_{d_2}(x, \delta_2) \subset L^k_1 \cap L^k_2 \cap L^k_3$. Proceeding like this, we get a metric $d_{n_k - 1} \in \Omega$ and $\delta_{n_k - 1} > 0$ such that $B_{d_{n_k-1}}(x, \delta_{n_k - 1}) \subset L^k_1 \cap L^k_2 \cap L^k_3 \cap \ldots . \cap L^k_{n_k}$. Then $B_{d_{n_k-1}}(x, \delta_{n_k-1}) \subset P_k$ and so $B_{d_{n_k-1}}(x, \delta_{n_k-1}) \subset L$. Therefore, $L \in \tilde{\mu}_{\Omega_X}$ and so $\mu_{\Omega_X} = \mu^*_{\Omega_X}$.

(b) Since $\mu^*_{\Omega_X}$ is closed under finite intersection, μ_{Ω_X} is closed under finite intersection, by (a). □

5. Hyperconnected spaces

Now, we deal with the most well-studied space, hyperconnected space. First, we discuss the nature of cliquish, lower, and upper semi-continuous functions in a hyperconnected space and give some properties of this space in GMSs.

First, we prove a few properties about hyperconnected spaces in a GTS.

Theorem 5.1. *Let (X, μ) be a hyperconnected space. If $X \in \mu^{**}$, then*
(a) Every non-null μ^*-open set is μ-dense.
(b) Every subset of X in $\tilde{\mu}^*$ is μ-residual.

Proof. Let $P \in \tilde{\mu}^*$. Then $P = \bigcup_t (P_1^t \cap P_2^t \cap \ldots . \cap P_{n_t}^t)$ where $P_i^t \in \mu$ for $i = 1$ to n_t. Take

$$Q_k = P_1^k \cap P_2^k \cap \ldots . \cap P_{n_k}^k \tag{5.1}$$

with $Q_k \neq \emptyset$ for some k. Since $P_i^k \in \tilde{\mu}$ for $i = 1$ to n_k and by hypothesis, each $P_i^k \in \mathcal{D}(\mu)$. Since X is Baire we have $\bigcap_{n \in \mathbb{N}} P_n \in \mathcal{D}(\mu)$ where each $P_n \in \mu; P_n \in \mathcal{D}(\mu)$. Therefore, $Q_k \in \mathcal{D}(\mu)$.

(a) From (5.1), $Q_k \subset P$ so that $P \in \mathcal{D}(\mu)$.

(b) By (5.1), Q_k is a μ-G_δ-set. Then $Q_k = \bigcap_{n=1}^{\infty} H_n$ where $H_n \in \tilde{\mu}$ for every $n \in \mathbb{N}$. Since $Q_k \subset H_n$ and $Q_k \in \mathcal{D}(\mu)$, each $H_n \in \mathcal{D}(\mu)$. Thus, $i_\mu(X - H_n) = \emptyset$ and $X - H_n$ is μ-closed for all $n \in \mathbb{N}$, since each $H_n \in \tilde{\mu}$. Therefore, each $X - H_n \in \mathcal{N}(\mu)$ which implies $X - Q_k \in \mathcal{M}(\mu)$. Hence Q_k is a μ-residual set. Since $Q_k \subset P$ we have P is μ-residual.

□

Theorem 5.1 is also true if we replace "(X, μ) is a hyperconnected and X is of μ-II category" by the condition "(X, μ^*) is a hyperconnected space", since $\mu^* \supset \mu$.

Theorem 5.2 provides an interesting property for dense sets in a hyperconnected space.

Theorem 5.2. *In a hyperconnected space (X, μ), $\tilde{\mu}^{**} \subset \mathcal{D}(\mu)$.*

Proof. Let $Q \in \tilde{\mu}^{**}$. Then $Q \notin \mathcal{M}(\mu)$ and so $Q \notin \mathcal{N}(\mu)$. So that, $i_\mu c_\mu(Q) \neq \emptyset$. By hypothesis, $i_\mu c_\mu(Q) \in \mathcal{D}(\mu)$ which implies $c_\mu A \in \mathcal{D}(\mu)$. Take $K \in \tilde{\mu}$ we get $K \cap c_\mu Q \neq \emptyset$ and so $K \cap Q \neq \emptyset$, by Lemma 2.3. Therefore, $Q \in \mathcal{D}(\mu)$. □

Corollary 5.1. *Let (X, ζ) be a hyperconnected space, $\eta \in \{\zeta, \zeta^*, \zeta^{**}\}$. If $C_\eta(h) \in \zeta^{**}$, then h is a (ζ, η)-cliquish function in X.*

Proof. Suppose that $C_\eta(h) \in \zeta^{**}$. Then $C_\eta(h)$ is ζ-II category. By hypothesis and Theorem 5.2, $C_\eta(h) \in \mathcal{D}(\mu)$. Hence h is (μ, η)-cliquish. □

The below Theorem 5.3 provides an easy way to prove a given space is a strong Baire space.

