Abstract. In this study, a novel operation is introduced, which creates a local function of $A$ with respect to $I$ and $\tau$ respectively denoted as $A_{D^*}(I, \tau) = \{y \in Y \mid V \cap A < I, \text{ for each } V \in \tau^D(y)\}$ where $\tau^D(y) = \{V \in \tau \mid y \in V\}$. We then look into some of the fundamental characteristics and attributes of $A_{D^*}(I, \tau)$. Additionally, we look into an operator $\eta : P(Y) \rightarrow \tau$ provides $\eta(E) = Y - [Y - E]_{D^*}$ for all $E \in P(Y)$. Then the closure operator $cl_{D^*}(E) = E_{D^*} \cup E$ which forms the topology and the relation $\tau_{D^*} = \{V \subseteq Y \mid cl_{D^*}(Y - V) = Y - V\}$.

1. Introduction and Preliminaries

Ideals in a topological space $(Y, \tau)$ were studied by Kuratowski in [5]. He had also defined local function for each subset of $Y$ with regards to an ideal $I$ and $\tau$. In [9], Vaidyanathaswamy extended this study of ideals and local functions. In 1990, Jankovic and Hamlett [2] found more characteristics of ideal topological spaces. Assume that, $(Y, \tau)$ is a space without separation axioms. Then $cl(E); int(E)$ indicate the closure and interior of $E$ respectively, in an ideal space $(Y, \tau, I)$. A nonempty set of $Y$ that satisfies the given conditions in $(Y, \tau)$ is defined ideal [2];

(a) $E \in I, F \subseteq E \Rightarrow F \in I$

(b) $E \in I$ and $F \in I \Rightarrow$ union of $E$ and $F$ belongs to $I$.

In 1960, Vaidyanathaswamy [9] gave the new local function which is defined by $P(Y)$ of $Y$ with a set operator $(.)^* : P(Y) \rightarrow P(Y)$. For a set $A$ in $Y$, $A^*(I, \tau) = \{y \in Y \mid V \cap A \not\in I, \text{ for each } V \in \tau(y)\}$ where in $\tau(y) = \{V \in \tau \mid y \in V\}$. Moreover, we will just denote $A^*(I, \tau)$ by $A^*$ and $\tau^*(I, \tau)$ by $\tau^*$.

A Kuratowski closure operator [10] denoted by $cl^*(E)$ for $\tau^*$ finer than $\tau$ is defined as $cl^*(E) = E \cup E^*(I, \tau)$. In 2013, Ahmad Al-Omairi et. al [1] defined local closure functions in ideal spaces

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and studied their various properties. $\delta^*\text{-local closure functions}$ were analyzed and their various characterizations were studied in 2020 by P. Periyasamy and P. Rock Ramesh [7]. In 1966 N. Velicko [8] studied about $\theta$-open sets and defined $cl_\theta(E)$ as $cl_\theta(E) = \{ y \in Y : cl(V) \cap E = \phi, \text{ for each } V \in \tau(y) \}$ also, a set $E$ of $Y$ is a $\theta$-closed set if $cl_\theta(E) = E$. Similarly, many authors have defined local functions using various open sets and have studied them.

Analogous to that in this paper, we have defined local function using $\Delta$-open sets which was first introduced by M. Veera Kumar in [4]. If a set $E$ of $Y$ in $(Y, \tau)$ equals to $(B - C) \cup (C - B)$, where $B$ and $C$ are sets of $Y$ which are open if so $E$ is defined as $\Delta$-open. All $\Delta$-open set collection satisfies the topology criterion and is given by $\tau^D$ for $Y$.

We generally, get $\Delta$-closed sets from the complement of $\Delta$-open sets. The set $E$ of $Y$ is said to be $\Delta$-closed if $E = cl_D(E)$ where $cl_D(E) = \{ y \in Y : V \cap E \neq \phi, \text{ for every } V \in \tau^D(y) \}$.

In $(Y, \tau, I)$, $\tau$ is defined as compatible with $I$, expressed as $\tau \sim I$, if the axioms given below is true for all $E \subseteq Y$, if for all $y \in E$, where $V \in \tau(y)$ in such a way that $V \cap E \in I, \Rightarrow E \in I$. $T_\delta$ spaces were introduced by P.S. Alexandroff and H. Hopf in 1935. A space is a $T_\delta$ space [3] if and only if for two points that are not equal, precisely one will be contained in an open set. In 2007 M.N. Mukherjee et al. [6] introduced free ideal and defined it in the following way: Consider the space $(Y, cl^*)$ and $y \in Y, I_{cl^*}(y) = \{ A \subseteq Y : y \notin cl^*(A) \}$ is an ideal on $Y$ called a free ideal on $(Y, cl^*)$.

