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Statistical Convergence with Rough I_3-Lacunary and Wijsman Rough I_3-Statistical Convergence in 2-Normed Spaces

M.H.M. Rashid*, **Sameer A. Al-Subh**

Department of Mathematics, Faculty of Science P.O. Box (7), Mutah University, Al-Karak, Jordan

**Corresponding author:* mrash@mutah.edu.jo, malik_okasha@yahoo.com

Abstract. In this paper, we have introduced the concept of the set of rough I_3-lacunary limit points for triple sequences in 2-normed spaces. We have established statistical convergence requirements associated with this set. Furthermore, we have introduced the idea of rough I_3-lacunary statistical convergence for triple sequences. Additionally, we have demonstrated that this set of rough I_3-lacunary limit points is both convex and closed within the context of a 2-normed space. We have also explored the relationships between a sequence's rough I_3-lacunary statistical cluster points and its rough I_3-lacunary statistical limit points in the same 2-normed space. Expanding upon the concept of triple sequence spaces, we have introduced the notion of Wijsman I_3-Cesáro summability for triple sequences. In doing so, we have investigated the connections between Wijsman strongly I_3-Cesáro summability and Wijsman statistical I_3-Cesáro summability. Furthermore, we have introduced the concepts of Wijsman rough strongly p-lacunary summability of order α and Wijsman rough lacunary statistical convergence of order α for triple sequences. These new concepts have been subjected to a thorough examination to understand their characteristics, and we have explored potential connections between them. Additionally, we have investigated how these newly introduced concepts relate to existing notions in the literature.

1. Introduction

Fast [22] and Schoenberg [42] independently extended the concept of series of real numbers converging to statistical convergence. Mursaleen and Edely [34] further extended this idea to double sequences. Fridy and Orhan's definition of lacunary statistical convergence can be found in [25], and Akan and Altay [9] presented multidimensional analogs of their findings.

I-convergence, a generalization of statistical convergence based on the ideal *I* of subsets of natural numbers, was initially proposed by Kostyrko et al. [28]. Kostyrko et al. [29] also conducted research on extremal *I*-limit points and the concept of *I*-convergence. Das et al. [12] defined *I*-convergence

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for double sequences in a metric space and explored some of its properties. Subsequent to the work of [22,30,35,42], there have been substantial advancements in the fields of statistical convergence, I-convergence, and triple sequences.

In [44], Tripathy et al. introduced the concept of lacunary ideal convergence for real sequences. The ideas of I-statistical convergence and I-lacunary statistical convergence were introduced by Das et al. [12] and Savaş et al. [41] using the ideal. Belen et al. [11] developed the concept of ideal statistical convergence for double sequences, offering a new generalization of statistical convergence and classical convergence. Kumar et al. [31] were the first to describe I-lacunary statistical convergence for double sequences, and further research and applications in this direction can be found in [21].

In the 1960s, Gähler [15] introduced the concept of 2-normed spaces, which has since been explored by various authors. Gürdal and Pehlivan [18] examined statistical convergence, statistical Cauchy sequences, and other aspects of statistical convergence in 2-normed spaces. Gürdal and Açk [19] studied I-Cauchy and I^*-Cauchy sequences in 2-normed spaces. Arslan and Dündar [6] investigated I-convergence, I^*-convergence, I-Cauchy, and I^*-Cauchy sequences of functions in 2-normed spaces. Significant developments in this field have also occurred (see [30,35]).

Phu [46] was the first to delve into rough convergence in finite-dimensional normed spaces. In his work [46], he demonstrated the closedness, convexity, and boundedness of the set LIM_x^r and introduced the concept of a rough Cauchy sequence. He also explored the relationships between rough convergence, various forms of convergence, and the dependence of LIM_x^r on the degree of roughness r. In a related study [48], he established the rough continuity of linear operators and proved that, given $\dim Y > 0$ and $r > 0$, with X and Y being normed spaces, every linear operator $f : X \to Y$ is r-continuous at every point $x \in X$. He extended these findings to infinite-dimensional normed spaces in [47].

Aytar [4] investigated rough statistical convergence and identified the set of rough statistical limit points of a sequence. He then derived two statistical convergence criteria related to this set and showed that it is both closed and convex. Aytar's [5] research revealed that the r-limit set of the sequence equals the intersection of these sets, while the r-core of the sequence is equal to the union of these sets.

The concepts of rough I-convergence and the set of rough I-limit points for a sequence were recently introduced by Dündar and Çakan [13], and Dündar [14] examined the concepts of rough convergence, I_2-convergence, and the sets of rough limit points and rough I_2-limit points for double sequences. In the context of 2-normed spaces, Arslan and Dündar [7,8] developed several concepts related to rough convergence.

The relationship between the strongly Cesáro summable sequences space $|\sigma_1|$ and the strongly lacunary summable sequences space N_θ defined by a lacunary sequence was demonstrated by Freedman et al. in [26]. Subsequently, Fridy and Orhan [25] introduced the concept of lacunary statistical convergence using the concept of lacunary sequences. Engül and Et [43] recently explored the ideas of substantially p-lacunary summability of order and lacunary statistical convergence of order α (see also [2]).

The notion of convergence for double sequences was initially presented by Pringsheim in [49] and expanded to include statistical convergence by Mursaleen and Edely [34]. Moreover, Patterson and Savaş [45] investigated the concept of lacunary statistical convergence using the concept of double lacunary sequences.

Many authors have extended the concepts of convergence from number sequences to set sequences. Two notable extensions are the ideas of Wijsman convergence and Hausdorff convergence (see [1, 23, 36, 37]). Nuray and Rhoades [32] extended Wijsman convergence and Hausdorff convergence to statistical convergence for set sequences and provided several fundamental theorems. Ulusu and Nuray introduced the concept of lacunary statistical convergence for set sequences using the concept of lacunary sequences.

Recently, Savaş [40] and Şengül and Et [43] independently explored the notion of Wijsman I-lacunary statistical convergence of order utilizing the concept of ideals.

In this paper, we introduce the concept of rough I_3-lacunary statistical convergence for triple sequences in normed linear spaces and conduct a thorough investigation of it. We examine the properties of rough I_3-lacunary statistical cluster points and rough I_3-lacunary statistical limit points in 2-normed spaces. Additionally, we establish a standard statistical convergence criterion associated with rough I_3-lacunary statistical cluster points for sequences in 2-normed spaces.

Furthermore, we explore the concepts of Wijsman rough I_3-lacunary statistical convergence, Wijsman rough I_3-lacunary statistical convergence, and Wijsman extremely rough I_3-lacunary convergence for triple sequences in this paper. We also investigate the relationships between these novel concepts. The introduction of lacunary triple sequences serves as the basis for these definitions. Following the definitions, we present natural inclusion theorems.

2. Definitions and Notions

Before delving deeper, let's familiarize ourselves with the concept of a 2-normed space, rough convergence, and several fundamental concepts and notations that will be employed in the following sections (Refer to citations such as [3–8, 15, 17, 20, 21, 25, 36, 37, 50, 51]).

The concept of a 2-normed space was first introduced by Gähler [15].

Definition 2.1. *Let $\mathscr{X}$ is a linear space of a dimension d, where $2 \leq d < \infty$. A 2-norm on $\mathscr{X}$ is a function $\|\cdot,\cdot\| : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ satisfying the following conditions: for every $\xi, \zeta \in X$, (i) $\|\xi, \zeta\| = 0$ if and only if ξ and ζ are linearly dependent; (ii) $\|\xi, \zeta\| = \|\zeta, \xi\|$; (iii) $\|\alpha\xi, \zeta\| = |\alpha| \|\xi, \zeta\|$, $\alpha \in \mathbb{R}$; (iv) $\left\|\xi + \zeta, \eta\right\| \leq \left\|\xi, \eta\right\| + \left\|\zeta, \eta\right\|$. In this case, $(\mathscr{X}, \|.,.\|$ is called a 2-normed space.*

Example 2.1. *Take $\mathscr{X} = \mathbb{R}^2$ being equipped with the 2-norm $\|\xi, \zeta\|$ = the area of the parallelogram spanned by the vectors ξ and ζ, which may be given explicitly by the formula*

$$\|\xi, \zeta\| = |\xi_1\zeta_2 - \xi_2\zeta_1|, \text{ where } \xi = (\xi_1, \xi_2), \zeta = (\zeta_1, \zeta_2).$$

A triple sequence $x = \{\varrho_{mnk}\}$ in 2-normed space $(\mathscr{X}, \|.,.\|)$ is said to be convergent to ξ in $\mathscr{X}$ if $\lim_{m,n,k\to\infty} \left\|\varrho_{mnk} - \xi, z\right\| = 0$ for each $z \in \mathscr{X}$. In such a case, we write $\lim_{m,n,k\to\infty} \varrho_{mnk} = \xi$ and ξ is called the limit of x.

Example 2.2. *Let* $x = \{\varrho_{mnk}\} = \left\{\left(\frac{2nmk}{mnk+3}, \frac{(-1)^{mnk}}{nmk+1}\right)\right\}$ *and* $\xi = (2,0)$. *Then clearly that* $x = \{\varrho_{mnk}\}$ *is convergent to* $\xi = (2,0)$ *in 2-normed space* $\mathscr{X}$.

Let r be a non-negative real number. A triple sequence $x = \{\varrho_{mnk}\}$ is said to be r-convergent to ξ in a 2-normed space X, denoted by $x \xrightarrow{r} \xi$, provided that for each $z \in X$

$$\forall \epsilon > 0\ \exists m_\epsilon, n_\epsilon, k_\epsilon \in \mathbb{N} : m > m_\epsilon, n > n_\epsilon \text{ and } k > k_\epsilon \Longrightarrow \|x - \xi, z\| < r + \epsilon$$

for each $z \in X$.

The set

$$LIM_x^r = \left\{\xi \in X : x \xrightarrow{r} \xi\right\}$$

is called the r-limit set of the triple sequence $x = \{\varrho_{mnk}\}$. A triple sequence $x = \{\varrho_{mnk}\}$ is said to be rough convergent (r-convergent) if $LIM_x^r \neq \emptyset$. In this case, r is called the rough convergence degree of the sequence $x = \{\varrho_{mnk}\}$. For $r = 0$, we get the ordinary convergence.

