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On the Affine and Affine Polarized $k-$Symplectic Manifolds

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Abstract. We are interested in studying the affine and the affine polarized $k-$symplectic manifolds exploiting the action of the polarized $k-$symplectic group $Sp(k,n;\mathbb{R})$, especially when it acts properly and discontinuously without fixed point in order to construct a particular class of affine polarized $k-$symplectic manifolds of odd dimension $2k^{'}+1$ with $k^{'} \in \mathbb{N}^{*}$ which will be a generalization of the case of dimension 3 studied in [8].

1. Introduction

Physical and mathematical considerations led to the introduction of the notion of polarized k-symplectic structure [3, 4, 10], indeed, polarized k-symplectic geometry is a generalization of polarized symplectic geometry formalizing the mechanics of Numbu like symplectic geometry which formalizes Hamiltonian mechanics.

An affine manifold is a manifold provided with an atlas whose map changes are affine transformations. Map changes of an affine manifold of dimension m are affine transformations $x \rightarrow Ax + B$ of $\mathbb{R}^m$. When we impose on the matrix A to belong to a subgroup of $GL(m,\mathbb{R})$, we obtain a particular class of affine manifolds. One can cite for example the flat Riemannian manifolds: A is orthogonal; flat Lorentz manifolds: $A \in O(m-1,1)$; flat affine manifolds with a parallel volume (we also say unimodular): $A \in SL(m,R)$. Thus from the point of view of related manifolds, this work can be considered as an enrichment of the classes of manifolds studied. We will see that the study of affine polarized $k-$symplectic manifolds leads naturally to the study of affine transformations $x \rightarrow Ax + B$ with A being an element of the polarized $k-$symplectic group $Sp(k,n;\mathbb{R})$, and

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by the end we manage to construct a particular class of affine polarized $k-$symplectic manifolds of odd dimension $2k^{'}+1$ with $k^{'}\in\mathbb{N}^{*}$ which will be a generalization of the case of dimension 3 developed in [8].

2. Preleminaries

2.1. **Polarized $k-$symplectic manifolds.**

Let M be a differentiable manifold of dimension $n(k+1)$.

Definition 2.1. *[7] We say that the triple $(M,\omega,\mathfrak{F})$ is a polarized $k-$symplectic manifold, if:*

(1) *ω is a differential $2-$form over M with values in $\mathbb{R}^{k}$: $\omega\in A^{2}(M)\otimes\mathbb{R}^{k}$.*
(2) *ω is closed and non degenerate.*
(3) *$\mathfrak{F}$ is a foliation of codimension n and $\omega(X,Y)=0\ \forall X,Y\in T(\mathfrak{F})$ $(X,Y$ tangent to $\mathfrak{F})$.*

Remark 2.1. *Let E be the sub-bundle of TM defined by the vectors tangent to the leaves of $\mathfrak{F}$, then the couple (ω,E) is called a polarized $k-$symplectic structure on M.*

Definition 2.2. *[7] Let $(M,\omega,\mathfrak{F})$, $(N,\theta,\mathfrak{F}')$ two polarized $k-$symplectic manifolds and $\varphi:M\to N$.*

We say that φ is a polarized $k-$symplectomorphism if φ is a diffeomorphism of M on N that exchanges the polarized $k-$symplectic structures $(\omega,\mathfrak{F})$ and $(\theta,\mathfrak{F}')$,
i.e:

$$\varphi^{*}\theta=\omega \text{ and } \varphi(\mathfrak{F})=\mathfrak{F}'$$

2.2. **The polarized $k-$symplectic group.**

Let V be a vector space of dimension $n(k+1)$ over $\mathbb{K}$, and let $(\omega;E)$ with $\omega\in A^{2}(M)\otimes\mathbb{R}^{k}$ be a polarized $k-$symplectic structure on V and φ be an endomorphism of V.