2. $D^*$-Local Functions

In this section, we introduce one new tool namely, $D^*$-local function and analyze its nature in ideal topological space.

**Definition 2.1.** Assume that $Y$ be an ideal space, $A \subseteq Y$. The operator $A_{D^*}(I, \tau) = \{ y \in Y : V \cap A \notin I, \forall V \in \tau^D(y) \}$ where $\tau^D(y) = \{ V \in \tau^D : y \in V \}$ is known as $D^*$-local function in $A$ related to $I, \tau$.

**Lemma 2.1.** $A_{D^*}(I, \tau) \subseteq A^*(I, \tau)$ always holds where $Y$ is in ideal topological space.

**Proof.** Suppose $y \in A_{D^*}$. Then $V \cap A \notin I \forall V \in \tau^D(y)$. However, we know that each open set is $\Delta$-open which implies $V \cap A \notin I \forall V \in \tau(y)$. Hence, $y \notin A^*$.

**Example 2.1.** Consider, $Y = \{ i, a, e, g \}$, 

$$\tau = \{ \phi, Y, \{ a \}, \{ e \}, \{ a, e \} \}$$

and

$$I = \{ \phi, \{ i \} \}.$$ 

Consider, $A = \{ a \} \Rightarrow A^* = \{ i, a \}$ and $A_{D^*} = \{ a \}$.

**Remark 2.1.** Generally, neither $A \subseteq A_{D^*}$ nor $A_{D^*} \subseteq A$.

**Example 2.2.** Consider, $Y = \{ k, i, e, f \}$, 

$$\tau = \{ \phi, Y, \{ k \}, \{ i \}, \{ k, i \}, \{ k, e, f \} \}$$

and.
\[ I = \{ \phi, \{ k \}, \{ f \}, \{ k, f \} \}. \]

Take \( A = \{ k, e \} \). Then \( A_D^* = \{ e, f \} \).

**Theorem 2.1.** Suppose \( E \) and \( F \) are subsets of \( Y \), an ideal space. If so the given conditions can be proved:

1. If \( E \subseteq F \), then \( E_D^* \subseteq F_D^* \).
2. \( (E_D^*)_D^* \subseteq E_D^* \).
3. \( E_D^* = \text{cl}_D(E_D^*) \subseteq \text{cl}_\theta(E) \) also \( E_D^* \) is closed.
4. If \( E \in I \), then \( E_D^* = \phi \).
5. \( E_D^* \subseteq \text{cl}_D(E) \).

**Proof.** (1) Assume that, \( x \notin F_D^* \). Then we can find a \( V \in \tau^D(x) \) provided \( F \cap V \in I \). \( E \) is a subset of \( F \Rightarrow E \cap V \subseteq F \cap V \), if so \( E \cap V \in I \). Therefore, \( x \notin E_D^* \Rightarrow E_D^* \subseteq F_D^* \).

(2) Let \( x \in (E_D^*)_D^* \). Then for all \( V \in \tau^D(x) \), \( V \cap E_D^* \notin I \Rightarrow V \cap E_D^* \neq \phi \). Assume \( z \in V \cap E_D^* \). Then \( V \in \tau^D(z) \) with \( z \in E_D^* \). Thus, \( V \cap E \notin I \) and \( x \in E_D^* \). This brings out \( (E_D^*)_D^* \subseteq A_D^* \).

(3) We know \( E_D^* \subseteq \text{cl}_D(E_D^*) \). We can find an element \( x \in \text{cl}_D(E_D^*) \Rightarrow E_D^* \cap V \neq \phi \) for all \( V \in \tau^D(x) \). Then we can choose one \( t \in E_D^* \cap V \) with \( V \in \tau^D(t) \). Perhaps \( t \in E_D^* \Rightarrow x \in E_D^* \Rightarrow \text{cl}_D(E_D^*) \subseteq E_D^* \) and we get \( E_D^* = \text{cl}_D(E_D^*) \). Again, let \( x \in E_D^* \), then \( E \cap V \notin I \) for all \( V \) belongs to \( \tau^D(x) \). Then for all \( V \) belonging to \( \tau^D(x) \) we get \( E \cap V \neq \phi \). Hence \( E \cap \text{cl}(V) \) is non-empty for all open set \( V \). Hence, \( x \in \text{cl}_\theta(E) \). This proves that \( E_D^* = \text{cl}(E_D^*) \subseteq \text{cl}_\theta(E) \).