We recall that a subset E of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(K) = \lim_{m,n,k\to\infty} \frac{K(m,n,k)}{mnk},$$

where $E(m,n,k) = \left|\{(i,j,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : i \leq m, j \leq n, l \leq k\}\right|$.

Let $x = \{\varrho_{mnk}\}$ be a triple sequence in a 2-normed space $(X, \|.,.\|)$ and r be a non negative real number. x is said to be r-statistically convergent to ξ, denoted by $x \xrightarrow{st_3^r} \xi$, if for $\epsilon > 0$ and each $z \in X$ we have $\delta(K(\epsilon)) = 0$, where $K(\epsilon) = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x - \xi, z\| \geq r + \epsilon\}$. In this case, ξ is called the r-statistical limit of x.

A family $I \subset 2^{\mathbb{N}}$ is said to be an ideal provided the following conditions hold:

(i) $\emptyset \in I$;

(ii) $A, B \in I$ imply $A \cup B \in I$;

(iii) $A \in I$, $B \subset A$ imply $B \in I$.

An ideal is called non-trivial if $\mathbb{N} \neq I$ and a non-trivial ideal is called admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter if the following conditions are hold:

(i) $\emptyset \notin F$;

(ii) $A, B \in F$ imply $A \cap B \in F$;

(iii) $A \in F$, $A \subset B \subset Y$ imply $B \in F$.

Definition 2.2. *[16] A non trivial ideal* I_3 *of* $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ *is said to be* strongly admissible *if* $\{i\}\times\mathbb{N}\times\mathbb{N}$, $\mathbb{N}\times\{i\}\times\mathbb{N}$ *and* $\mathbb{N}\times\mathbb{N}\times\{i\}$ *belong to* I_3 *for each* $i \in \mathbb{N}$. *It is clear that a strongly admissible ideal is an admissible ideal.*

If $I_3^0 = \{A \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})\,(i,j,k \geq m(A)) \Rightarrow (i,j,k) \notin A\}$. Then, I_3^0 is a non-trivial strongly admissible ideal and we can see that I_3 is a strongly admissible ideal if and only if $I_3^0 \subset I_3$.

Let $x = \{\varrho_{mnk}\}$ be a triple sequence in a 2-normed space $(X, \|.,.\|)$ and r be a non negative real number. x is said to be rough I_3-convergent (I_3^r-convergent) to ξ with the roughness degree r, denoted by $x \xrightarrow{I_3^r} \xi$ provided that

$$\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x-\xi, z\| \geq r+\epsilon\} \in I_3$$

for every $\epsilon > 0$ and each $z \in X$; or equivalently, if the condition

$$I_3 - \limsup \|x-\xi, z\| \leq r$$

is satisfied for each $z \in X$. Moreover, we can write $x \xrightarrow{I_3^r} \xi$ if and only the inequality $\|x-\xi,z\| < r+\epsilon$ holds for every $\epsilon > 0$ and each $z \in X$ and almost all (m,n,k).

A subset $E \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to be have I_3-asymptotic density $\delta_{I_3}(E)$ if

$$\delta_{I_3}(E) = I_3 - \lim_{m,n,k\to\infty} \frac{\left|E(m,n,k)\right|}{mnk}$$

where $E(m,n,k) = \{(i,j,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : i \leq m, j \leq n, l \leq k; (i,j,l) \in E\}$ and $\left|E(m,n,k)\right|$ denotes number of elements of the set $E(m,n,k)$.

A triple sequence $x = \{\varrho_{mnk}\}$ in a 2-normed space $(X, \|.,.\|)$ is I_3-statistically convergent to ξ, and we write $x \xrightarrow{I_3 - st_3} \xi$, provided that for any $\epsilon > 0, \delta > 0$ and each $z \in X$

$$\left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{mnk}\left|\{(i,j,l) : \|x-\xi,z\| \geq \epsilon, i \leq m, j \leq n, l \leq k\}\right| \geq \delta\right\} \in I_3.$$

Let $x = \{\varrho_{mnk}\}$ be a triple sequence in a 2-normed linear space $(X, \|.,.\|$ and r be a non-negative real number. Then x is said to be rough I_3-statistical convergent to ξ or I_3^r-statistical convergent to ξ if for any $\epsilon > 0, \delta > 0$ and each $z \in X$

$$\left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{mnk}\left|\{(i,j,l), i \leq m, j \leq n, l \leq k : \|x-\xi,z\| \geq \epsilon + r\}\right| \geq \delta\right\} \in I_3.$$

In this case, ξ is called the rough I_3-statistical limit of $x = \{\varrho_{mnk}\}$ and we denote it by $x \xrightarrow{I_3^r - st_3} \xi$. The triple sequence $\theta_3 = \theta_{r,s,t} = \{(i_r, j_s, l_t)\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that

$$\begin{aligned} i_0 &= 0, \hbar_u = i_u - i_{u-1} \to \infty \text{ as } r \to \infty, \\ j_0 &= 0, \hbar_v = j_v - j_{v-1} \to \infty \text{ as } s \to \infty \text{ and} \\ l_0 &= 0, \hbar_w = l_w - l_{w-1} \to \infty \text{ as } t \to \infty. \end{aligned}$$

Let $z_{uvw} = i_u j_v l_w, \hbar_{uvw} = \hbar_u \hbar_v \hbar_w$ and $\theta_{u,v,w}$ is determined by

$$\wp_{uvw} = \{(i,j,l) : i_{u-1} < i \leq i_u, j_{v-1} < j \leq j_v, l_{w-1} < l \leq l_w\},$$

$$q_u = \frac{i_u}{i_{u-1}}, q_v = \frac{j_v}{j_{v-1}}, q_w = \frac{l_w}{l_{w-1}} \text{ and } q_{uvw} = q_u q_v q_w.$$

Throughout the study, $\theta_3 = \{(i_u, j_v, l_w)\}$ will be taken as a triple lacunary sequence.

A triple sequence $x = \{\varrho_{mnk}\}$ in a 2-normed space $(X, \|.,.\|)$ is said to be I_3-lacunary statistical convergent or $\mathcal{S}_{\theta_3}(I_3)$-convergent to ξ, if for each $\epsilon > 0, \delta > 0$ and each $z \in X$,

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon\right\}\right| \geq \delta\right\} \in I_3.$$

In this case, we write $x \to \xi(\mathcal{S}_{\theta_3}(I_3))$ or $\mathcal{S}_{\theta_3}(I_3)$-$\lim_{m,n,k\to\infty} x = \xi$.

3. Rough I_3-Lacunary Statistical Convergence of Triple Sequences

In this section, we delve into the concept of rough I_3-lacunary statistical convergence within 2-normed linear spaces for triple sequences. Furthermore, we provide a definition for the rough I_3-lacunary statistical limit set of a triple sequence and explore some of its key characteristics.

Definition 3.1. *A triple sequence $x = \{\varrho_{mnk}\}$ in a 2-normed space $(X, \|.,.\|)$ and r be a non-negative real number. Then x is said to be rough lacunary statistical convergent to ξ or r-lacunary statistical convergent to ξ if for any $\epsilon > 0$ and each $z \in X$,*

$$\lim_{u,v,w\to\infty} \frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| = 0.$$

In this case ξ is called the rough lacunary statistical limit of $x = \{\varrho_{mnk}\}$ and we denote it by

$$x \xrightarrow{\mathcal{S}^r_{\theta_3}} \xi$$

Definition 3.2. *A triple sequence $x = \{\varrho_{mnk}\}$ in a 2-normed space $(X, \|.,.\|)$ and r be a non-negative real number. Then x is said to be rough I_3-lacunary statistical convergent to ξ or I^r_3-lacunary statistical convergent to ξ if for any $\epsilon > 0, \delta > 0$ and each $z \in X$,*

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| \geq \delta\right\} \in I_3.$$

In this case, ξ is called the rough I_3-lacunary statistical limit of $x = \{\varrho_{mnk}\}$ and we denote it by $x \xrightarrow{I^r_{\theta_3}-st_3} \xi$.

In the aforementioned definition, we designate the degree of roughness for rough I_3-lacunary statistical convergence as r. When r equals zero, we obtain the concept of I_3-lacunary convergence. However, our primary focus lies in cases where r is greater than zero. It is conceivable that a triple sequence denoted as $y = y_{mnk}$ satisfies the conditions of being I_3-lacunary statistically convergent and meeting the requirement $\left\|\varrho_{mnk} - y_{mnk}, z\right\| \leq r$ for all (m,n,k) and each $z \in X$, yet it does not conform to the conventional notion of I_3-lacunary statistical convergence. In such cases, statistically, x is roughly I_3-lacunary and converges to the same limit. As described earlier, the rough I_3-lacunary statistical limit of a triple sequence is not unique.

To denote the set of approximate I_3-lacunary statistical limits for a triple sequence x, we use the notation I_{θ_3}-st_3-LIM^r_x. This set represents all potential approximate upper statistical bounds for I_3-lacunary statistical convergence of a triple sequence x. If a triple sequence x is not unique, and

I_{θ_3}-st_3-$LIM^r_x \neq \emptyset$, we refer to it as roughly I_3-lacunary statistically convergent. In this paper, we consistently represent a 2-normed linear space as $X = (X, \|\cdot,\cdot\|)$, and we use the symbol x to refer to the triple sequence denoted as ϱ_{mnk} within this space.