Definition 2.3. *[5] We say that φ preserves the polarized $k-$symplectic structure $(\omega;E)$ if it leaves invariant both the form ω and the subspace E.*

The set of all automorphisms of V leaving (ω,E) invariant, is a Lie group, denoted by $Sp(k,n;V)$ and called polarized $k-$symplectic group of V.
Let $Sp(k,n;\mathbb{R})$ be the group of all matrices of polarized $k-$symplectic automorphisms of V expressed in the polarized $k-$symplectic basis $(e_{p_i},e_i)_{1\le p\le k,1\le i\le n}$. The group $Sp(k,n;\mathbb{R})$ consists of the matrices of the type:

$$\begin{pmatrix} T & 0 & \cdots 0 & S_1 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & T & S_k \\ 0 & \cdots & 0 & (T^{-1})^{t} \end{pmatrix}=[(T,S_1,\dots,S_k)],$$

with $T\in GL_n(\mathbb{R})$, $S_1,...,S_k\in\mathfrak{gl}_n(\mathbb{R})$ and $S_pT^{t}\in S_n(\mathbb{R})$ for each $p(p=1,\dots,k)$.
The product in $Sp(k,n;\mathbb{R})$ is given by:

$$[(T,S_1,\dots,S_k)]\times[(Q,R_1,\dots,R_k)]=\Big[\Big(TQ,TR_1+S_1(Q^{-1})^{t},\dots,TR_k+S_k(Q^{-1})^{t}\Big)\Big]$$

2.3. **Group action and quotient manifold.**

Definition 2.4. *[11] A group Γ acts properly and discontinuously on a manifold M if:*

(1) *Every point $p \in M$ has a neighborhood U such that:*

$$card\ \{g \in \Gamma; gU \cap V \neq 0\} < \infty$$

(2) *If $p, q \in M$ are not in the same -orbit, there exist neighborhoods U of p and V of q such that:*

$$U \cap \Gamma V = \phi$$

Remark 2.2. *[12] When the action is free, the condition (1) is equivalent to:*
Every point $p \in M$ has a neighborhood U such that:

$$gU \cap U \neq \phi \Rightarrow g = identity\ of\ \Gamma.$$

Recall that, if G is a Lie group and Γ a discrete subgroup, then Γ acts properly and discontinuously on G. In addition, a complete locally affine manifold M of dimension n is the quotient of $\mathbb{R}^n$ by a subgroup of the affine group $A(n)$ of $\mathbb{R}^n$, which acts properly and discontinuously without fixed point on $\mathbb{R}^n$. Thus

$$M = \mathbb{R}^n/\Gamma, \quad \Gamma = \pi_1(M)$$

Furthermore, two compact complete locally affine manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

3. Affine manifolds

Let M be a real smooth manifold of dimension n. We say that M admit an affine structure if there exists a smooth atlas $\left\{(U_\alpha, \varphi_\alpha)_{\alpha \in I}\right\}$ of M such that the coordintes changes are affine transformations of $\mathbb{R}^n$.

Definition 3.1. *A manifold equipped with an affine structure is called an affine manifold.*
This means that a manifold is affine if it admit a smooth atlas $\mathfrak{A} = \left\{(U_\alpha, \varphi_\alpha)_{\alpha \in I}\right\}$ such as for all $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, there exists an affine transformation $\sigma_{\alpha\beta} \in GA(\mathbb{R}^n)$ such that:

$$\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)_{|\varphi_\alpha\left(U_\alpha \cap U_\beta\right)} = \sigma_{\alpha\beta|\varphi_\alpha\left(U_\alpha \cap U_\beta\right)},$$

where $GA(\mathbb{R}^n)$ is the affine group of $\mathbb{R}^n$ [2].

We denote by $\Gamma(TM)$ the space of differentiable vector fields over the manifold M. The tensors of curvature and torsion a given connection ∇, are given by: [1]

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \Gamma(TM)$.
A characterization of affine manifolds is given by the following classical theorem.

Theorem 3.1. *An affine structure on M is equivalent to a given connection*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

without curvature and torsion.

Proof. Let $\mathfrak{A}$ be an affine structure on M. There is a unique connection ∇ on TM whose Christoffel symbols vanish on each $(U;(x_1,\cdots,x_n)) \in \mathfrak{A}$. This connection is given on U by:

$$\nabla_X Y = X(Y^i)\frac{\partial}{\partial x_i}$$

where $X, Y = Y^i\frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$.