(4) Choose \( x \in E_D^* \). Then we can find any \( V \in \tau^D(x) \), \( E \cap V \) does not belong to \( I \). But we have \( E \) belongs to \( I \Rightarrow E \cap V \in I \) \( \forall \ V \in \tau^D(x) \) which is absurd. Therefore, \( E_D^* = \phi \).

(5) Let \( x \in E_D^* \). Consequently, for each \( V \in \tau^D(x) \), \( V \cap E \notin I \) with \( \forall \ V \in \tau^D(x) \), \( V \cap E \neq \phi \). Thus, \( x \in \text{cl}_D(E) \). \( \square \)

**Theorem 2.2.** The ideal space \( Y \) containing ideals \( J_1, J_2 \) as well as and \( C \subseteq Y \). If so the following conditions are true:

1. If \( J_1 \) is a subset of \( J_2 \) then \( C_D^*(J_2) \) is a subset of \( C_D^*(J_1) \).
2. \( C_D^*(J_1 \cap J_2) \) equals to \( C_D^*(J_1) \cup C_D^*(J_2) \).

**Proof.** (1) Let \( x \in C_D^*(J_2) \) and \( J_1 \subseteq J_2 \). For each \( C \cap U \notin J_2 \) follows so does \( C \cap U \notin J_1 \). Therefore \( x \in C_D^*(J_1) \).

(2) We know \( C_D^*(J_1) \) is a subset of \( C_D^*(J_1 \cap J_2) \) and \( C_D^*(J_2) \subseteq C_D^*(J_1 \cap J_2) \) by (1). Hence, union of \( C_D^*(J_1) \) and \( C_D^*(J_2) \) is a subset of \( C_D^*(J_1 \cap J_2) \). We take \( x \in C_D^*(J_1 \cap J_2) \), for each \( \Delta \)-open \( U \), \( U \cap C \notin J_1 \cap J_2 \Rightarrow U \cap C \notin J_1 \) or \( U \cap C \notin J_2 \). Then \( x \in C_D^*(J_1) \) or \( x \in C_D^*(J_2) \). So \( x \in C_D^*(J_1) \cup C_D^*(J_2) \). \( \square \)
**Lemma 2.2.** Assume $Y$ be of $(Y, \tau, I)$ with $\Delta$-open set $V$. If so $V \cap C_{D'} = V \cap (V \cap C)_{D'} \subseteq (V \cap C)_{D'}$ for each subset $C$ of $Y$.

**Proof.** Let’s assume that when $V$ is a $\Delta$-open set with $y \in V \cap C_{D'} \Rightarrow y \in V$ and $y \in C_{D'}$ then $V \cap C \notin I$ for each $\Delta$-open set containing $y$. Assume $W$ to be any $\Delta$-open set comprising $y$ that is $W \in \tau^D(y)$. So $W \cap V \in \tau^D(y)$ and $W \cap (V \cap C) = (W \cap V) \cap C \notin I$. This expresses that $y \in (V \cap C)_{D'}$ which leads to the conclusion that $V \cap C_{D'}$ is a subset of $(V \cap C)_{D'}$. Additionally, $V \cap C_{D'} \subseteq V \cap (V \cap C)_{D'}$ and by Theorem [2.2] $(V \cap C)_{D'} \subseteq C_{D'}$ and $V \cap (V \cap C)_{D'} \subseteq V \cap C_{D'}$. Hence, $V \cap C_{D'} = V \cap (V \cap C)_{D'}$. □

**Theorem 2.3.** Assume that $Y$ is an ideal space with $E, G$ are any two sets of $Y$. If so, the given observations are true:

1. $\phi_{D'} = \phi$.
2. $E_{D'} \cup G_{D'} = (E \cup G)_{D'}$.
3. $(E \cap G)_{D'} \subseteq E_{D'} \cap G_{D'}$.

**Proof.** (1) It is obvious.

(2) From Theorem [2.2] it is evident that $(E \cup G)_{D'} \supseteq E_{D'} \cup G_{D'}$. To get the reverse part, assume $y \in (E \cup G)_{D'}$. Then for all $\Delta$-open set containing $y$, $V \cap (E \cup G) \notin I$. From this $(V \cap E) \notin I$ or $(V \cap G) \notin I$ for each $\Delta$-open set containing $y$. Hence $y \in E_{D'} \cup G_{D'} \Rightarrow (E \cup G)_{D'} \subseteq E_{D'} \cup G_{D'}$.