Theorem 5.3. *If $X \in \mu^{**}$ and if (X, μ^*) is hyperconnected, then (X, μ) is a sBS.*

Proof. Let $L_1, L_2, \ldots ., L_n \in \mu$ with $L_1 \cap L_2 \cap \ldots . \cap L_n \neq \emptyset$. Take $Q = L_1 \cap L_2 \cap \ldots . \cap L_n$. Then $Q \in \tilde{\mu}^*$ and also Q is μ-G_δ-set. By hypothesis, $Q \in \mathcal{D}(\mu^*)$ and so $Q \in \mathcal{D}(\mu)$, since $\mu \subset \mu^*$. Therefore, Q is a μ-residual set. Hence $Q \in \mu^{**}$, by Theorem 3.2(b) which states that (X, μ) is a strong Baire Space. □

Theorem 5.3 is true if we replace "(X, μ^*) is a hyperconnected space" by the condition "(X, μ) is a hyperconnected space", since $\mu^* \supset \mu$.

The Converse part of Theorem 5.3 need not be true as shown by Example 5.1.

Example 5.1. Take $X = [0, 5]$ and

$$\mu = \{\emptyset, [0, 2), [2, 4], [0, 3), [0, 4]\}.$$

clearly, (X, μ) is a strong Baire space. Now

$$\mu^* = \{\emptyset, [0,2), [2,4], [0,3), [2,3), [0,4]\}.$$

Choose $H = [2,3)$. Then $H \in \tilde{\mu}^*$. But $c_{\mu^*}H = [2,5] \neq X$. Thus, (X, μ^*) is not hyperconnected.

Proposition 5.1. *In a GTS (X, μ), the below results are true.*
(a) $h \in \mathfrak{L}(\eta) \Longleftrightarrow h^{-1}((\beta, \infty)) \in \eta$ for all $\beta \in \mathbb{R}$.
(b) $h \in \mathfrak{U}(\eta) \Longleftrightarrow h^{-1}((-\infty, \beta)) \in \eta$ for every $\beta \in \mathbb{R}$; $\eta \in \{\mu^*, \mu^{**}\}$.

Proof. (a) Assume that, $h \in \mathfrak{L}(\eta)$. Take $t \in h^{-1}((\beta, \infty))$ we get $\beta < h(t)$. By hypothesis, there is $L \in \eta(t)$ such that $h(L) \subset (\beta, \infty)$. This implies $L \subset h^{-1}(h(L)) \subset h^{-1}((\beta, \infty))$ which in turn implies that $h^{-1}((\beta, \infty)) \in \eta$ for any $\beta \in \mathbb{R}$. Conversely, suppose that $h^{-1}((\beta, \infty)) \in \eta$ for any $\beta \in \mathbb{R}$. Let $r \in X$ and $\beta < h(r)$. Then $h(r) \in (\beta, \infty)$ and so $r \in h^{-1}((\beta, \infty))$. By hypothesis, there is $P \in \eta(r)$ such that $P \subset h^{-1}((\beta, \infty))$. This implies $h(P) \subset h(h^{-1}((\beta, \infty))) \subset (\beta, \infty)$ which implies that $h \in \mathfrak{L}(\eta)$.

(b) Use the same argument in (a) for the upper semi-continuous function.

□

Theorem 5.4. *In a hyperconnected space (X, μ), the below results are true.*
(a) *If $h \in \mathfrak{L}(\mu)$, then $h^{-1}((\beta, \infty))$ is μ-residual for any $\beta < h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$.*
(b) *If $h \in \mathfrak{U}(\mu)$, then $h^{-1}((-\infty, \beta))$ is μ-residual for any $\beta > h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$.*

Proof. (a) Let $\beta < h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$. We get some $K \in \mu(t_0)$ such that $h(K) \subset (\beta, \infty)$. Thus, $h^{-1}((\beta, \infty)) \neq \emptyset$. Also, $h^{-1}((\beta, \infty)) \in \mu$ implies that $h^{-1}((\beta, \infty)) \in \tilde{\mu}$. by hypothesis, $h^{-1}((\beta, \infty)) \in \mathcal{D}(\mu)$. Also, $h^{-1}((\beta, \infty))$ is μ-G_δ-set. Therefore, $h^{-1}((\beta, \infty))$ is μ-residual.