Theorem 3.1. *Let $x = \{\varrho_{mnk}\}$ be a triple sequence and $r \geq 0$. Then I_{θ_3}-st_3-$LIM^r_x \leq 2r$. In particular if x is rough I_3-lacunary statistically convergent to ξ, then I_{θ_3}-st_3-$LIM^r_x = \overline{B}_r(\xi)$, where $\overline{B}_r(\xi) = \{y \in X : \left\|y - \xi, z\right\| \leq r\}$ for eah $z \in X$ and so $diam(I_{\theta_3} - st_3 - LIM^r_x) = 2r$.*

Proof. Let $diam(I_{\theta_3} - st_3 - LIM^r_x) > 2r$. Then there exist $y, p \in I_{\theta_3} - st_3 - LIM^r_x$ such that $\left\|y - p, z\right\| > 2r$ for each $z \in X$. Now, we select $\epsilon > 0$ so that $\epsilon < \dfrac{\left\|y - p, z\right\|}{2} - r$ for each $z \in X$. Let

$$\begin{aligned} A &= \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq r + \epsilon\right\} \text{ and} \\ B &= \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - p, z\right\| \geq r + \epsilon\right\} \end{aligned}$$

for each $z \in X$. Then

$$\begin{aligned} \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A \cup B\}\right| &\leq \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A\}\right| \\ &+ \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in B\}\right| \end{aligned}$$

and so by the property of I_3-convergence

$$\begin{aligned} &\lim_{u,v,w\to\infty} \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A \cup B\}\right| \\ &\leq \lim_{u,v,w\to\infty} \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A\}\right| \\ &+ \lim_{u,v,w\to\infty} \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in B\}\right| = 0 \end{aligned}$$

Hence

$$\left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A \cup B\}\right| \geq \delta\right\} \in I_3$$

for each $\delta > 0$ and each $z \in X$. Let

$$M = \left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A \cup B\}\right| \geq \frac{1}{2}\right\}$$

clearly $M \in I_3$, so choose $(u_0, v_0, w_0) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus M$. Then

$$\frac{1}{\hbar_{u_0v_0w_0}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A \cup B\}\right| < \frac{1}{2}.$$

Consequently,

$$\frac{1}{\hbar_{u_0v_0w_0}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \notin A \cup B\}\right| \geq \frac{1}{2},$$

that is, $\{(m,n,k) \in \wp_{uvw} : (m,n,k) \notin A \cup B\} \neq \emptyset$.
Take $(m_0, n_0, k_0) \in \wp_{uvw}$ such that $(m_0, n_0, k_0) \notin A \cup B$. Then $(m_0, n_0, k_0) \in A^c \cap B^c$ and so $\left\|\varrho_{m_0n_0k_0} - y, z\right\| < r + \epsilon$ and $\left\|\varrho_{m_0n_0k_0} - p, z\right\| < r + \epsilon$ for each $z \in X$. Hence, we have

$$\left\|y - p, z\right\| \leq \left\|\varrho_{m_0n_0k_0} - y, z\right\| + \left\|\varrho_{m_0n_0k_0} - p, z\right\| < 2r + 2\epsilon \leq \left\|y - p, z\right\|$$

for each $z \in X$. This is impossible and so $diam(I_{\theta_3} - st_3 - LIM_x^r) \leq 2r$.
If $I_{\theta_3} - st_3 - LIM_x^r = \xi$, then we proceed as follows. Let $\epsilon > 0$ and $\delta > 0$ be given. Then for each $z \in X$

$$A = \left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon\right\}\right| \geq \delta\right\} \in I_3.$$

Then for $(u,v,w) \notin A$ we have

$$\frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon\right\}\right| < \delta,$$

i.e.,

$$\frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon\right\}\right| \geq 1 - \delta.$$

Now for each $y \in \overline{B}_r(\xi)$ we have

$$\left\|\varrho_{mnk} - y, z\right\| \leq \left\|\varrho_{mnk} - \xi, z\right\| + \left\|\xi - y, z\right\| \leq \left\|\varrho_{mnk} - \xi, z\right\| + r$$

for each $z \in X$. Let $B_{uvw} = \{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| < \epsilon\}$ for each $z \in X$. Then for $(m,n,k) \in B_{uvw}$ we have $\left\|\varrho_{mnk} - y, z\right\| < r + \epsilon$ for each $z \in X$. Hence we have

$$B_{uvw} = \{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| < r + \epsilon\}$$

for each $z \in X$. This yields

$$\frac{|B_{uvw}|}{\hbar_{uvw}} \leq \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| < r + \epsilon\}\right|$$

i.e.,

$$\frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| < r + \epsilon\}\right| \geq 1 - \delta.$$

Thus, for all $(m,n,k) \notin A$ we have

$$\frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq r + \epsilon\}\right| \geq 1 - (1 - \delta) = \delta.$$

Therefore

$$B = \left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq r + \epsilon\}\right| \geq \delta\right\} \subset A.$$

Since $A \in I_3$ we get $B \in I_3$. This shows that $y \in I_{\theta_3} - st_3 - LIM_x^r$ and so $\overline{B}_r(\xi) \subset I_{\theta_3} - st_3 - LIM_x^r$.
Conversely, let $y \in I_{\theta_3} - st_3 - LIM_x^r$, $\left\|y - \xi, z\right\| > r$ and $\epsilon = \dfrac{\left\|y - \xi, z\right\| - r}{2}$ for each $z \in X$. Now, we take for each $z \in X$

$$M_1 = \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq r + \epsilon\right\}$$

and

$$M_2 = \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq \epsilon\right\}.$$

Then

$$\begin{aligned} &\frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in M_1 \cup M_2\}\right| \\ &\leq \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in M_1\}\right| + \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in M_2\}\right| \end{aligned}$$

and by the property of I_3-convergence

$$I_3-\lim_{u,v,w\to\infty}\frac{1}{\hbar_{uvw}}\left|\{(m,n,k)\in\wp_{uvw}:(m,n,k)\in M_1\cup M_2\}\right|$$
$$\leq I_3-\lim_{u,v,w\to\infty}\frac{1}{\hbar_{uvw}}\left|\{(m,n,k)\in\wp_{uvw}:(m,n,k)\in M_1\}\right|$$
$$+I_3-\lim_{u,v,w\to\infty}\frac{1}{\hbar_{uvw}}\left|\{(m,n,k)\in\wp_{uvw}:(m,n,k)\in M_2\}\right|=0$$

Now,let

$$M=\left\{(u,v,w)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{\hbar_{uvw}}\left|\{(m,n,k)\in\wp_{uvw}:(m,n,k)\in M_1\cup M_2\}\right|\geq\frac{1}{2}\right\}.$$

Clearly $M\in I_3$ and we choose $(u_0,v_0,w_0)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}\setminus M$. Then we have

$$\frac{1}{\hbar_{uvw}}\left|\{(m,n,k)\in\wp_{uvw}:(m,n,k)\in M_1\cup M_2\}\right|<\frac{1}{2}$$

and so

$$\frac{1}{\hbar_{uvw}}\left|\{(m,n,k)\in\wp_{uvw}:(m,n,k)\notin M_1\cup M_2\}\right|\geq 1-\frac{1}{2}=\frac{1}{2},$$

i.e., $\{(m,n,k):(m,n,k)\notin M_1\cup M_2\}$ is a nonempty set. Let $(m_0,n_0,k_0)\in\wp_{uvw}$ such that $(m_0,n_0,k_0)\notin M_1\cup M_2$. Then $(m_0,n_0,k_0)\in M_1^c\cap M_2^c$ and hence $\left\|\varrho_{m_0n_0k_0}-y,z\right\|<r+\epsilon$ and $\left\|\varrho_{m_0n_0k_0}-\xi,z\right\|<\epsilon$ for each $z\in X$. Hence

$$\left\|y-\xi,z\right\|\leq\left\|\varrho_{m_0n_0k_0}-y,z\right\|+\left\|\varrho_{m_0n_0k_0}-\xi,z\right\|\leq r+2\epsilon\leq\left\|y-\xi,z\right\|$$

for each $z\in X$, which is impossible. Consequently, $\left\|y-\xi,z\right\|\leq r$ and so $y\in\overline{B}_r(\xi)$ and so $I_{\theta_3}-st_3-LIM_x^r=\overline{B}_r(\xi)$. □

Theorem 3.2. *Let $x=\{\varrho_{mnk}\}$ be a triple sequence and $r\geq 0$ be a real number. Then the rough I_3-lacunary statistical limit set of the triple sequence x, i.e., the set $I_{\theta_3}-st_3-LIM_x^r$ is closed.*

Proof. If $I_{\theta_3}-st_3-LIM_x^r=\emptyset$, then there is nothing to prove. Let us assume that $I_{\theta_3}-st_3-LIM_x^r\neq\emptyset$. Now, consider a double sequence $\{y_{mnk}\}$ in $I_{\theta_3}-st_3-LIM_x^r$ with $\lim_{m,n,k\to\infty}y_{mnk}=y$. Choose $\epsilon>0$ and $\delta>0$. Then there exists $i_{\epsilon/2}$ such that for all $m,n,k\geq i_{\epsilon/2}$ and each $z\in X$

$$\left\|y_{mnk}-y,z\right\|<\frac{\epsilon}{2}.$$

Let $m_0,n_0,k_0>i_{\epsilon/2}$. Then $y_{m_0n_0k_0}\in I_{\theta_3}-st_3-LIM_x^r$. Therefore, we have

$$A=\left\{(u,v,w)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k)\in\wp_{uvw}:\left\|\varrho_{mnk}-y_{m_0n_0k_0,z}\right\|\geq r+\frac{\epsilon}{2}\right\}\right|\geq\delta\right\}\in I_3.$$

Clearly $M=\mathbb{N}\times\mathbb{N}\times\mathbb{N}\setminus A$ is nonempty, choose $(u,v,w)\in M$. We have

$$\frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k)\in\wp_{uvw}:\left\|\varrho_{mnk}-y_{m_0n_0k_0,z}\right\|\geq r+\frac{\epsilon}{2}\right\}\right|<\delta$$

and so

$$\frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k)\in\wp_{uvw}:\left\|\varrho_{mnk}-y_{m_0n_0k_0,z}\right\|<r+\frac{\epsilon}{2}\right\}\right|\geq 1-\delta.$$

Put

$$B_{uvw} = \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y_{m_0n_0k_0,z}\right\| < r + \frac{\epsilon}{2}\right\}$$

and select $(m,n,k) \in B_{uvw}$. Then we have for each $z \in X$

$$\left\|\varrho_{mnk} - y, z\right\| \leq \left\|\varrho_{mnk} - y_{m_0n_0k_0,z}\right\| + \left\|y - y_{m_0n_0k_0,z}\right\| < r + \epsilon$$

and so for each $z \in X$

$$B_{uvw} \subset \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| < r + \epsilon\right\} = K$$

which implies that

$$1 - \delta \leq \frac{|B_{uvw}|}{\hbar_{uvw}} \leq \frac{|K|}{\hbar_{uvw}}.$$

Therefore,

$$\frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq r + \epsilon\right\}\right| < 1 - (1 - \delta) = \delta$$

and so we have

$$\left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq r + \epsilon\right\} \geq \delta\right|\right\} \subset A \in I_3.$$