(3) $E \cap G$ is a subset of $E \Rightarrow (E \cap G)_{D'} \subseteq E_{D'}$ Similarly, $(E \cap G)_{D'}$ is a subset of $G_{D'}$ by Theorem [2.2]. So $(E \cap G)_{D'}$ is a set of $E_{D'} \cap G_{D'}$. □

**Remark 2.2.** The reverse implication of Theorem [2.5] (3) doesn’t always apply, as may be seen by an example.

**Example 2.3.** Consider, $Y = \{l, c, e, g\}$,

$\tau = \{\phi, Y, (l, c)\}$

and

$I = \{\phi, (l), (c), (l, c)\}$.

Suppose $C = \{e\}$ and $D = \{c, g\}$. Then $C_{D'} \cap D_{D'} = \{e, g\}$ and $(C \cap D)_{D'} = \phi$.

**Lemma 2.3.** Suppose $Y$ be an ideal space with $E, H \subseteq Y$. If so $E_{D'} - H_{D'}$ equals to $(E - H)_{D'} - H_{D'}$ is a set of $(E - H)_{D'}$.

**Proof.** We have by Theorem [2.5], $E_{D'} = [(E - H) \cup (E \cap H)]_{D'} = (E - H)_{D'} \cup (E \cap H)_{D'} \subseteq (E - H)_{D'} \cup H_{D'}$. Thus $E_{D'} - H_{D'} \subseteq (E - H)_{D'} - H_{D'}$. We have by Theorem [2.2], $(E - H)_{D'} \subseteq E_{D'}$ it follows $(E - H)_{D'} - H_{D'}$ is a subset of $E_{D'} - H_{D'}$. Henceforth, $E_{D'} - H_{D'}$ equals $(E - H)_{D'} - H_{D'} \subseteq (E - H)_{D'}$. □
Corollary 2.1. Assuming $Y$ is an ideal topological space with two subsets $G$ and $H$ in $Y$, $H \in I$. Then \((G \cup H)_D = G_D = (G - H)_D\) holds.

Proof. Though $H \in I$, by Theorem [2.2], we get $H_D = \phi$. By Previous Lemma 2.3, $G_D = (G - H)_D$ and by Theorem [2.5], we get $(G \cup H)_D = G_D \cup H_D = G_D$. \hfill \Box

Theorem 2.4. $E_D \subseteq \Gamma(E)$

Proof. By lemma 2.1, $E_D \subseteq E^*$ and we know that, $E^* \subseteq \Gamma(E)$ [1]. Therefore, $E_D \subseteq \Gamma(E)$. \hfill \Box

Theorem 2.5. $E_D \subseteq E_{\delta^*}$

Proof. By lemma 2.1, $E_D \subseteq E^*$ and we know that, $E^* \subseteq E_{\delta^*}$ [7]. Therefore, $E_D \subseteq E_{\delta^*}$.

3. The Open Sets of $\tau_D$

A closure operator $cl_D(E) = E \cup E_D$ is defined in this section and we prove the Kuratowski closure operator.

Theorem 3.1. Assume $Y$ be an ideal space, $cl_D(B) = B \cup B_D$ where $B$ and $C$ are sets of $Y$. If so the given observations hold:

1. $cl_D(\phi) = \phi$ and $cl_D(Y) = Y$.
2. $B \subseteq cl_D(B)$.
3. $cl_D(B \cup C) = cl_D(B) \cup cl_D(C)$.
4. $cl_D(cl_D(B)) = cl_D(B)$.

Proof. (1) Since $\phi \in Y, cl_D(\phi) = \phi_D \cup \phi = \phi, cl_D(Y) = Y_D \cup Y = Y$.

(2) We know that $B \subseteq B \cup B_D = cl_D(B)$.

(3) As a result of Theorem [2.5], $cl_D(B \cup C) = (B \cup C)_D = (B \cup C) = (B_D \cup C_D) \cup (B \cup C) = (B_D \cup B) \cup (C_D \cup C) = cl_D(B) \cup cl_D(C)$.

(4) $cl_D(cl_D(B)) = cl_D(B_D \cup B) = (B_D \cup B)_D \cup (B_D \cup B) = ((B_D)_D \cup B_D) \cup (B_D \cup B) = B_D \cup B = cl_D(B)$, by Theorem [2.5]. \hfill \Box

Remark 3.1. According to Theorem [3.1], $cl_D(E) = E \cup E_D$ is a Kuratowski closure operator. The open sets in $\Delta$ are referred to as $\tau_D$-open sets while its complement is referred to as $\tau_D$-closed sets. This topology is represented by $\tau_D$ and defined as $\tau_D = \{V \subseteq Y/ cl_D(Y - V) = Y - V\}$.