(b) By the same arguments in (a), we get the proof.

□

Theorem 5.5. *Let (X, μ) be a hyperconnected space. If $X \in \mu^{**}$, then the below results are true.*
(a) *$h^{-1}((\beta, \infty))$ is μ-residual for any $\beta < h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$, if $h \in \mathfrak{L}(\mu^*)$.*
(b) *$h^{-1}((-\infty, \beta))$ is μ-residual for any $\beta > h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$, if $h \in \mathfrak{U}(\mu^*)$.*

Proof. (a) If $h \in \mathfrak{L}(\mu^*)$ and $\beta < h(t_0)$ where $\beta \in \mathbb{R}, t_0 \in X$, then there exists $K \in \mu^*(t_0)$ such that $h(K) \subset (\beta, \infty)$ and so $h^{-1}((\beta, \infty)) \neq \emptyset$. Thus, $h^{-1}((\beta, \infty)) \in \mu^*$, by Proposition 5.1. Hence $h^{-1}((\beta, \infty)) \in \tilde{\mu}^*$. By Theorem 5.1, $h^{-1}((\beta, \infty))$ is μ-residual for any $\beta < h(t_0)$ where $\beta \in \mathbb{R}, t_0 \in X$.

(b) It follows from the similar considerations in (a).

□

Theorem 5.6. *If $X \in \mu^{**}$ and $\eta \in \{\mu, \mu^*\}$, then*
(a) *If $h \in \mathfrak{L}(\mu, \eta)$, then $\eta \cap \mu^{**} \neq \emptyset$ and $h^{-1}((\beta, \infty)) \in \mu^{**}$ for any $\beta < h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$.*
(b) *If $h \in \mathfrak{U}(\mu, \eta)$, then $\eta \cap \mu^{**} \neq \emptyset$ and $h^{-1}((-\infty, \beta)) \in \mu^{**}$ for any $\beta > h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$.*

Proof. (a) Given $h \in \mathfrak{L}(\mu, \eta)$. Let $\beta < h(t_0)$ where $\beta \in \mathbb{R}$ and $t_0 \in X$. Then there exists $E \in \eta(t_0)$ being a μ-residual set such that $h(E) \subset (\beta, \infty)$. By Theorem 3.2(b), $E \in \mu^{**}$ so that $\eta \cap \mu^{**} \neq \emptyset$. Since $E \subset h^{-1}((\beta, \infty))$ and E is μ-residual, $h^{-1}((\beta, \infty))$ is in μ^{**}, by Corollary 3.1(b). Therefore, $h^{-1}((\beta, \infty)) \in \mu^{**}$ for any $\beta < h(t_0)$ where $\beta \in \mathbb{R}, t_0 \in X$.

(b) By the same arguments in (a), we get the proof.

□

Theorem 5.7 is a special case of the result: if $\eta \subset \zeta$ and Q is η-dense, then Q is ζ-dense.

Theorem 5.7. *Let (X, Ω_X) be a generalized metric space, $\Omega_0 \subset \Omega_X$. Then the below results are true.*
(a) $\mu_{\Omega_0} \subset \mu_{\Omega_X}$.
(b) *If Ω_0 is the kernel, then $Q \in \mathcal{D}(\mu_{\Omega_0})$ if and only if $Q \in \mathcal{D}(\mu_{\Omega_X})$.*

Proof. (a) Let $G \in \mu_{\Omega_0}$ and $r \in G$. Then there is $\sigma \in \Omega_0$ and $\varepsilon > 0$ such that $B_\sigma(r, \varepsilon) \subset G$. By hypothesis, $\sigma \in \Omega_X$. Thus, there is $\sigma \in \Omega_X, \varepsilon > 0$ such that $B_\sigma(x, \varepsilon) \subset G$. Therefore, $G \in \mu_{\Omega_X}$. Hence $\mu_{\Omega_0} \subset \mu_{\Omega_X}$.