Conversely, Let ∇ be a flat torsion free connection on M. The set $\mathfrak{A}$ of coordinates, for which the Christoffel symbols of ∇ vanish, gives an affine structure on M. In fact, the existence of these local coordinates is ensured by the existence of normal coordinates, see for example [9].

Now, let $(U;(x_1,\cdots,x_n))$, $\left(U';\left(x'_1,\cdots,x'_n\right)\right) \in \mathfrak{A}$. The transformation laws of Christoffel symbols of ∇ on $U \cap U'$ give:

$$\Gamma'^{u}_{ij} = \frac{\partial x'_u}{\partial x_r}\frac{\partial^2 x_r}{\partial x'_i \partial x'_j} + \frac{\partial x'_u}{\partial x_r}\frac{\partial x_k}{\partial x'_j}\frac{\partial x_l}{\partial x'_i}\Gamma^r_{lk}.$$

Since $\Gamma'^{u}_{ij} = 0 = \Gamma^r_{lk}$, then $\frac{\partial^2 x_r}{\partial x'_i \partial x'_j} = 0$, consequently, $x_r = a^i_r x'_j + b_r$, where $\left(a^i_r\right) = \left(\frac{\partial x_r}{\partial x'_j}\right) \in GL(n,\mathbb{R})$ and $b_r \in \mathbb{R}$.

□

Example 3.1. *The circle $\mathbb{S}^1$ can be equipped by an atlas whose mappings of transition are affine.*

Let α be the parametrization: $t \longrightarrow \alpha(t) = (\cos t; \sin t)$.

We remark that $\alpha_{|]0;2\pi[}$ and $\alpha_{|]-\pi;\pi[}$ are respectively homeomorphisms on $V_1 = \mathbb{S}^1 - \{(1;0)\}$ and $V_2 = \mathbb{S}^1 - \{(-1;0)\}$.

Denote by φ_1 and φ_2 their inverse homeomorphisms.

Note that $V_1 \bigcap V_2$ has two connected components V^+ and V^-. Therefore:

$$\varphi_1(V_1 \bigcap V_2) = \varphi_1(V^+) \bigcup \varphi_1(V^-) =]0;\pi[\bigcup]\pi;2\pi[$$

and

$$\varphi_2 \circ \varphi_1^{-1} = t \text{ if } t \in]0;\pi[\text{ or } t - 2\pi \text{ if } t \in]\pi;2\pi[.$$

In fact, $\mathbb{S}^1$ is the only unique affine sphere, according to the proposition below.

Let M and N two affine manifolds of the same dimension n.

A mapping $f : M \to N$ is called affine if for any affine charts (V,ψ) of N and (U,φ) of M, $\psi \circ f \circ \varphi^{-1}$ is the restriction of an affine transformation of $\mathbb{R}^n$. Such a mapping is obviously a local diffeomorphism.

Let $p : \widetilde{M} \longrightarrow M$ be a universal covering of a connected manifold M. If M admits an affine structure, then there exists a unique affine structure on $\widetilde{M}$ for which p is a morphism of affine manifolds. Moreover, there exists a canonical affine mapping $D : \widetilde{M} \to \mathbb{R}^n$, called developing of M, and a representation $h : \pi_1(M) \to GA\left(\mathbb{R}^n\right)$ such that:

$$D \circ \gamma = h(\gamma) \circ D,$$

for all γ in $\pi_1(M)$. The mapping h is called a representation of holonomy of M. We recall the construction of the developing of an affine manifold M as follows: let x_0 be a fixed point of M and let $x \in M$. We fix a continuous path $(x_t)_{0\leq t\leq 1}$ joining x_0 to $x_1 = x$. Also, we cover this path by domains of affine charts U_i, $0 \leq i \leq n$, and let real numbers $0 < a_1 < b_0 < a_2 < b_1 < \cdots < a_n < b_{n-1} < 1 < b_n$, such that, $x_t \in U_i$, for all $t \in]a_i, b_i]$. We denote by $\psi_i : U_i \to \mathbb{R}^n$ the affine coordinates and let σ_i the unique affine transformation of $\mathbb{R}^n$ such as $\sigma_i \circ \psi_i$ and ψ_{i-1} coincide on the connected component of $U_i \cap U_{i-1}$ containing the curve $(x_t)_{(a_i<t<b_{i-1}).}$, we take

$$D(x) = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \sigma_n \psi(x) \in \mathbb{R}^n.$$

We verify that D is well defined, i.e. $D(x)$ does not depend on the cover (U_i) nor the chosen path $(x_t)_{0\leq t\leq 1}$.