Theorem 3.2. Suppose $Y$ is an ideal space with $R, C$ as sets of $Y$. If so the given observations are true:

1. If $R$ is a set of $C$ $\Rightarrow cl_D(R) \subseteq cl_D(C)$.
2. $cl_D(R \cap C) \subseteq cl_D(R) \cap cl_D(C)$.
3. $cl_D(R) \subseteq cl^*(R)$. 
Proof. (1) Assume \( R \) is a set of \( C \) where \( \text{cl}_{D^r}(R) = R \cup R_{D^r} \subseteq C \cup C_{D^r} = \text{cl}_{D^r}(C) \) by Theorem [2.2].

(2) We had, \( R \cap C \) is a set of \( R \) and \( R \cap C \subseteq C \) then by (1), \( \text{cl}_{D^r}(R \cap C) \subseteq \text{cl}_{D^r}(R) \) and \( \text{cl}_{D^r}(R \cap C) \subseteq \text{cl}_{D^r}(C) \Rightarrow \text{cl}_{D^r}(R \cap C) \subseteq \text{cl}_{D^r}(R) \cap \text{cl}_{D^r}(C) \).

(3) \( \text{cl}_{D^r}(R) = R \cup R_{D^r} \subseteq R \cup R^* = \text{cl}^*(R) \) since \( R_{D^r} \subseteq R^* \). \( \square \)

**Definition 3.1.** Consider, the space \( (Y, \text{cl}_{D^r}) \) and for \( y \in Y \), \( I_{\text{cl}_{D^r}}(y) = \{ E \subseteq Y : y \notin \text{cl}_{D^r}(E) \} \) is an ideal on \( Y \) called a free ideal on \( (Y, \text{cl}_{D^r}) \)

**Theorem 3.3.** A space \( (Y, \text{cl}_{D^r}) \) is a \( T_o \) space \( \iff \) for any two points \( w, z \) of \( Y \) and \( w \neq z \), \( I_{\text{cl}_{D^r}}(w) \neq I_{\text{cl}_{D^r}}(z) \).

**Proof.** Assume \( (Y, \text{cl}_{D^r}) \) is a \( T_o \) space. Suppose \( w \notin \text{cl}_{D^r}(E) \) then \( z \in \text{cl}_{D^r}(E) \). But if \( w \notin \text{cl}_{D^r}(E) \) \( \Rightarrow E \in I_{\text{cl}_{D^r}}(w) \). Also \( z \in \text{cl}_{D^r}(E) \Rightarrow E \notin I_{\text{cl}_{D^r}}(z) \). Therefore, \( I_{\text{cl}_{D^r}}(w) \neq I_{\text{cl}_{D^r}}(z) \). Conversely, let \( I_{\text{cl}_{D^r}}(w) \neq I_{\text{cl}_{D^r}}(z) \). Suppose \( E \subseteq Y \) such that \( E \in I_{\text{cl}_{D^r}}(w) \) but \( E \notin I_{\text{cl}_{D^r}}(z) \). Implies \( w \notin \text{cl}_{D^r}(E) \) and \( z \in \text{cl}_{D^r}(E) \) for some \( E \subseteq Y \). Therefore, \( w \in Y - \text{cl}_{D^r}(E) \) but \( z \notin Y - \text{cl}_{D^r}(E) \Rightarrow (Y, \text{cl}_{D^r}) \) is a \( T_o \) space. \( \square \)

**Remark 3.2.** \( I_{\text{cl}^*}(y) \subseteq I_{\text{cl}_{D^r}}(y) \)

**Example 3.1.** Consider, \( Y = \{ p, b, s, d \} \) and \( \tau = \{ \phi, Y, \{ p \}, \{ s \}, \{ d \}, \{ p, s \}, \{ p, d \}, \{ s, d \}, \{ p, s, d \} \} \).

Then \( I_{\text{cl}^*}(b) = \{ \phi, \{ p \} \} \) and \( I_{\text{cl}_{D^r}}(b) = \{ \phi, \{ s \}, \{ d \}, \{ p, s \}, \{ p, d \}, \{ s, d \}, \{ p, s, d \} \} \).

4. \( D^* \)-Compatibility

**Definition 4.1.** Assume \( Y \) is an ideal space. If the given conditions are true for all \( E \subseteq Y \) then \( F \in I \). If for each \( y \in F \), then \( \exists \) a \( \Delta \)-open set in a way that \( V \cap F \in I \) then \( \tau \) is \( D^* \)-compatible with the ideal \( I \), indicated by \( \tau \sim_D I \).