(b) Given that $Q \in \mathcal{D}(\mu_{\Omega_0})$. Let $G \in \tilde{\mu}_{\Omega_X}$. Since Ω_0 is a kernel, there exists $\sigma_0 \in \Omega_0$ such that $i_{\sigma_0}G \neq \emptyset$ and so $i_{\sigma_0}G \in \tilde{\mu}_{\Omega_0}$. This implies that $Q \cap i_{\sigma_0}G \neq \emptyset$ which in turn implies that $Q \cap G \neq \emptyset$. Hence $Q \in \mathcal{D}(\mu_{\Omega_X})$. Converse part is trivial.

□

Theorem 5.8. *Let (X, Ω_X) be a GMS with kernel $\Omega_0 \subset \Omega_X$. Then (X, μ_{Ω_0}) is hyperconnected $\Leftrightarrow$ (X, μ_{Ω_X}) is hyperconnected.*

Proof. Suppose that, (X, μ_{Ω_0}) is hyperconnected. Let $G \in \tilde{\mu}_{\Omega_X}$. Since Ω_0 is a kernel, there exist $\sigma_0 \in \Omega_0$ such that $i_{\sigma_0}G \neq \emptyset$ and hence $i_{\sigma_0}G \in \tilde{\mu}_{\Omega_0}$. By hypothesis, $i_{\sigma_0}G \in \mathcal{D}(\mu_{\Omega_0})$ and so $i_{\sigma_0}G \in \mathcal{D}(\mu_{\Omega_X})$, by Theorem 5.7 (b). Therefore, $G \in \mathcal{D}(\mu_{\Omega_X})$. Hence (X, μ_{Ω_X}) is hyperconnected. Conversely, assume that (X, μ_{Ω_X}) is hyperconnected. Let $G \in \tilde{\mu}_{\Omega_0}$. Then $G \in \tilde{\mu}_{\Omega_X}$, by Theorem 5.7 (a). Therefore, $G \in \mathcal{D}(\mu_{\Omega_X})$. By Theorem 5.7 (b), $G \in \mathcal{D}(\mu_{\Omega_0})$. Hence (X, μ_{Ω_0}) is hyperconnected. □

Theorem 5.9. *Let (X, Ω_X) be a GMS and $(W, \Omega_X|_W)$ is an open subspace of X. If Ω_X satisfy the $\mathcal{V}$-property and (X, μ_{Ω_X}) is hyperconnected, then $(W, \mu_{\Omega_X|_W})$ is hyperconnected.*

Proof. Assume that (X, μ_{Ω_X}) is hyperconnected. Let $G \in \tilde{\mu}_{\Omega_X|_W}$. By Theorem 4.9 (a), $G \in \tilde{\mu}_{\Omega_X}$. Also, by hypothesis, $G \in \mathcal{D}(\mu_{\Omega_X})$. By Lemma 2.2, $c_W G = cG \cap W = X \cap W = W$. Thus, G is a $\mu_{\Omega_X|_W}$-dense in W. Therefore, $(W, \mu_{\Omega_X|_W})$ is a hyperconnected space. □

Theorem 5.10. *Let (X, Ω_X) be a generalized metric space with a perfect kernel $\Omega_0 \subset \Omega_X$. Then the below results are true.*
(a) *If $Q \in \mathcal{D}(\mu_{\Omega_X})$, then $Q \in \mathcal{D}(\mu^*_{\Omega_X})$.*
(b) *$(X, \mu^*_{\Omega_X})$ is hyperconnected if (X, μ_{Ω_X}) is hyperconnected.*

Proof. (a) Given Q in $\mathcal{D}(\mu_{\Omega_X})$. Let K in $\tilde{\mu}^*_{\Omega_X}$. Then $D = \bigcup_t (D^t_1 \cap D^t_2 \cap \ldots . \cap D^t_{n_t})$ where $D^t_i \in \mu, i = 1$ to n_t. Take $B_k = D^k_1 \cap D^k_2 \cap \ldots . \cap D^k_{n_k}$ with $B_k \neq \emptyset$ where $D^k_i \in \mu, i = i$ to n_k. By hypothesis, there exists $\sigma \in \Omega_0$ such that $i_\sigma B_k \neq \emptyset$ and so $i_\sigma B_k \in \tilde{\mu}_{\Omega_X}$. This implies that $Q \cap i_\sigma B_k \neq \emptyset$ for that $Q \cap B_k \neq \emptyset$. Thus, $Q \cap G \neq \emptyset$. Therefore, $Q \in \mathcal{D}(\mu^*_{\Omega_X})$.