Consequently, $y \in I_{\theta_3} - st_3 - LIM_x^r$ and so $I_{\theta_3} - st_3 - LIM_x^r$ is closed. □

Theorem 3.3. *Let $x = \{\varrho_{mnk}\}$ be a triple sequence and $r \geq 0$ be a real number. Then the rough I_3-lacunary statistical limit set $I_{\theta_3} - st_3 - LIM_x^r$ of the triple sequence x is a convex set.*

Proof. Let $y_0, y_1 \in I_{\theta_3} - st_3 - LIM_x^r$ and $\epsilon > 0$ be given. Define

$$\begin{aligned} A_1 &= \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y_0, z\right\| \geq r + \epsilon\right\} \text{ and} \\ A_2 &= \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y_1, z\right\| \geq r + \epsilon\right\} \end{aligned}$$

for each $z \in X$. Then by Theorem 3.1, for $\delta > 0$ and each $z \in x$ we have

$$\left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A_1 \cup A_2\}\right| \geq \delta\right\} \in I_3.$$

Now, we choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$ and let

$$A = \left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A_1 \cup A_2\}\right| \geq 1 - \delta_1\right\}.$$

Then $A \in I_3$. For all $(u,v,w) \notin A$, we have

$$\frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A_1 \cup A_2\}\right| < 1 - \delta_1$$

and so

$$\frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \notin A_1 \cup A_2\}\right| \geq 1 - (1 - \delta_1) = \delta_1.$$

Therefore, $\{(m,n,k) \in \wp_{uvw} : (m,n,k) \notin A_1 \cup A_2\}$ is a nonempty set. Let us take $(m_0,n_0,k_0) \in A_1^c \cap A_2^c$ and $\mu \in [0,1]$. Then for each $z \in X$

$$\begin{aligned} \left\|\varrho_{m_0n_0k_0} - (\mu y_0 + (1-\mu)y_1),z\right\| &= \left\|\mu\varrho_{m_0n_0k_0} + (1-\mu)\varrho_{m_0n_0k_0} - (\mu y_0 + (1-\mu)y_1),z\right\| \\ &\le \mu\left\|\varrho_{m_0n_0k_0} - y_0,z\right\| + (1-\mu)\left\|\varrho_{m_0n_0k_0} - y_0,z\right\| \\ &< \mu(r+\epsilon) + (1-\mu)(r+\epsilon) = r+\epsilon. \end{aligned}$$

Let

$$M = \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - (\mu y_0 + (1-\mu)y_1),z\right\| \ge r+\epsilon\right\}$$

for all $z \in X$. Then clearly $A_1^c \cap A_2^c \subset M^c$. So for $(u,v,w) \notin A^c$, we have

$$\delta_1 \le \frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \notin A_1 \cup A_2\}\right| \le \frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \notin M\}\right|$$

and so

$$\frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in M\}\right| < 1-\delta_1 < \delta.$$

Consequently,

$$A^c \subset K = \left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in M\}\right| < \delta\right\}.$$

Since $A^c \in F(I_3)$, we have $K \in F(I_3)$ and so

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in M\}\right| \ge \delta\right\} \in I_3.$$

Therefore, $I_{\theta_3} - st_3 - LIM_x^r$ is convex. □

Theorem 3.4. *A triple sequence $x = \{\varrho_{mnk}\}$ is rough I_3-lacunary statistical convergent to ξ if and only if there exists a triple sequence $y = \{y_{mnk}\}$ such that $I_{\theta_3} - st_3 - y = \xi$ and $\left\|\varrho_{mnk} - y_{mnk},z\right\| \le r$ for all $(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and each $z \in X$.*

Proof. Let $y = \{y_{mnk}\}$ be a triple sequence in X, which is I_3-lacunary statistically convergent to ξ and $\left\|\varrho_{mnk} - y_{mnk},z\right\| \le r$ for all $(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and each $z \in X$. Then for any $\epsilon > 0, \delta > 0$ and each $z \in X$

$$A = \left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi,z\right\| \ge \epsilon\right\}\right| \ge \delta\right\} \in I_3.$$

Let $(u,v,w) \notin A$. Then we have for each $z \in X$

$$\begin{aligned} &\frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi,z\right\| \ge \epsilon\right\}\right| < \delta \\ &\Longrightarrow \frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi,z\right\| < \epsilon\right\}\right| \ge 1-\delta. \end{aligned}$$

Now, we let for each $z \in X$

$$B_{uvw} = \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi,z\right\| < \epsilon\right\}.$$

Then, for $(m,n,k) \in B_{uvw}$, we have

$$\left\|\varrho_{mnk} - \xi,z\right\| \le \left\|\varrho_{mnk} - y_{mnk},z\right\| + \left\|y_{mnk} - \xi,z\right\| < r+\epsilon,$$

for each $z \in X$. So, for each $z \in X$

$$\begin{aligned} B_{uvw} &\subset E = \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < r + \epsilon\right\} \\ &\Longrightarrow \frac{|B_{uvw}|}{\hbar_{uvw}} \leq \frac{|E|}{\hbar_{uvw}} \\ &\Longrightarrow \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < r + \epsilon\right\}\right| \geq 1 - \delta \\ &\Longrightarrow \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right| < \delta. \end{aligned}$$

Thus, we have for each $z \in X$

$$Q = \left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right| \geq \delta\right\} \subset A$$

and since $A \in I_3$, we have $Q \in I_3$. Therefore, $I_{\theta_3} - st_3 - y = \xi$.

Conversely, suppose that $I_{\theta_3} - st_3 - y = \xi$. Then, for $\epsilon > 0, \delta > 0$ and each $z \in X$,

$$Q = \left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right| \geq \delta\right\} \in I_3.$$

Let $(u,v,w) \notin Q$. Then we have for $z \in X$,

$$\frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right| < \delta$$

and so for $z \in X$,

$$\frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < r + \epsilon\right\}\right| \geq 1 - \delta.$$

Let

$$B_{uvw} = \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| < r + \epsilon\right\}.$$

Now, we define a triple sequence $y = \{y_{mnk}\}$ for each $z \in X$ as follows:

$$y_{mnk} = \begin{cases} \xi, & \text{if } \left\|\varrho_{mnk} - \xi, z\right\| \leq r\,; \\ \varrho_{mnk} + r\dfrac{\xi - \varrho_{mnk}}{\left\|\varrho_{mnk} - \xi, z\right\|}, & \text{otherwise.} \end{cases}$$

Then for each $z \in X$

$$\left\|y_{mnk} - \xi, z\right\| = \begin{cases} 0, & \text{if } \left\|\varrho_{mnk} - \xi, z\right\| \leq r\,; \\ \left\|\varrho_{mnk} - \xi, z\right\| - r, & \text{otherwise.} \end{cases}$$

Let $(u,v,w) \in B_{uvw}$. Then for each $z \in X$, we have

$$\left\|y_{mnk} - \xi, z\right\| = \begin{cases} 0, & \text{if } \left\|\varrho_{mnk} - \xi, z\right\| \leq r; \\ < \epsilon, & \text{if } r < \left\|\varrho_{mnk} - \xi, z\right\| < r + \epsilon\,. \end{cases}$$

and so for each $z \in X$

$$B_{uvw} \subset E = \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\}.$$

This implies

$$\frac{|B_{uvw}|}{\hbar_{uvw}} \leq \frac{|E|}{\hbar_{uvw}}$$

for each $z \in X$. Hence we have

$$\frac{|E|}{\hbar_{uvw}} \geq 1 - \delta \Longrightarrow \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| \geq \epsilon\right\} < \delta,$$

and so for each $z \in X$

$$\left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| \geq \epsilon\right\}\right| \geq \delta\right\} \subset Q.$$

Since $Q \in I_3$, we have for each $z \in X$

$$\left\{(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| \geq \epsilon\right\}\right| \geq \delta\right\} \in I_3.$$

Therefore, $I_{\theta_3} - st_3 - y = \xi$. □

The next result provides a relationship between boundedness and rough I_{θ_3}-statistical convergence of triple sequences.

Theorem 3.5. *If a triple sequence $x = \{\varrho_{mnk}\}$ is bounded then there exists $r \geq 0$ such that $I_{\theta_3} - st_3 - LIM_x^r \neq \emptyset$.*

Proof. Let $x = \{\varrho_{mnk}\}$ be bounded triple sequence. There exists a positive real number M such that $\left\|\varrho_{mnk}, z\right\| < M$, for all $(m,n,k) \in \wp_{uvw}$ and each $z \in X$. Let $\epsilon > 0$ be given. Then for each $z \in X$

$$\left\{(u,v,w) \in \wp_{uvw} : \left\|\varrho_{mnk}, z\right\| \geq M + \epsilon\right\} = \emptyset.$$

Consequently, $0 \in I_{\theta_3} - st_3 - LIM_x^M$ and so $I_{\theta_3} - st_3 - LIM_x^M \neq \emptyset$. □

The converse of Theorem 3.5 is not true as shown by the following example.

Example 3.1. *Consider the triple sequence $x = \{\varrho_{mnk}\}$ in $\mathbb{R}$ defined by*

$$\varrho_{mnk} = \begin{cases} mnk, & \text{if } m, n \text{ and } k \text{ are square;} \\ 1, & \text{otherwise.} \end{cases}$$

Then $I_{\theta_3} - st_3 - LIM_x^0 = \{1\}$. But $x = \{\varrho_{mnk}\}$ is unbounded.

Definition 3.3. *A point $c \in X$ is said to be an I_3-lacunary statistical cluster point of a triple sequence $x = \{\varrho_{mnk}\}$ in X if for any $\epsilon > 0$ and each $z \in X$*

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - c, z\right\| < \epsilon\right\}\right) \neq 0$$

where

$$\delta_{I_3}(A) = I_3 - \lim_{u,v,w \to \infty} \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A\}\right|,$$

if it exists. The set of I_3-lacunary statistical cluster points of x is denoted by $\Lambda_x^{S_{\theta_3}}(I_3)$.

Definition 3.4. *A point $c \in X$ is said to be an I_3-lacunary rough statistical cluster point of a triple sequence $x = \{\varrho_{mnk}\}$ in X for a non-negative real number $r \geq 0$ if for any $\epsilon > 0$ and each $z \in X$*

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - c, z\right\| < r + \epsilon\right\}\right) \neq 0$$

where

$$\delta_{I_3}(A) = I_3 - \lim_{u,v,w\to\infty} \frac{1}{\hbar_{uvw}} \left|\{(m,n,k) \in \wp_{uvw} : (m,n,k) \in A\}\right|,$$

if it exists. The set of I_3-lacunary rough statistical cluster points of x is denoted by $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$.