**Theorem 4.1.** Suppose \( Y \) be an ideal space, if so the observations given below can be equivalent:

1. \( \tau \sim_D I \),
2. \( E \in I \), if \( E \) a set of \( Y \) has a \( \Delta \)-open cover, which intersects \( E \) is in \( I \),
3. For all \( E \) set of \( Y \), \( E_{D^r} \cap E = \phi \Rightarrow E \) belongs to \( I \),
4. For each \( E \subseteq Y \), \( E - E_{D^r} \in I \),
5. For all \( E \subseteq Y \) does not contains nonempty subset \( D \) with \( D \subseteq D_{D^r} \), then \( E \in I \).

**Proof.** (1) \( \Rightarrow \) (2): The result is true obviously.

(2) \( \Rightarrow \) (3): Suppose, \( y \in E \) with \( E \subseteq Y \). If \( y \notin E_{D^r} \), we can find \( V \in \tau^D(y) \) provided \( V \cap E \in I \). Hence \( E \in I \) since we have \( E \subseteq \cup \{ V_y : y \in E \} \) for \( V_y \) is in \( \tau^D(x) \) containing \( y \).
(3) \(\Rightarrow\) (4): For each \(E\) which is a set of \(Y\), \(E - E_{D'}\) is a set of \(E\) if so, \((E - E_{D'}) \cap (E - E_{D'})_{D'}\) set of \((E - E_{D'}) \cap E_{D'} = \phi\), since \(E - E_{D'} \in I\).

(4) \(\Rightarrow\) (5): For each \(E \subseteq Y, E - E_{D'} \in I\) by (4). Consider \(E - E_{D'} = J \in I\), then \(E = J \cup (E \cap E_{D'})\) and according to Theorem [2.5] and [2.2], \(E_{D'} = J_{D'} \cup (E \cap E_{D'})_{D'} = (E \cap E_{D'})_{D'}\). Then, we have \(E \cap E_{D'} = E \cap (E \cap E_{D'})_{D'}\) is a subset of \((E \cap E_{D'})_{D'}\) and \(E \cap E_{D'}\) is a set of \(E\). On the basis of the assumption \(E \cap E_{D'} = \phi\) and hence \(E = E_{D'} \in I\).

(5) \(\Rightarrow\) (1): Let \(E \subseteq Y\) with each \(y \in E, \exists V \in \tau^D(y)\) in such a way that \(V \cap E \in I \Rightarrow E \cap E_{D'} = \phi\). Also assume that \(E\) contains \(D\) such that \(D \subseteq D_{D'}\). Then \(D = D \cap D_{D'} \subseteq E \cap E_{D'} = \phi\). Hence, \(E\) does not contain nonempty subset \(D\) with \(D \subseteq D_{D'}\). Hence, \(E \in I\). \(\Box\)

**Theorem 4.2.** The given statements can be equivalent for an ideal space \(Y\) and if \(\tau\) is \(D'\) compatible in \(I\).

1. If for all \(E \subseteq Y, E \cap E_{D'} = \phi \Rightarrow E_{D'} = \phi\),
2. If every \(E \subseteq Y\), then \((E - E_{D'})_{D'} = \phi\),
3. If every \(E \subseteq Y\), then \((E \cap E_{D'})_{D'} = E_{D'}\).

**Proof.** (1) \(\Rightarrow\) (2): For each \(E \subseteq Y, E \cap E_{D'} = \phi \Rightarrow E_{D'} = \phi\). Let \(F = E - E_{D'}\), so \(F \cap F_{D'} = (E \cap (Y - E_{D'})) \cap (E \cap (Y - E_{D'}))_{D'} \subseteq [E \cap (Y - E_{D'})] \cap [E_{D'} \cap (Y - E_{D'})_{D'}] = \phi\). By (1), we have \(F_{D'} = \phi \Rightarrow (E - E_{D'})_{D'} = \phi\).

(2) \(\Rightarrow\) (3): For each \(E \subseteq Y, E = (E - E_{D'}) \cup (E \cap E_{D'})\). \(E_{D'} = (E - E_{D'})_{D'} \cup (E \cap E_{D'})_{D'} = (E \cap E_{D'})_{D'}\).

(3) \(\Rightarrow\) (1): For all \(E \subseteq Y, E \cap E_{D'} = \phi\) and \((E \cap E_{D'})_{D'} = E_{D'} \Rightarrow \phi = \phi_{D'} = E_{D'}\). \(\Box\)

**Corollary 4.1.** Suppose \(Y\) be an ideal space with \(E\) set of \(Y\) and \(\tau\) is \(D'\)-compatible in ideal \(I \Rightarrow E_{D'} = (E_{D'})_{D'}\).