(b) Assume that, (X, μ_{Ω_X}) is a hyperconnected space. Let $H \in \tilde{\mu}^*_{\Omega_X}$. Then $H = \bigcup_t (H^t_1 \cap H^t_2 \cap \ldots . \cap H^t_{n_t})$ where $H^t_i \in \mu, i = 1$ to n_t. Choose $D_k = H^k_1 \cap H^k_2 \cap \ldots . \cap H^k_{n_k}$ with $D_k \neq \emptyset$. Since Ω_0 is a perfect kernel, there exists a metric $\sigma_0 \in \Omega_0$ such that $i_{\sigma_0} D_k \neq \emptyset$. By hypothesis, $i_{\sigma_0} D_k \in \mathcal{D}(\mu_{\Omega_X})$. Since $i_{\sigma_0} D_k \subset H, H \in \mathcal{D}(\mu_{\Omega_X})$. Therefore, $H \in \mathcal{D}(\mu^*_{\Omega_X})$, by (a). Hence $(X, \mu^*_{\Omega_X})$ is hyperconnected.

□

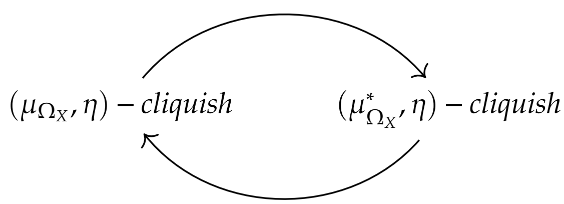
Example 5.2 describes that the condition "Ω_0 is a perfect kernel" is necessary in Theorem 5.10.

Example 5.2. Consider the GMS (X, Ω), $X = [0, 1], Q = [0, \frac{1}{2}], K = [\frac{1}{2}, 1]$ and $\Omega = \{\sigma_Q, \sigma_K, \sigma_E\}$ where σ_E is the Euclidean metric for $\mathbb{R}$,

$$\sigma_Q = \begin{cases} \sigma_E(r,s) & \text{if} \quad r, s \in Q \quad \text{or} \quad [0,1] - Q, \\ 1 & \text{otherwise.} \end{cases}$$

$$\sigma_K = \begin{cases} \sigma_E(r,s) & \text{if} \quad r, s \in K \quad \text{or} \quad [0,1] - K, \\ 1 & \text{otherwise.} \end{cases}$$

Then $Q, K \in \mu_\Omega$ such that $Q \cap K \neq \emptyset$. But $i_{\mu_\Omega}(Q \cap K) = \emptyset$. Therefore, (X, Ω) has no perfect kernel. Choose $G = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Then $G \in \mathcal{D}(\mu_\Omega)$. Take $Q = \{2\}$. Then $\{2\} \in \tilde{\mu}^*_\Omega$. But $G \cap A = \emptyset$. Therefore, $G \notin \mathcal{D}(\mu^*_\Omega)$.



Theorem 5.11 describes the above diagram.

Theorem 5.11. *Let (X, Ω_X) be a GMS with perfect kernel $\Omega_0 \subset \Omega_X$. If $\eta \in \{\mu_{\Omega_X}, \mu^*_{\Omega_X}, \mu^{**}_{\Omega_X}\}$, then h is a (μ_{Ω_X}, η)-cliquish function $\Leftrightarrow$ h is $(\mu^*_{\Omega_X}, \eta)$-cliquish for $h : X \to \mathbb{R}$.*

Proof. If h is (μ_{Ω_X}, η)-cliquish, then $C_\eta(h) \in \mathcal{D}(\mu_{\Omega_X})$. By Theorem 5.10(a), $C_\eta(h) \in \mathcal{D}(\mu^*_{\Omega_X})$. Therefore, h is $(\mu^*_{\Omega_X}, \eta)$-cliquish. Conversely, suppose that h is $(\mu^*_{\Omega_X}, \eta)$-cliquish. Then $C_\eta(h) \in \mathcal{D}(\mu^*_{\Omega_X})$. Since $\mu_{\Omega_X} \subset \mu^*_{\Omega_X}$, $C_\eta(h) \in \mathcal{D}(\mu_{\Omega_X})$. Therefore, h is (μ_{Ω_X}, η)-cliquish. □

A map $h : (X, \mu) \to (W, \zeta)$ is said to be *feebly (μ, ζ)-continuous* [21] if $i_\mu h^{-1}(B) \neq \emptyset$ for every $B \subset W$ with $i_\zeta B \neq \emptyset$. h is called *(μ, η)-open* (resp. *feebly (μ, η)-open*, *(μ, η)-closed*) if $h(K) \in \eta$ for each $K \in \mu$ (resp. $i_\eta h(K) \neq \emptyset$ for each $K \in \tilde{\mu}$, $h(K)$ is η-closed for each K is μ-closed) [17].