Theorem 3.6. *For any arbitrary $\nu \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ of a triple sequence $x = \{\varrho_{mnk}\}$ we have $\|\xi - \nu, z\| \leq r$ for all $\xi \in I_{\theta_3} - st_3 - LIM_x^r$ and each $z \in X$.*

Proof. Assume that there exists a point $\nu \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ and $\xi \in I_{\theta_3} - st_3 - LIM_x^r$ such that $\|\xi - \nu, z\| > r$ for each $z \in X$. Let $\epsilon = \dfrac{\|\nu - \xi, z\| - r}{3}$ for each $z \in X$. Then for each $z \in X$,

$$\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \nu, z\right\| < \epsilon\right\} \subset \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}.$$

Since $\nu \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ we have

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \nu, z\right\| < \epsilon\right\}\right) \neq 0$$

and so

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right) \neq 0$$

which contradicts that $\xi \in I_{\theta_3} - st_3 - LIM_x^r$ and consequently, $\|\xi - \nu, z\| \leq r$. □

Theorem 3.7. *Let $x = \{\varrho_{mnk}\}$ be a triple sequence in X. Then, for every $r \geq 0$, the set $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$ is closed.*

Proof. If $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) = \emptyset$ there is nothing to prove. Assume that $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) \neq \emptyset$ and consider a sequence $\{y_{mnk}\} \subset \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$ such that $y_{mnk} \to \xi$. Let us show that

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| < r + \epsilon\right\}\right) \neq 0$$

for every $\epsilon > 0$ and $z \in X$. Fix $\epsilon > 0$. Since $y_{mnk} \to \xi$, there exists an $(m_0, n_0, k_0) = (m_0(\epsilon), n_0(\epsilon), k_0(\epsilon)) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\left\|y_{mnk} - \xi, z\right\| < \frac{\epsilon}{2}$$

for all $(m,n,k) > (m_0, n_0, k_0)$ and every $z \in X$. Fix p_0, q_0, s_0 such that $(p_0, q_0, s_0) > (m_0, n_0, k_0)$. Then, we have

$$\left\|y_{p_0,q_0 s_0} - \xi, z\right\| < \frac{\epsilon}{2}$$

for every $z \in X$. Let (p,q,s) be any point of the set

$$\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y_{p_0,q_0 s_0}, z\right\| < r + \frac{\epsilon}{2}\right\}.$$

Since $\left\|\varrho_{pqs} - y_{p_0,q_0 s_0}, z\right\| < r + \dfrac{\epsilon}{2}$, we have

$$\begin{aligned} \left\|\varrho_{pqs} - \xi, z\right\| &\leq \left\|\varrho_{pqs} - y_{p_0,q_0 s_0}, z\right\| + \left\|y_{p_0,q_0 s_0} - \xi, z\right\| \\ &< r + \frac{\epsilon}{2} + \frac{\epsilon}{2} = r + \epsilon \end{aligned}$$

and so,

$$(p,q,s) \in \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{pqs} - \xi, z\right\| < r + \epsilon\right\},$$

for every $z \in X$. Hence, we have

$$\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{pqs} - y_{p_0,q_0s_0}, z\right\| < r + \frac{\epsilon}{2}\right\} \subseteq \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{pqs} - \xi, z\right\| < r + \epsilon\right\}. \tag{3.1}$$

Since

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{pqs} - y_{p_0,q_0s_0}, z\right\| < r + \frac{\epsilon}{2}\right\}\right) \neq 0$$

by (3.1), we obtain

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{pqs} - \xi, z\right\| < r + \epsilon\right\}\right) \neq 0,$$

for every $z \in X$. Consequently, $\xi \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$. □

Theorem 3.8. *Let $r > 0$. For a triple sequence $x = \{\varrho_{mnk}\}$ in X, we have $\xi \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$ if and only if there exists a sequence $y = \{y_{mnk}\}$ such that $\xi \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ and $\left\|\varrho_{mnk} - y_{mnk}, z\right\| \leq r$ for every $z \in X$ and almost all (m,n,k).*

Proof. Necessity: Fix r and ϵ and suppose that $\xi \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$. Thus, we have $\delta_{I_3}(A) \neq 0$, where

$$A := \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| < r + \epsilon\right\},$$

for every $z \in X$. Define

$$y_{mnk} := \begin{cases} \xi, & \text{if } \left\|\varrho_{mnk} - \xi, z\right\| \leq r \text{ and } (m,n,k) \in A; \\ \varrho_{mnk} + r\dfrac{\xi - \varrho_{mnk}}{\left\|\varrho_{mnk} - \xi, z\right\|}, & \text{if } \left\|\varrho_{mnk} - \xi, z\right\| > r \text{ and } (m,n,k) \in A; \\ t_{mnk}, & \text{if } (m,n,k) \notin A \end{cases} \tag{3.2}$$

where the sequence $t = \{t_{mnk}\}$ is arbitrary. It is clear that

$$\left\|y_{mnk} - \xi, z\right\| = \begin{cases} 0, & \text{if } \left\|\varrho_{mnk} - \xi, z\right\| \leq r; \\ \left\|\varrho_{mnk} - \xi, z\right\| - r, & \text{otherwise.} \end{cases} \tag{3.3}$$

and $\left\|\varrho_{mnk} - y_{mnk}, z\right\| \leq r$, for every $(m,n,k) \in A$ and $z \in X$. Now let us show that the inclusion

$$A \subseteq \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\} \tag{3.4}$$

holds, for every $z \in X$. If $(m_0,n_0,k_0) \in A$, then we have

$$\left\|\varrho_{m_0n_0k_0} - \xi, z\right\| < r + \epsilon,$$

for every $z \in X$. Hence the following two cases are possible:

(i) If $\left\|\varrho_{m_0n_0k_0} - \xi, z\right\| \leq r$, then from (3.3), we have

$$\left\|y_{m_0n_0k_0} - \xi, z\right\| = 0,$$

that is,

$$(m_0,n_0,k_0) \in \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\},$$

for every $z \in X$.

(ii) If $\left\|\varrho_{m_0n_0k_0} - \xi, z\right\| > r$, then from (3.3), we have

$$\left\|y_{m_0n_0k_0} - \xi, z\right\| = \left\|\varrho_{m_0n_0k_0} - \xi, z\right\| - r < r + \epsilon - r = \epsilon,$$

that is,

$$(m_0, n_0, k_0) \in \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\},$$

for every $z \in X$.

Since $\delta_{I_3}(A) \neq 0$, by the inclusion (3.4), we have

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\}\right) \neq 0$$

for every $z \in X$.

Sufficiency: Assume that $\xi \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ and fix $\epsilon > 0$. Then, we have

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\}\right) \neq 0$$

for every $z \in X$. Now, we let $(p,q,s) \in \left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\}$ and so, we can write

$$\left\|\varrho_{mnk} - \xi, z\right\| \leq \left\|\varrho_{mnk} - y_{mnk}, z\right\| + \left\|y_{mnk} - \xi, z\right\| < r + \epsilon,$$

for every $z \in X$. Therefore, we have

$$(p,q,s) \in \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| < r + \epsilon\right\}$$

and so, for every $z \in X$.

$$\left\{(m,n,k) \in \wp_{uvw} : \left\|y_{mnk} - \xi, z\right\| < \epsilon\right\} \subseteq \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| < r + \epsilon\right\}$$

holds. From this inclusion, we have

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| < r + \epsilon\right\}\right) \neq 0$$

and so $\xi \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$. □

Theorem gives a straightforward approach to finding the set $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$

Theorem 3.9. *Let $r > 0$. For a triple sequence $x = \{\varrho_{mnk}\}$ in X, we have*

$$\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) = \bigcup_{c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c),$$

where $\overline{B}_r(c) = \left\{y \in X : \left\|y - c, z\right\| \leq r\right\}$ for every $z \in X$.

Proof. Let $\eta \in \bigcup_{c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c)$. Then, there exists a point $c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ such that $\eta \in \overline{B}_r(c)$, that is, $\left\|\eta - c, z\right\| \leq r$ for every $z \in X$. Fix $\epsilon > 0$. Since $c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$, there exists a set

$$A(\epsilon) := \left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - c, z\right\| < \epsilon\right\}$$

with $\delta_{I_3}(A(\epsilon)) \neq 0$. Hence, we have

$$\left\|\varrho_{mnk} - \eta, z\right\| \leq \left\|\varrho_{mnk} - c, z\right\| + \left\|\eta - c, z\right\| < r + \epsilon,$$

and so,

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \eta, z\right\| < r + \epsilon\right\}\right) \neq 0$$

for every $(m,n,k) \in A(\epsilon)$ and every $z \in X$. Therefore, $\eta \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$ and so,

$$\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) \supseteq \bigcup_{c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c).$$

For the converse inclusion, take $\eta \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r)$. Then, we have

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \eta, z\right\| < r + \epsilon\right\}\right) \neq 0 \tag{3.5}$$

for every $\epsilon > 0$ and every $z \in X$. We must show that $\eta \in \bigcup_{c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c)$. Suppose that this is not satisfied. Then, it is clear that $\eta \notin \overline{B}_r(c)$, that is, $\left\|\eta - c, z\right\| > r$ for every $c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ and every $z \in X$. Since the set $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ is closed, there exists a vector $\tilde{c} \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ such that

$$\left\|\eta - \tilde{c}, z\right\| = \min\left\{\left\|\eta - c, z\right\| : c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)\right\}.$$

We can write $\nu := \left\|\eta - \tilde{c}, z\right\| > r$, because $\left\|\eta - c, z\right\| > r$, for all $c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ and every $z \in X$. Define $\tilde{\epsilon} := \dfrac{\nu - r}{3}$. Then, we get

$$X \setminus B_{\tilde{\epsilon}}(\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)) \supseteq \left\{y \in X : \left\|\eta - y, z\right\| < \tilde{\epsilon} + r\right\} \tag{3.6}$$

for every $z \in X$, where

$$B_{\tilde{\epsilon}}(\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)) = \left\{y \in X : \min\left\{\left\|y - c, z\right\| : c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)\right\} < \tilde{\epsilon}\right\}.$$