**Proof.** Assume \(E \subseteq Y\), using Theorem [4.2] and the result in Theorem [2.2], we get \(E_{D'} = (E \cap E_{D'})_{D'} \subseteq E_{D'} \cap (E_{D'})_{D'} = (E_{D'})_{D'}\). Hence we could have \(E_{D'} \subseteq (E_{D'})_{D'}\). Then by the result of Theorem [2.3], \((E_{D'})_{D'}\) is a subset of \(E_{D'} \Rightarrow E_{D'} = (E_{D'})_{D'}\). \(\Box\)

**Theorem 4.3.** The results can be equivalent to an ideal space \(Y\)

1. \(\tau^D(y) \cap I = \phi\), such that \(\tau^D(y) = \{V \in \tau^D / y \in V\}\),
2. When \(J \in I \Rightarrow int_{\theta}(J) = \phi\),
3. \(F \subseteq F_{D'}\), For each clopen \(F\),
4. \(Y = Y_{D'}\).

**Proof.** (1) \(\Rightarrow\) (2): Assume \(J \in I\) and \(\tau^D(y) \cap I = \phi\). Also, assume \(y \in int_{\theta}(I)\). Then \(\exists W\) belongs to \(\tau^D(y)\) in such a way that \(y \in W \subseteq cl(W) \subseteq I\). Since \(J \in I \Rightarrow \phi \neq [y] \subseteq cl(W) \in \tau^D(y) \cap I\). This is a contradiction to \(\tau^D(y) \cap I = \phi\). Hence, \(int_{\theta}(J) = \phi\).
(2) ⇒ (3): Let $y \in F$. Assume $y \notin F_{D^*}$, such that $V \in \tau_{D^*}(y)$ in a way that $F \cap V \in I$. As $F$ is clopen, by [1], $y \in F \cap V = \text{int}_0(F \cap V) = \text{int}_0(F \cap (V)) = \phi$. This becomes a contradiction ⇒ $y \in F_{D^*}$. Hence $F \subseteq F_{D^*}$.

(3) ⇒ (4): When $Y$ is clopen ⇒ $Y \subseteq Y_{D^*}$ then by (3) $Y = Y_{D^*}$.

(4) ⇒ (1): Assume $Y = Y_{D^*} = \{y \in Y \mid V \cap Y = V \notin I \text{ where } V \in \tau_{D^*}(y)\}$. So $\tau_{D^*}(y) \cap I = \phi$. □

**Theorem 4.4.** Let $Y$ be in $(Y, \tau, I)$ and $\tau$ is of $D^*$-compatible of $I$. If so for each $S \in \tau_{D^*}$, any set $T$ of $Y$, $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*} = \text{cl}_D(S \cap T_{D^*})$.

**Proof.** Let $S \in \tau_{D^*}$. Then by Lemma [2.4] and by Theorem[2.2] $(S \cap T_{D^*})_{D^*} \subseteq ((S \cap T)_{D^*})_{D^*} \subseteq (S \cap T)_{D^*}$. Also, $(S \cap (T - T_{D^*}))_{D^*} \subseteq S_{D^*} \cap (T - T_{D^*})_{D^*} = S_{D^*} \cap \phi = \phi$ by Theorem [2.2] and [4.2]. Moreover, $(S \cap T)_{D^*} - (S \cap T_{D^*})_{D^*} \subseteq ((S \cap T) - (S \cap T_{D^*}))_{D^*} = (S \cap (T - T_{D^*}))_{D^*} = \phi \Rightarrow (S \cap T)_{D^*} \subseteq (S \cap T)_{D^*}$. Hence, $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*}$. Also, $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*} \subseteq \text{cl}_D(S \cap T_{D^*})$, by Theorem [2.2]. By Lemma [2.4], $S \cap T_{D^*} \subseteq (S \cap T)_{D^*} \Rightarrow \text{cl}_D(S \cap T_{D^*}) \subseteq \text{cl}_D((S \cap T)_{D^*}) = (S \cap T)_{D^*}$. Thus $(S \cap T)_{D^*} = (S \cap T_{D^*})_{D^*} = \text{cl}_D(S \cap T_{D^*})$. □

5. $\eta$-Operator

**Definition 5.1.** An operator $\eta: \mathcal{P}(Y) \to \tau$ in an ideal space $Y$ is given by the following: for each $E$ belongs to $Y$, $\eta(E) = \{y \in Y : \exists$ a $\Delta$-open set $V$ in a way that $V - E \in I\}$ and also we have $\eta(E) = Y - (Y - E)_{D^*}$.