Lemma 5.1. [23, Theorem 4.4] If $h : (X, \mu) \to (W, \eta)$ is a feebly (μ, η)-continuous, injective mapping and if Q is μ-codense, then $h(Q)$ is η-codense.

Lemma 5.2. [23, Theorem 4.6] If $h : (X, \mu) \to (W, \eta)$ is a feebly (μ, η)-open map and if Q is η-codense, then $h^{-1}(Q)$ is μ-codense.

Theorem 5.12. *Let (X, μ) be a generalized submaximal space and (W, η) be a hyperconnected space. Then every feebly (μ, η)-open map from X to W is (μ, η)-continuous.*

Proof. Let $Q \in \tilde{\eta}$. Since (W, η) is a hyperconnected space, Q is a η-dense set in W. Then $W - Q$ is a η-codense set in W. By Lemma 5.2, $h^{-1}(W - Q) = X - h^{-1}(Q)$ is μ-codense which implies $h^{-1}(Q) \in \mathcal{D}(\mu)$. Therefore, $h^{-1}(Q) \in \mu$. Hence h is a (μ, η)-continuous function. □

$$
\begin{array}{ccc}
(\mu,\eta)-open \quad map & \longleftarrow & feebly \quad (\mu,\eta)-continuous, \quad bijective \quad map \\
 & & \downarrow \\
 & & (\mu,\eta)-closed \quad map
\end{array}
$$

The following Theorem 5.13 establishes the above diagram.

Theorem 5.13. *Let (X, μ) be a hyperconnected space and (W, η) be a generalized submaximal space. Then every feebly (μ, η)-continuous, bijective map from X to W is both (μ, η)-closed and (μ, η)-open.*

Proof. Given that h is a feebly (μ, η)-continuous, bijective map from X to W. Assume that,

$$(X, \mu) \quad is \quad a \quad hyperconnected \quad space \tag{5.2}$$

and

$$(W, \eta) \quad is \quad a \quad generalized \quad submaximal \quad space \tag{5.3}$$

Let $c_\mu(K) = K$. If $X - K = \emptyset$, then $K = X$ and so $h(K) = W$. Thus, $h(K)$ is η-closed. If $X - K \neq \emptyset$, then $X - K \in \mathcal{D}(\mu)$, by equation (5.2) so that K is μ-codense. Thus, $h(K)$ is a η-codense set in W, by Lemma 5.1 and so $W - h(K) \in \mathcal{D}(\eta)$. By equation (5.3), $W - h(K) \in \eta$. Therefore, $h(K)$ is η-closed. Hence h is (μ, η)-closed.

Let $M \in \mu$. If $M = \emptyset$, then there is nothing to prove. If $M \neq \emptyset$, then $M \in \mathcal{D}(\mu)$, by equation (5.2) so that $X - M$ is μ-codense. By Lemma 5.1, $h(X - M)$ is η-codense in W. Now $h(X - M) = h(X) - h(M) = W - h(M)$, since h is bijective. Thus, $W - h(M)$ is η-codense in W and hence $h(M) \in \mathcal{D}(\eta)$. From equation (5.3), $h(M) \in \eta$. Therefore, h is a (μ, η)-open map. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M.R. Ahmadi Zand, R. Khayyeri, Generalized G_δ-Submaximal Spaces, Acta Math. Hungar. 149 (2016), 274–285. https://doi.org/10.1007/s10474-016-0627-9.

[2] S. Al Ghour, A. Al-Omari, T. Noiri, On Homogeneity and Homogeneity Components in Generalized Topological Spaces, Filomat, 27 (2013), 1097–1105. https://www.jstor.org/stable/24896442.

[3] S.P. Arya, M.P. Bhamini, Some Weaker Forms of Semi-Continuous Functions, Ganita, 33 (1982), 124–134.