We give here a direct consequence of the existence of a developing mapping.

Theorem 3.2. *If M is simply connected compact manifold, then M does not admit an affine structure.*

Proof. By above results, if M admits an affine structure, then it admits a developing mapping $D : M \to \mathbb{R}^n$ which is a local diffeomorphism. As M is compact, $D(M)$ would be an open compact of $\mathbb{R}^n$, which is absurd.

□

3.1. **Complete affine manifolds and crystallographic groups.** A geodesic of an affine manifold M is an affine application $\mathbb{R} \to M$. The manifold M is said to be complete if any segment of a geodesic can be extended into a geodesic or equivalently if the universal covering $\widetilde{M}$ of M is affinely diffeomorphic to $\mathbb{R}^m$. It is then known (see [13]) that the fundamental group $\pi_1(M)$ of M acts on $\mathbb{R}^m$ as a group Γ of affine transformatios, that this action is properly discontinuous and without fixed points and thus that the manifold M is the quotient $M = \mathbb{R}^m/\Gamma$.

For non-complete affine manifolds the situation is more delicate. Giving oneself an affine structure amounts to giving oneself a homomorphism $h : \pi_1(M) \to Aff(\mathbb{R}^m)$ and an immersion $D : \widetilde{M} \to \mathbb{R}^n$ equivariant, i.e.

$$D \circ \gamma = h(\gamma) \circ D,$$

for all $\gamma \in \pi_1(M)$. The immersion D is called the development of the affine manifold M and h its holonomy.

4. Affine polarized $k-$symplectic manifolds

Let $A_k(k,n;\mathbb{R})$ be the group of all affine transformations of $\mathbb{R}^{n(k+1)}$ leaving invariant the canonical polarized $k-$symplectic structure of $\mathbb{R}^{n(k+1)}$. Then:

$$A_k(k,n;\mathbb{R}) = \{X \longmapsto AX + B; \;\; A \in Sp(k,n;\mathbb{R})\}$$

with $X \in \mathbb{R}^{n(k+1)}$ and B a column vector. [6]

Furthermore, let M be a differentiable manifold of dimension $n(k+1)$. M is a complete connected affine polarized $k-$symplectic manifold,

then:

$$M = \mathbb{R}^{n(k+1)}/\Gamma, \quad \text{with } \pi_1(M) = \Gamma$$

where Γ is a subgroup of $A_k(k,n;\mathbb{R})$ acting properly and discontinuously without fixed point on $\mathbb{R}^{n(k+1)}$.

4.1. Case where the foliation $\mathfrak{F}$ is of codimension 1.

Let $H_p(k,n;\mathbb{R})$ be the group of all matrices

$$\begin{pmatrix} I_n & & & S_1 & R_1 \\ & \ddots & & \vdots & \vdots \\ 0 & & I_n & S_k & R_k \\ 0 & \cdots & 0 & I_n & Q \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

where $S_1, \cdots, S_n \in S(\mathbb{R})$ and $R_1, \ldots, R_k, Q$ are column vectors of length n.
We denote by (S,Q,R) the matrices of the previous form where $S = (S_1, \ldots, S_k)$, $R = (R_1, \ldots, R_k)$.
Then, we have the following proposition:

Proposition 4.1. *[6] if M is a differentiable manifold of dimension $(k+1)$. M is a complete connected affine polarized k−symplectic manifold, implies that:*

$$M = \mathbb{R}^{(k+1)}/\Gamma, \quad \textit{with} \quad \pi_1(M) = \Gamma$$

where Γ is a subgroup of $H_p(k,1;\mathbb{R})$ acting properly and discontinuously without fixed point on $\mathbb{R}^{(k+1)}$.

4.2. A particular class of affine polarized k-symplectic manifolds.

Based on the proposition above, and by following the same approach in our article [8], we manage to construct a class of affine polarized $k-$symplectic manifolds of odd dimension $2k^{'}+1$ with $k^{'} \in \mathbb{N}^*$ which will be a generalization of the case of dimension 3. by constructing a family of subgroups of $Hp(2k^{'},1;\mathbb{Z})$ which act freely and properly discontinuously on $\mathbb{R}^{2k^{'}+1}$.