**Theorem 5.1.** Suppose $Y$ is an ideal space. If so then the given statements are true:

1. If $M \subseteq Y$, then $\eta(M)$ is open.
2. If $M$ is a subset of $E \Rightarrow \eta(M) \subseteq \eta(E)$.
3. If $M, E \in \mathcal{P}(Y)$, then $\eta(M \cap E)$ equals $\eta(M) \cap \eta(E)$.
4. If $M \subseteq Y$, then $\eta(M)$ equals $\eta(\eta(M))$ if and only if $(Y - M)_{D^*} = ((Y - M)_{D^*})_{D^*}$.
5. If $M \in I$, then $\eta(M)$ equals $Y - Y_{D^*}$.
6. If $M \subseteq Y$ and $J \in I \Rightarrow \eta(M - J) = \eta(M)$.
7. If $M$ set of $Y$, $J \in I \Rightarrow \eta(M \cup J) = \eta(M)$.
8. If $(M - E) \cup (E - M) \in I$, then $\eta(M) = \eta(E)$.

**Proof.** (1) By Theorem [2.2], $M_{D^*}$ is closed ⇒ $(Y - M)_{D^*}$ is closed ⇒ $\eta(M)$ is open.

(2) If $M \subseteq E$ in such a way that $(Y - E)$ is a set of $(Y - M) \Rightarrow (Y - E)_{D^*} \subseteq (Y - M)_{D^*}$. Hence $Y - (Y - M)_{D^*}$ is a set of $Y - (Y - E)_{D^*}$, that is $\eta(M) \subseteq \eta(E)$.

(3) $\eta(M \cap E) = Y - ((Y - (M \cap E))_{D^*} = Y - [(Y - M) \cup (Y - E)]_{D^*} = Y - [(Y - M)_{D^*} \cup (Y - E)_{D^*}] = [Y - (Y - M)_{D^*}] \cap [Y - (Y - E)_{D^*}] = \eta(M) \cap \eta(E)$. 


(4) If \((Y - M)_{D^r} = ((Y - M)_{D^r})_{D^r}\) then, \(\eta(M) = (Y - (Y - M)_{D^r})_{D^r} = Y - (Y - (Y - M)_{D^r})_{D^r} = \eta(M)\).

(5) By Corollary [2.6.1], we get \((Y - M)_{D^r} = Y_{D^r}\). Therefore, \(\eta(M) = Y - Y_{D^r}\).

(6) \(\eta(M - J) = Y - [(Y - (M - J)]_{D^r} = Y - [(Y - M) - J]_{D^r} = Y - (Y - M)_{D^r} = \eta(M)\), by using Corollary [2.6.1].

(7) \(\eta(M \cup J) = Y - [(Y - (M \cup J)]_{D^r} = Y - [(Y - M) \cap (Y - J)]_{D^r} = Y - [(Y - M) - J]_{D^r} = Y - (Y - M)_{D^r} = \eta(M)\), by using Corollary [2.6.1].

(8) Assume that, union of \((M - E)\) and \((E - M)\) belongs to \(I\). Let \(M - E = I\) and \(E - M = J\). We observed that \(I, J\) belongs to \(I\) by the property heredity. Also, we can have \(E = (M - I) \cup J\). Thus, \(\eta(M) = \eta(M - I) = \eta[(M - I) \cup J] = \eta(E)\) by (6) and (7). \(\square\)

**Corollary 5.1.** Suppose \(Y\) is in \((Y, \tau, I)\). For each \(\theta\)-open \(W\) set of \(Y\), \(W \subseteq \eta(W)\).

**Proof.** We have \(\eta(W) = Y - (Y - W)_{D^r}\). Then \((Y - W)_{D^r} \subseteq \text{cl}_\theta(Y - W) = Y - W\), though \(Y - W\) is \(\theta\)-closed. We can have \(W = Y - (Y - W) \subseteq Y - (Y - W)_{D^r} = \eta(W)\). \(\square\)

6. **Conclusion**

In this paper, we have defined the local function using \(\Delta\)-open sets and studied its various properties in ideal topological space. We defined a closure operator and determined whether it was a Kuratowski closure operator. Also, the compatibility property of the topology with the ideal was verified. We then defined a new operator \(\eta\) and its characterizations were studied. Future work in this topic would be to generalize the \(\Delta\)-closed set, compare it with the already existing generalized closed sets, and verify some of its properties.

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**References**