[4] M. Beddow, D. Rose, Collectionwise Weak Continuity Duals, Acta Math. Hung. 124 (2009), 189–200. https://doi.org/10.1007/s10474-009-8190-2.

[5] N. Biswas, On Characterization of Semi-Continuous Function, Atti Accad. Naz. Lincei Ser. Ottava Rend. Cl. Sci. Fis. Mat. Nat. 48 (1970), 399–402.

[6] J. Borsík, Generalized Oscillations for Generalized Continuities, Tatra Mt. Math. Publ. 49 (2011), 119–125. https://doi.org/10.2478/v10127-011-0031-3.

[7] Á. Császár, Generalized Open Sets, Acta Math. Hung. 75 (1997), 65–87. https://doi.org/10.1023/a:1006582718102.

[8] E. Ekici, Generalized Hyperconnectedness, Acta Math Hung. 133 (2011), 140–147. https://doi.org/10.1007/s10474-011-0086-2.

[9] E. Ekici, Generalized Submaximal Spaces, Acta Math. Hung. 134 (2011), 132–138. https://doi.org/10.1007/s10474-011-0109-z.

[10] J. Ewert, Note on Limits of Simply Continuous and Cliquish Functions, Int. J. Math. Math. Sci. 17 (1994), 447–450. https://doi.org/10.1155/s0161171294000645.

[11] J. Ewert, Characterization of Cliquish Functions, Acta Math. Hung. 89 (2000), 269–276. https://doi.org/10.1023/a:1006762702077.

[12] Z. Frolík, Baire Spaces and Some Generalizations of Complete Metric Spaces, Czechoslovak Math. J. 11 (1961), 237–248. http://dml.cz/dmlcz/100457.

[13] L.A. Fudali, On Cliquish Functions on Product Spaces, Math. Slovaca, 33 (1983), 53–58. http://dml.cz/dmlcz/129052.

[14] Z. Grande, E. Strońska, Some Continuous Operations on Pairs of Cliquish Functions, Tatra Mt. Math. Publ. 44 (2009), 15–25. https://doi.org/10.2478/v10127-009-0044-3.

[15] S. Jafari, T. Noiri, On Strongly θ-Semi-Continuous Functions, Indian J. Pure Appl. Math. 29 (1988), 1195–1201.

[16] E. Korczak-Kubiak, A. Loranty, R.J. Pawlak, Baire Generalized Topological Spaces, Generalized Metric Spaces and Infinite Games, Acta Math. Hung. 140 (2013), 203–231. https://doi.org/10.1007/s10474-013-0304-1.

[17] Z. Li, F. Lin, Baireness on Generalized Topological Spaces, Acta Math. Hung. 139 (2012), 320–336. https://doi.org/10.1007/s10474-012-0284-6.

[18] W.K. Min, Weak Continuity on Generalized Topological Spaces, Acta Math. Hung. 124 (2008), 73–81. https://doi.org/10.1007/s10474-008-8152-0.

[19] W.K. Min, Generalized Continuous Functions Defined by Generalized Open Sets on Generalized Topological Spaces, Acta Math. Hung. 128 (2009), 299–306. https://doi.org/10.1007/s10474-009-9037-6.

[20] A. Neubrunnová, On Quasicontinuous and Cliquish Functions, Čas. Pěst. Mat. 99 (1974), 109–114. http://dml.cz/dmlcz/117825.

[21] V. Renukadevi, T. Muthulakshmi, Weak Baire Spaces, Kyungpook Math. J. 55 (2015), 181–189. https://doi.org/10.5666/KMJ.2015.55.1.181.

[22] H.P. Thielman, Types of Functions, Amer. Math. Mon. 60 (1953), 156–161. https://doi.org/10.1080/00029890.1953.11988260.

[23] S. Vadakasi and V. Renukadevi, *Properties of Nowhere Dense Sets in GTSs*, Kyungpook Math. J. **57** (2017), 199 - 210.

[24] P. Yupapin, S. Vadakasi, Y. Farhat, On Nowhere Dense Sets, Eur. J. Pure Appl. Math. 15 (2022), 403–414. https://doi.org/10.29020/nybg.ejpam.v15i2.4283.

[25] Y. Farhat, S. Vadakasi, Generalized Dense set in Bigeneralized Topological Spaces, Eur. J. Pure Appl. Math. 16 (2023), 2049–2065. https://doi.org/10.29020/nybg.ejpam.v16i4.4911.