By definition of $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$ we can say that the set

$$\left\{(m,n,k) : \varrho_{mnk} \notin B_{\tilde{\epsilon}}(\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3))\right\}$$

has density zero. Then, by the inclusion (3.6), we have

$$\left\{(m,n,k) : \varrho_{mnk} \notin B_{\tilde{\epsilon}}(\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3))\right\} \supseteq \left\{(m,n,k) : \left\|\varrho_{mnk} - \eta, z\right\| < \tilde{\epsilon} + r\right\} \tag{3.7}$$

for every $z \in X$. Thus, from the inclusion (3.7), for every $z \in X$ we have that the set $\left\{(m,n,k) : \left\|\varrho_{mnk} - \eta, z\right\| < \tilde{\epsilon} + r\right\}$ has natural density zero, which contradicts to (3.5) and so,

$$\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) \subseteq \bigcup_{c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c).$$

Therefore,

$$\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) = \bigcup_{c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c).$$

□

Theorem 3.10. *Let $x = \{\varrho_{mnk}\}$ be a triple sequence and $r \geq 0$. Then*

$$I_{\theta_3} - st_3 - LIM_x^r = \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r).$$

Proof. Necessity. Assume that the sequence $x = \{\varrho_{mnk}\}$ rough I_3-lacunary statistically convergent to ξ. Then, $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3) = \{\xi\}$. By Theorem 3.9, we can write $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) = \overline{B}_r(\xi)$. Therefore, from Theorem 3.1, we get

$$\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3^r) = \overline{B}_r(\xi) = I_{\theta_3} - st_3 - LIM_x^r.$$

Sufficiency. First, we will show that $I_{\theta_3} - st_3 - LIM_x^r = \bigcap_{c\in\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)}$. To do this, let $\xi \in I_{\theta_3} - st_3 - LIM_x^r$ and $c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$. Then, by Theorem 3.6, $\|\xi - c, z\| \leq r$, otherwise, we get

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right) \neq 0$$

for $\epsilon := \dfrac{\|\xi - c, z\| - r}{3}$ for each $z \in X$. This contradicts the fact $\xi \in I_{\theta_3} - st_3 - LIM_x^r$ and therefore,

$$I_{\theta_3} - st_3 - LIM_x^r \subseteq \overline{B}_r(c). \tag{3.8}$$

Now, it follows by inclusion (3.8) that

$$I_{\theta_3} - st_3 - LIM_x^r \subseteq \bigcap_{c\in\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c). \tag{3.9}$$

Now let $y \in \bigcap_{c\in\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c)$. Then for each nonzero $z \in X$, we have

$$\left\|y - c, z\right\| \leq r,$$

for all $c \in \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)$, which is equivalent to

$$\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3) \subseteq \overline{B}_r(y),$$

that is,

$$\bigcap_{c\in\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3)} \overline{B}_r(c) \subseteq \left\{\xi \in X : \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3) \subseteq \overline{B}_r(\xi).\right\} \tag{3.10}$$

Now let $y \notin I_{\theta_3} - st_3 - LIM_x^r$. Then, there exists an $\epsilon > 0$ such that for each nonzero $z \in X$,

$$\delta_{I_3}\left(\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - y, z\right\| \geq r + \epsilon\right\}\right) \neq 0,$$

which implies the existence of anI_3-lacunary rough statistical cluster point c of the sequence x with $\left\|y - c, z\right\| \geq r + \epsilon$, that is $\Lambda_x^{\mathcal{S}_{\theta_3}}(I_3) \not\subseteq \overline{B}_r(y)$ and $y \notin \left\{\xi \in X : \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3) \not\subseteq \overline{B}_r(\xi)\right\}$. Hence, $y \in I_{\theta_3} - st_3 - LIM_x^r$ follows from $y \in \left\{\xi \in X : \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3) \not\subseteq \overline{B}_r(\xi)\right\}$, that is,

$$\left\{\xi \in X : \Lambda_x^{\mathcal{S}_{\theta_3}}(I_3) \not\subseteq \overline{B}_r(\xi)\right\} \subseteq I_{\theta_3} - st_3 - LIM_x^r. \tag{3.11}$$

Therefore, the inclusions (3.9)-(3.11) ensure that (3.8) holds. So it follows by Theorem 3.6 that

$$\bigcap_{c\in\Lambda_x^{S_{\theta_3}}(I_3)} \overline{B}_r(c) = \bigcup_{c\in\Lambda_x^{S_{\theta_3}}(I_3)} \overline{B}_r(c) \tag{3.12}$$

The equality (3.12) is valid if and only if, either the set $\Lambda_x^{S_{\theta_3}}(I_3)$ is empty or it is a singleton. Since

$$I_{\theta_3} - st_3 - LIM_x^r = \bigcap_{c\in\Lambda_x^{S_{\theta_3}}(I_3)} \overline{B}_r(c) = \overline{B}_r(\xi)$$

we have $I_{\theta_3} - st_3 - LIM_x^r = \{\xi\}$. □

4. Lacunary Statistical-Convergence for Triple Sequences via Ideals

In this section, we utilize lacunary sequences and triple sequences to introduce novel concepts related to Wijsman rough I_3-statistical convergence. Subsequently, we derive equivalent results based on these new definitions.

Definition 4.1. *Let r be a non-negative real number. We say that the triple sequence $x = \{\varrho_{mnk}\}$ is Wijsman rough I_3-statistically-convergent to ξ, if for each $\epsilon > 0, \delta > 0$ and for each $z \in X$,*

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{uvw}\left|\left\{m \le u, n \le v, k \le w : \left\|\varrho_{mnk} - \xi, z\right\| \ge \epsilon + r\right\}\right| \ge \delta\right\} \in I_3.$$

In this case we write I_3^r-st-lim $_{W(S)}\varrho_{mnk} = \xi$.

The set of Wijsman rough I_3-statistically-convergent triple sequences will be denoted by

$$W_3S(I_3^r) := \left\{\{\varrho_{mnk}\} : I_3^r - st - \lim{}_{W(S)}\varrho_{mnk} = \xi\right\}.$$

Definition 4.2. *Let $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ be a lacunary triple sequence and r be a non-negative real number. We say that the triple sequence $x = \{\varrho_{mnk}\}$ is Wijsman rough I_3-lacunary statistically convergent to ξ, if for each $\epsilon > 0, \delta > 0$, and for each $z \in X$,*

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}}\left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \ge r + \epsilon\right\}\right| \ge \delta\right\} \in I_3.$$

In this case, we write we write I_3^r-st-lim $_{W_\theta(S_\theta)} = \xi$. The set of Wijsman rough I_3-lacunary statistically convergent triple sequences will be denoted by

$$W_{\theta_3}S_\theta(I_3^r) := \left\{\{\varrho_{mnk}\} : I_3^r - st - \lim{}_{W_\theta(S_\theta)} = \xi\right\}.$$

Definition 4.3. *Let r be a non-negative real number.We say that the triple $x = \{\varrho_{mnk}\}$ is Wijsman rough I_3-statistically convergent of order α to ξ, where $\alpha \in (0,1]$ if for each $\epsilon > 0, \delta > 0$, and for each $z \in X$,*

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{(uvw)^\alpha}\left|\left\{m \le u, n \le v, k \le w : \left\|\varrho_{mnk} - \xi, z\right\| \ge r + \epsilon\right\}\right| \ge \delta\right\} \in I_3.$$

In this case we write I_3^r-st-lim $_{W^\alpha(S)}\varrho_{mnk} = x$.

The set of Wijsman rough I_3-statistically convergent triple sequences of order α will be denoted by

$$W_3^{\alpha}S(I_3^r) := \left\{\{\varrho_{mnk}\} : I_3^r - st - \lim{}_{W^{\alpha}(S)}\varrho_{mnk} = x\right\}.$$

Definition 4.4. *Let $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ be a lacunary triple sequence and r be a non-negative real number. We say that the triple sequence $x = \{\varrho_{mnk}\}$ is Wijsman rough I_3-lacunary statistically convergent of order α to ξ, where $\alpha \in (0, 1]$, if for each $\epsilon > 0, \delta > 0$, and for each $z \in X$,*

$$\left\{(u, v, w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}^{\alpha}} \left|\left\{(u, v, w) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| \geq \delta\right\} \in I_3.$$

In this case, we write we write I_3^r-st-$\lim{}_{W_{\theta}^{\alpha}(S_{\theta})}\varrho_{mnk} = \xi$.

The set of Wijsman rough I_3-lacunary statistically convergent triple sequences of order α will be denoted by

$$W_{\theta_3}^{\alpha}S_{\theta}(I_3^r) = \left\{\{\varrho_{mnk}\} : I_3^r - st - \lim{}_{W_{\theta}^{\alpha}(S_{\theta})}\varrho_{mnk} = \xi\right\}.$$

Theorem 4.1. *Let $0 < \alpha \leq \beta \leq 1$. Then $W_3^{\alpha}S(I_3^r) \subseteq W_3^{\beta}S(I_3^r)$.*

Proof. Let $0 < \alpha \leq \beta \leq 1$. Then for each $z \in X$,

$$\frac{1}{(uvw)^{\beta}} \left|\left\{m \leq u, n \leq v, k \leq w : \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right|$$
$$\leq \frac{1}{(uvw)^{\alpha}} \left|\left\{m \leq u, n \leq v, k \leq w : \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right|$$

and so for each $\delta > 0$ and each $z \in X$,

$$\left\{(u, v, w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{(uvw)^{\beta}} \left|\left\{m \leq u, n \leq v, k \leq w : \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{(u, v, w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{(uvw)^{\alpha}} \left|\left\{m \leq u, n \leq v, k \leq w : \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\}\right| \geq \delta\right\}$$

Hence if the set on the right hand side belongs to the ideal I_3 then obviously the set on the left hand side also belongs to I_3. We obtain the desired result. □

Corollary 4.1. *If a triple sequence is Wijsman rough I_3-statistically-convergent of order α to ξ for some $\alpha \in (0, 1]$ then it is Wijsman rough I_3-statistically-convergent.*

Similarly we can show that

Theorem 4.2. *Let $0 < \alpha \leq \beta \leq 1$. Then $W_{\theta_3}^{\alpha}S_{\theta}(I_3^r) \subseteq W_{\theta_3}^{\beta}S_{\theta}(I_3^r)$ and in particular $W_{\theta_3}^{\alpha}S_{\theta}(I_3^r) \subseteq W_{\theta_3}S_{\theta}(I_3^r)$.*

Definition 4.5. *Let $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ be a lacunary triple sequence and r be a non-negative real number. We say that the triple sequence $x = \{\varrho_{mnk}\}$ is Wijsman strongly rough I_3-lacunary convergent to ξ, if for each $\epsilon > 0$ and each $z \in X$,*

$$\left\{(u, v, w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{\hbar_{uvw}} \sum_{(u,v,w) \in \wp_{uvw}} \left\|\varrho_{mnk} - \xi, z\right\| \geq r + \epsilon\right\} \in I_3.$$

In this case, we write we write I_3^r-lim $_{W_\theta(N_\theta)} = \xi$.