The group $Hp(2k^{'},1;\mathbb{Z})$ is formed by the matrices of the form:

$$\begin{pmatrix} 1 & & & a_1 & c_1 \\ 0 & \ddots & & \vdots & \vdots \\ 0 & & 1 & a_{2k'} & c_{2k'} \\ 0 & \cdots & 0 & 1 & b \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

We denote by (A,b,C) the matrices above, where

$$A=(a_1;\ a_2\ldots;a_{2k'}),C=(c_1;\ c_2;\ldots;c_{2k'})\in\mathbb{Z}^{2k^{'}}\ and\ b\in\mathbb{Z}$$

For all $g=(A,b,C)$, $g^{'}=(A^{'},b^{'},C^{'})\in H_p(2k^{'},1;\mathbb{Z})$, as in [8], we have:

$$gg^{'}=(A+A^{'},b+b^{'},b^{'}A+C+C^{'})$$

$$g^n=(nA,nb,P_nb+nC)$$

where $n\in\mathbb{Z}$ and $P_n=\frac{n(n-1)}{2}$.

$$g^{-1}=(-A,-b,bA-C)$$

$$\left[g,g^{'}\right]=(0,0,b^{'}A-bA^{'}).$$

which allows us to generalize the same propositions of the 3− dimensional affine polarized 2−symplectic manifolds to the case of the affine polarized k−symplectic manifolds of odd dimension where $n=1$ and $k=2k^{'}$ with $k^{'}\in\mathbb{N}^*$. Therefore:

Proposition 4.2. *The subgroups Γ of $Hp(2k^{'},1;\mathbb{Z})$ of the type:*

$$\Gamma=<(A^0,0,C^0),\ (m_iA^0,0,C^i),\ldots,\ (A^{2k^{'}},b_{2k'},C^{2k^{'}})>$$

with $A^0,C^0,C^i,A^{2k^{'}},C^{2k^{'}}\in\mathbb{Z}^{2k^{'}}$, $b_{2k'}\in\mathbb{N}^*$, $m\in\mathbb{Q}$, and $1\le i<2k^{'}$ satisfying:

$$det(C^0,C^i)\neq 0,\ \det(A^0,C^i)=m_i\ det(A^0,C^0)$$

and their subgroups are the all subgroups of $Hp(2k^{'},1;\mathbb{Z})$ act freely and properly discontinuously without fixed point on $\mathbb{R}^{2k^{'}+1}$.

Proposition 4.3. *For all $A^0,C^0,C^i,A^{2k^{'}},C^{2k^{'}}\in\mathbb{Z}^{2k^{'}}$, $b\in\mathbb{N}^*$, $m\in\mathbb{Q}$, and $1\le i<2k^{'}$ satisfying:*

$$det(C^0,C^i)\neq 0,\ \det(A^0,C^i)=m_i\ det(A^0,C^0)$$

we denote by $M(A^0,C^0,C^i,A^{2k^{'}},C^{2k^{'}})$ the quotient manifold:

$$M(A^0,C^0,C^i,A^{2k^{'}},C^{2k^{'}})=\mathbb{R}^{2k^{'}+1}/<(A^0,0,C^0),\ (m_iA^0,0,C^i),\ldots,\ (A^{2k^{'}},b_{2k'},C^{2k^{'}})>.$$

- The quotient $M(A^0, C^0, C^i, A^{2k'}, C^{2k'})$ is a locally affine compact and complete polarized $2k'$−symplectic manifold, and its fundamental group is given by:

$$\pi_1(M(A^0, C^0, C^i, A^{2k'}, C^{2k'})) =< (A^0, 0, C^0),\ (m_i A^0, 0, C^i), \dots,\ (A^{2k'}, b_{2k'}, C^{2k'}) >$$

- The manifold $M(A^0, C^0, C^i, A^{2k'}, C^{2k'})$ is homeomorphic to the torus $\mathbb{T}^{2k'+1}$ if and only if $A^0 = 0_{\mathbb{Z}^{2k'}}$.

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