The set of Wijsman strongly rough I_3-lacunary-convergent triple sequences will be denoted by

$$W_{\theta_3}N_\theta(I_3^r) := \left\{ \{\varrho_{mnk}\} : I_3^r - \lim{}_{W_\theta(N_\theta)} = \xi \right\}.$$

Theorem 4.3. *Let* $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ *be a lacunary triple sequence and* r *be a non-negative real number. Then* I_3^r-lim $_{W_\theta(N_\theta)} = \xi$ *implies* I_3^r-*st*-lim $_{W_\theta(S_\theta)} = \xi$.

Proof. Let $\epsilon > 0$ be given and r be a non-negative real number. Then for each $z \in X$,

$$\begin{aligned} \sum_{(u,v,w)\in\wp_{uvw}} \left\| \varrho_{mnk} - \xi, z \right\| &\geq \sum_{\substack{(u,v,w)\in\wp_{uvw} \\ \left\|\varrho_{mnk}-\xi,z\right\| \geq r+\epsilon}} \left\| \varrho_{mnk} - \xi, z \right\| \\ &\geq (r+\epsilon) \left| \left\{ (u,v,w) \in \wp_{uvw} : \left\| \varrho_{mnk} - \xi, z \right\| \geq r+\epsilon \right\} \right| \end{aligned}$$

and consequently,

$$\frac{1}{(r+\epsilon)\hbar_{uvw}} \sum_{(u,v,w)\in\wp_{uvw}} \left\| \varrho_{mnk} - \xi, z \right\| \geq \frac{1}{\hbar_{uvw}} \left| \left\{ (u,v,w) \in \wp_{uvw} : \left\| \varrho_{mnk} - \xi, z \right\| \geq r+\epsilon \right\} \right|$$

Then, for each $\delta > 0$ and each $z \in X$

$$\begin{aligned} &\left\{ (u,v,w) \in \wp_{uvw} : \frac{1}{\hbar_{uvw}} \left| \left\{ (u,v,w) \in \wp_{uvw} : \left\| \varrho_{mnk} - \xi, z \right\| \geq r+\epsilon \right\} \right| \geq \delta \right\} \\ &\subseteq \left\{ (u,v,w) \in \wp_{uvw} : \frac{1}{\hbar_{uvw}} \sum_{(u,v,w)\in\wp_{uvw}} \left\| \varrho_{mnk} - \xi, z \right\| \geq (r+\epsilon)\delta \right\} \in I_3. \end{aligned}$$

This ends the proof. □

Definition 4.6. *A triple sequence* $x = \{\varrho_{mnk}\}$ *is said to be bounded if there exists areal number* $M > 0$ *such that* $\left\| \varrho_{mnk}, z \right\| \leq M$ *for all* $m, n, k \in \mathbb{N}$ *and each* $z \in X$. *We denote the space of all bounded triple sequences by* ℓ_∞^3.

Theorem 4.4. *Let* $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ *be a lacunary triple sequence and* r *be a non-negative real number. If* $x = \{\varrho_{mnk}\} \in \ell_\infty^3$ *and* $x = \{\varrho_{mnk}\}$ *is Wijsman rough* I_3*-lacunary statistical-convergent to* ξ*, then* $x = \{\varrho_{mnk}\}$ *is Wijsman strongly rough* I_3*-lacunary-convergent to* ξ.

Proof. Suppose that $x = \{\varrho_{mnk}\}$ belongs to the space ℓ_∞^3 and I_3^r-st-lim $_{W_\theta(S_\theta)} \varrho_{mnk} = \xi$. Then, we can assume that $\left\| \varrho_{mnk} - \xi, z \right\| \leq M$, for each $z \in X$ and all $m, n, k \in \mathbb{N}$. Given $\epsilon > 0$ and each $z \in X$ we have

$$\begin{aligned} &\frac{1}{\hbar_{uvw}} \sum_{(m,n,k)\in\wp_{uvw}} \left\| \varrho_{mnk} - \xi, z \right\| \\ &= \frac{1}{\hbar_{uvw}} \sum_{\substack{(m,n,k)\in\wp_{uvw} \\ \left\|\varrho_{mnk}-\xi,z\right\| \geq r+\frac{\epsilon}{2}}} \left\| \varrho_{mnk} - \xi, z \right\| + \frac{1}{\hbar_{uvw}} \sum_{\substack{(m,n,k)\in\wp_{uvw} \\ \left\|\varrho_{mnk}-\xi,z\right\| < \frac{\epsilon}{2}}} \left\| \varrho_{mnk} - \xi, z \right\| \\ &\leq \frac{M}{\hbar_{uvw}} \left| \left\{ (m,n,k) \in \wp_{uvw} : \left\| \varrho_{mnk} - \xi, z \right\| \geq r + \frac{\epsilon}{2} \right\} \right| + \frac{\epsilon}{2}. \end{aligned}$$

Consequently, we have

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}} \sum_{(m,n,k)\in\wp_{uvw}} \left\|\varrho_{mnk}-\xi, z\right\| \geq \epsilon + r\right\}$$

$$\subseteq \left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk}-\xi, z\right\| \geq r + \frac{\epsilon}{2}\right\}\right| \geq \frac{\epsilon}{2M}\right\} \in I_3.$$

Consequently, I_3^r-lim $_{W_\theta(S_\theta)}\varrho_{mnk} = \xi$. □

From Theorem 4.3 and Theorem 4.4, we have following Corollary.

Corollary 4.2. *Let $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ be a lacunary triple sequence and r be a non-negative real number. Then we have*

$$W_{\theta_3}S_\theta(I_3^r) \cap \ell_\infty^3 = W_{\theta_3}N_\theta(I_3^r) \cap \ell_\infty^3.$$

We will now look at how the Wijsman rough I_3-statistical-convergence for triple sequence and the Wijsman rough I_3-lacunary statistical-convergence relate to one another.

Theorem 4.5. *Let $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ be a lacunary triple sequence and r be a non-negative real number with* $\liminf q_{uvw} > 1$. *Then, I_3^r-st-*lim $_{W(S)}\varrho_{mnk} = \xi$ *implies I_3^r-st-*lim $_{W_\theta(S_\theta)}\varrho_{mnk} = \xi$.

Proof. Suppose that $\liminf q_{uvw} > 1$. Then, there exists a $\eta > 0$ such that $q_{uvw} \geq \eta + 1$ for sufficiently large u, v, w, which implies

$$\frac{\hbar_{uvw}}{z_{uvw}} \geq \frac{\eta}{\eta+1}.$$

If I_3^r-st-lim $_{W(S)}\varrho_{mnk} = \xi$, then for every $\epsilon > 0$, for each $z \in X$ and for sufficiently large u, v, w, we have

$$\frac{1}{z_{uvw}} \left|\left\{m \leq i_u, n \leq \wp_v, k \leq l_w : \left\|\varrho_{mnk}-\xi, z\right\| \geq \epsilon + r\right\}\right|$$
$$\geq \frac{1}{z_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk}-\xi, z\right\| \geq \epsilon + r\right\}\right|$$
$$\geq \frac{\eta}{\eta+1}\frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk}-\xi, z\right\| \geq \epsilon + r\right\}\right|.$$

Then, for each $z \in X$ and for any $\delta > 0$, we get

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk}-\xi, z\right\| \geq \epsilon + r\right\}\right| \geq \delta\right\}$$
$$\subseteq \{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} :$$
$$\frac{1}{z_{uvw}} \left|\left\{m \leq i_u, n \leq \wp_v, k \leq l_w : \left\|\varrho_{mnk}-\xi, z\right\| \geq \epsilon + r\right\}\right| \geq \frac{\delta\eta}{\eta+1}\right\} \in I_3.$$

So, the result. □

Theorem 4.6. *Let $I_3 = I_3^{fin} = \{J : J$ is finite set$\}$ be a non-trivial ideal, and $\theta_3 = \theta_{uvw} = \{(m_u, n_v, k_w)\}$ be a lacunary triple sequence with* $\limsup q_{uvw} < \infty$ *and r be a non-negative real number. Then we have I_3^r-st-*lim $_{W_\theta(S_\theta)}\varrho_{mnk} = \xi$ *implies I_3^r-st-*lim $_{W(S)}\varrho_{mnk} = \xi$.

Proof. If $\limsup q_{uvw} < \infty$, then without loss of generality, we can assume that there exists a $K > 0$ such that $q_{uvw} < K$ for all $u, v, w \in \mathbb{N}$. Suppose that I_3^r-st-lim $_{W_\theta(S_\theta)} \varrho_{mnk} = \xi$ and for $\epsilon > 0, \delta > 0$ and for each $z \in X$ define the sets

$$\begin{aligned} G_{uvw} &= \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| \text{ and} \\ &\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{\hbar_{uvw}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| \geq \delta\right\} \\ &= \left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{G_{uvw}}{\hbar_{uvw}} \geq \delta\right\} \in I_3. \end{aligned}$$

and, therefore, it is a finite set. We choose integers $u_0, v_0, w_0 \in \mathbb{N}$ such that $\frac{G_{uvw}}{\hbar_{uvw}} < \delta$ for all $u > u_0, v > v_0, w > w_0$.

Let $G = \max\{G_{uvw} : 1 \leq u \leq u_0, 1 \leq v \leq v_0, 1 \leq w \leq w_0\}$ and p, q, y be any three integers with $m_{u-1} < p \leq m_u, n_{v-1} < q \leq n_v$ and $k_{w-1} < y \leq k_w$, then we have

$$\begin{aligned} &\frac{1}{pqy} \left|\left\{m \leq p, n \leq q, k \leq y : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| \\ &\leq \frac{1}{m_{u-1}n_{v-1}k_{w-1}} \left|\left\{m \leq m_u, n \leq n_v, k \leq k_w : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| \\ &= \frac{1}{m_{u-1}n_{v-1}k_{w-1}} \Bigg[\left|\left\{(m,n,k) \in \wp_{111} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right| \\ &+ \cdots + \frac{1}{m_{u-1}n_{v-1}k_{w-1}} \left|\left\{(m,n,k) \in \wp_{uvw} : \left\|\varrho_{mnk} - \xi, z\right\| \geq \epsilon + r\right\}\right|\Bigg] \\ &= \frac{1}{m_{u-1}n_{v-1}k_{w-1}} \left[G_{111} + G_{222} + \cdots + G_{u_0v_0w_0} + \cdots + G_{uvw}\right] \\ &= \frac{G}{m_{u-1}n_{v-1}k_{w-1}} u_0v_0w_0 + \frac{1}{m_{u-1}n_{v-1}k_{w-1}} \Bigg[\hbar_{(u_0+1)(v_0+1)(w_0+1)} \left(\frac{G_{(u_0+1)(v_0+1)(w_0+1)}}{\hbar_{(u_0+1)(v_0+1)(w_0+1)}}\right) \\ &+ \cdots + \hbar_{uvw} \frac{G_{uvw}}{\hbar_{uvw}}\Bigg] \\ &= \frac{G}{m_{u-1}n_{v-1}k_{w-1}} u_0v_0w_0 + \frac{1}{m_{u-1}n_{v-1}k_{w-1}} \left(\sup_{u>u_0, v>v_0, w>w_0} \frac{G_{uvw}}{\hbar_{uvw}}\right) \left(\hbar_{(u_0+1)(v_0+1)(w_0+1)} + \cdots + \hbar_{uvw}\right) \\ &\leq \frac{G}{m_{u-1}n_{v-1}k_{w-1}} u_0v_0w_0 + \delta \left(\frac{m_u n_v k_w - m_{u_0} n_{v_0} k_{w_0}}{m_{u-1}n_{v-1}k_{w-1}}\right) \\ &\leq \frac{G}{m_{u-1}n_{v-1}k_{w-1}} u_0v_0w_0 + \delta q_{uvw} \\ &\leq \frac{G}{m_{u-1}n_{v-1}k_{w-1}} u_0v_0w_0 + \delta K. \end{aligned}$$

Since $m_{u-1} \to \infty, n_{v-1} \to \infty, k_{w-1} \to \infty$, as $p \to \infty, q \to \infty, y \to \infty$, respectively, it follows that I_3^r-st-lim $_{W(S)} \varrho_{mnk} = \xi$. This completes the proof of the theorem. □

Definition 4.7. *A triple sequence $x = \{\varrho_{mnk}\}$ is said to be Wijsman rough I_3 Cesáro convergent to ξ, if for every $\epsilon > 0$ and for each $z \in X$,*

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \left|\frac{1}{uvw}\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|\right| \geq \epsilon + r\right\} \in I_3$$

In this case, we write $\varrho_{mnk} \xrightarrow{W_3C(I_3^r)} \xi$.

Definition 4.8. *A triple sequence $x = \{\varrho_{mnk}\}$ is said to be Wijsman strongly rough I_3 Cesáro convergent to ξ, if for every $\epsilon > 0$ and for each $z \in X$,*

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{uvw}\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\| \geq \epsilon + r\right\} \in I_3$$

In this case, we write $\varrho_{mnk} \xrightarrow{W_3NC(I_3^r)} \xi$.

Definition 4.9. *Let r be a non-negative number and p be a positive real number. A triple sequence $x = \{\varrho_{mnk}\}$ is said to be Wijsman p-strongly rough I_3 Cesáro convergent to ξ, if for every $\epsilon > 0$ and for each $z \in X$,*

$$\left\{(u,v,w) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{uvw}\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^p \geq \epsilon + r\right\} \in I_3$$

In this case, we write $\varrho_{mnk} \xrightarrow{W_3NC_p(I_3^r)} \xi$.

Definition 4.10. *A triple sequence $x = \{\varrho_{mnk}\}$ is said to be I_3-analytic if there exists a positive real number M such that $\left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \left\|\varrho_{mnk}, z\right\| \geq M\right\} \in I_3$ for each $z \in X$.*

Theorem 4.7. *Let r be a non-negative number, p be a positive real number and $x = \{\varrho_{mnk}\}$ be an I_3-analytic triple sequence. If $x = \{\varrho_{mnk}\}$ is Wijsman rough I_3-statistical convergent to ξ, then $\{\varrho_{mnk}\}$ is Wijsman p-strongly rough I_3- Cesáro convergent to ξ.*

Proof. Suppose that $\{\varrho_{mnk}\}$is I_3-analytic triple sequence and $\varrho_{mnk} \xrightarrow{W_3S(I_3^r)} \xi$ Then, there is an $M > 0$ such that $\left\|\varrho_{mnk}-\xi,z\right\|^{\frac{1}{m+n+k}} \leq M$ for each $z \in X$ and all $m,n,k \in \mathbb{N}$. Given $\epsilon > 0$, we get

$$\begin{aligned}
&\frac{1}{uvw}\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^{\frac{p}{m+n+k}} \\
&= \frac{1}{uvw}\sum_{\substack{m=1,n=1,k=1 \\ \left\|\varrho_{mnk}-\xi,z\right\| \geq r+\epsilon}}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^{\frac{p}{m+n+k}} + \frac{1}{uvw}\sum_{\substack{m=1,n=1,k=1 \\ \left\|\varrho_{mnk}-\xi,z\right\| < r+\epsilon}}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^{\frac{p}{m+n+k}} \\
&\leq \frac{M^{\frac{p}{m+n+k}}}{uvw}\left|\left\{(m,n,k) \leq (u,v,w) : \left\|\varrho_{mnk}-\xi,z\right\|^{\frac{p}{m+n+k}} \geq r+\epsilon\right\}\right| \\
&+ \frac{\epsilon^{\frac{p}{m+n+k}}}{uvw}\left|\left\{(m,n,k) \leq (u,v,w) : \left\|\varrho_{mnk}-\xi,z\right\|^{\frac{p}{m+n+k}} < r+\epsilon\right\}\right|
\end{aligned}$$

$$\leq \frac{M^{\frac{p}{m+n+k}}}{uvw}\left|\left\{(m,n,k)\leq(u,v,w):\left\|\varrho_{mnk}-\xi,z\right\|^{\frac{p}{m+n+k}}\geq r+\epsilon\right\}\right|+\epsilon^{\frac{p}{m+n+k}}.$$

Then for any $\delta>0$

$$\left\{(u,v,w)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{uvw}\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^{\frac{p}{m+n+k}}\geq\delta\right\}$$

$$\subseteq\left\{(u,v,w)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{uvw}\left|\left\{(m,n,k)\leq(u,v,w):\left\|\varrho_{mnk}-\xi,z\right\|\geq r+\epsilon\right\}\right|\geq\frac{\delta^{\frac{p}{m+n+k}}}{M^{\frac{p}{m+n+k}}}\right\}\in I_3.$$

Therefore, $\varrho_{mnk}\xrightarrow{WC_p(I_3^r)}\xi$. □

Theorem 4.8. *Let r be a non-negative number, p be a positive real number and $x=\{\varrho_{mnk}\}$ be a triple sequence. If $\{\varrho_{mnk}\}$ is Wijsman p-strongly rough I_3 Cesáro convergent to ξ, then $\{\varrho_{mnk}\}$ is Wijsman rough I_3-statistical convergent to ξ.*

Proof. Suppose that $\varrho_{mnk}\xrightarrow{WC_p(I_3^r)}\xi$ and given $\epsilon>0$. Then we have

$$\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^p\geq\sum_{\substack{m=1,n=1,k=1\\ \left\|\varrho_{mnk}-\xi,z\right\|^p\geq r+\epsilon}}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^p$$

$$\geq\epsilon^p\left|\left\{(m,n,k)\leq(u,v,w):\left\|\varrho_{mnk}-\xi,z\right\|\geq r+\epsilon\right\}\right|$$

for each $z\in X$ and hence

$$\frac{1}{(uvw)\epsilon^p}\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^p\geq\frac{1}{uvw}\left|\left\{(m,n,k)\leq(u,v,w):\left\|\varrho_{mnk}-\xi,z\right\|\geq r+\epsilon\right\}\right|.$$

Consequently, for each $\delta>0$ we have

$$\left\{(u,v,w)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{uvw}\left|\left\{(m,n,k)\leq(u,v,w):\left\|\varrho_{mnk}-\xi,z\right\|\geq r+\epsilon\right\}\right|\geq\delta\right\}$$

$$\subseteq\left\{(u,v,w)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{uvw}\sum_{m=1,n=1,k=1}^{u,v,w}\left\|\varrho_{mnk}-\xi,z\right\|^p\geq(r+\epsilon)^p\delta\right\}\in I_3$$

for each $z\in X$ and so $\varrho_{mnk}\xrightarrow{W_3S(I_3^r)}\xi$. □

5. Conclusion and Future Work

In this paper, we have introduced the concept of the set of rough I_3-lacunary limit points for triple sequences in 2-normed spaces. We have established statistical convergence criteria associated with this set and introduced the concept of rough I_3-lacunary statistical convergence for triple sequences. Furthermore, we have demonstrated that this set of rough I_3-lacunary limit points exhibits both convexity and closure within the context of a 2-normed space. We have also investigated the relationships between a sequence's rough I_3-lacunary statistical cluster points and its rough I_3-lacunary statistical limit points in the same 2-normed space.

Building upon the framework of triple sequence spaces, we have introduced the notion of Wijsman I_3-Cesáro summability for triple sequences and explored the connections between Wijsman strongly I_3-Cesáro summability and Wijsman statistical I_3-Cesáro summability. Additionally, we have introduced the concepts of Wijsman rough strongly p-lacunary summability of order α and Wijsman rough lacunary statistical convergence of order α for triple sequences. These novel concepts have been thoroughly examined to understand their properties, and we have explored potential relationships among them. Furthermore, we have investigated how these newly introduced concepts relate to existing notions in the literature.

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