Fixed Point Methodologies for $\psi$-Contraction Mappings in Cone Metric Spaces over Banach Algebra with Supportive Applications

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Abstract. The explicit aim of this manuscript is to obtain fixed point consequences under novel $\psi$-contraction mappings in a complete cone metric space over Banach algebra. We connect and relate different fixed point theorems by using the idea of $\psi$-contraction mappings, providing a thorough viewpoint that deepens our comprehension of this topic. Our theorems generalize and unify many results in the scientific literature. These prospective extensions offer intriguing research directions and have the potential to further advance the study of fixed point theory. The investigation of examples plays an extremely crucial role in verifying the effectiveness and validity of our theoretical results. Moreover, to support the theoretical results, some examples are investigated to emphasize these results. Ultimately, the existence and uniqueness of the solution to the Urysohn integral and nonlinear fractional differential equation are cooperated as applications to provide an authoritative basis for dealing with actual problems that include these equations.

1. Introduction

Topology and analysis both benefit from, depend on, and mutually supportive of fixed point (FP) theory in important ways. This theory provides fundamental tools and insights that expand our knowledge of various mathematical structures and functions, serving as an essential component for both fields. The basis for investigating the characteristics and behaviors of mappings in a variety...
of spaces is laid by the fixed point theory (FP), which enhances the fabric of topology and analysis by proving the presence of fixed points under particular circumstances. Mathematical modelling, mathematical physics, economics, chemistry, and biology are just a few of the disciplines where the FP theory is useful. In mathematical modelling, for instance, and not as a limitation, FP theory presents crucial resources for studying dynamic systems and identifying equilibrium states. FP analysis can be used to describe the stable states and long-term behaviors of these complex systems, whether researchers are simulating ecological interactions, population increase, or the spread of diseases.

Using Banach’s contraction principle (BCP) [1], the fixed point technique may solve a variety of ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations (IEs), fractional differential equations (FDEs) and current optimization problems (see [2–5]). The BCP is the most powerful and fundamental result in metric FP theory. By altering the axioms of the metric notion, the concept of metric has been expanded in a number of ways: quasi-metric, symmetric, dislocated, 2-metric, b-metric, D-metric, G-metric, S-metric, ultra-metric, partial metric, etc. We will concentrate on Banach-valued metric space, or more specifically, cone metric spaces (CMSs) over Banach algebra (BA).

In 1695, Leibniz introduced the idea of fractional calculus (FC) [6], which is one of the developments of ordinary calculus. Lately, the FC theory has an influential role in fluid mechanics, entropy, engineering and physics [7–10]. Some engineering techniques and physical models can be interpreted more practically and accurately using FC. As an instance, FC-based entropies may be used more widely than the entropy of Shannon [11]. Fractional entropy has been an extensively investigated topic because of how widely it is used [12]. In addition, the fractional differential equations (FDEs) are very useful for modeling and describing a variety of phenomena [13]. This is due to the fact that a system’s next state is determined by all of its previous circumstances, not simply its current form. Compared to integer-order differential equations, these equations may better reflect physical reality. It’s crucial to highlight that the theory and applications of FC have been extensively discussed within the literature [14–18]. FDEs have attracted a lot of attention in recent years due to their precise explanation of complex events in viscoelastic materials, system identification, control issues, signal processing, non-Brownian motion and polymers [19]. Recent studies have concentrated on fractional functional analysis and many applications have been investigated to fractional ordinary differential systems, fractional ODEs, and fractional PDEs [20–28].

In 2007, Zhang and Huang [29] presented the idea of CMSs using ordered Banach space instead of the set of real numbers. They discussed some properties of the convergence sequences and showed some FP results of contractive mappings in such spaces. Many articles have recently used the same methodology to generalize the results from the ordinary metric space to the CMSs; For more details, see [30–37].

On the other hand, Xu and Liu [38] presented the idea of a CMSs over BA, which generalizes and extends the BCP in ordinary metric spaces. Numerous authors used CMSs over Banach algebras
(BAs) to unify the BCP in many directions. Without making the assumption of normality, the authors established a number of FP results for generalized Lipschitz mappings in the new circumstances, which have no connection to metric space in relation to the existence of the mapping’s FP. Many studies have been written regarding FPs in spaces that resemble cone b-metrics over BA and other spaces; see [39–49] for more information.

Motivated by the previous results, the explicit purpose of this manuscript is to launch the notion of ψ-contraction mappings and examine various FP results in CMSs over BA. Our crucial results have been supported by many corollaries, illustrative examples and applications for the existence and uniqueness to IEs and nonlinear FDEs. Our results represent a generalization, development and an extension to the scientific research in the literature.

2. BASIC FACTS

This part is devoted to presenting basic concepts that help us for obtaining our goals.

Definition 2.1. [50] Suppose that $B$ denotes a real BA, yield to the following properties (for every $\omega, \varrho, \theta \in B, \gamma \in \mathbb{R}$):

(BA$_1$) $\omega (\varrho \theta) = (\omega \varrho) \theta$;
(BA$_2$) $\omega (\varrho + \theta) = \omega \varrho + \omega \theta$ and $(\omega + \varrho) \theta = \omega \theta + \varrho \theta$;
(BA$_3$) $\gamma (\omega \varrho) = (\gamma \omega) \varrho = \omega (\gamma \varrho)$;
(BA$_4$) $\|\omega \varrho\| \leq \|\omega\| \|\varrho\|$.

Proposition 2.1. [51] Let $(\Delta, d)$ be a complete metric space and let $d_B(f(\omega), f(\varrho)) \leq \psi(d_B(\omega, \varrho))$ for all $\omega, \varrho \in \Delta$ where $\psi : [0, \infty) \rightarrow [0, \infty)$ is any monotone non-decreasing function with $\lim_{n \rightarrow \infty} \psi^n(\tau) = 0$ for any fixed $\tau > 0$. Then $f$ has a unique FP.

Definition 2.2. [42] A subset $P$ of $B$ is called a cone if
1. $P$ is closed, non-empty and $\{\theta, I\} \in P$;
2. $\alpha P + \beta P \subset P$ for all $\alpha, \beta \in \mathbb{R}_+$;
3. $PP = P^2 \subset P$;
4. $(-P) \cap P = \{\theta\}$;
where $\theta$ denotes the null of the BA $B$.

For a given cone $P \subset B$, define a partial order $\leq$ with respect to $P$ by $\omega \leq \varrho$ if and only if $\varrho - \omega \in P$. $\omega < \varrho$ will denote $\omega \leq \varrho$ and $\omega \neq \varrho$. While $\omega < \varrho$ will denote $\varrho - \omega \in \text{int}P$ where $\text{int}P$ represents the interior of $P$. If $\text{int}P \neq \varnothing$, then $P$ is called a solid cone.

The cone $P$ is called normal if there exist a number $W > 0$ such that for all $\omega, \varrho \in B$,

$$\theta \leq \omega \leq \varrho \implies \|\omega\| \leq W \|\varrho\|.$$

The smallest positive number verifying the above inequality is called a normal constant [52].

Definition 2.3. [29] Let $\Delta$ be a non-empty set. Suppose that the mapping $d_B : \Delta \times \Delta \rightarrow B$ verifies, for all $\omega, \varrho, \theta \in \Delta$
(i) \( \theta < d_B(\omega, \varrho) \) with \( \omega \neq \varrho \) and \( d_B(\omega, \varrho) = \theta \) if \( \omega = \varrho \); 
(ii) \( d_B(\omega, \varrho) = d_B(\varrho, \omega) \); 
(iii) \( d_B(\omega, \varrho) \leq d_B(\omega, \delta) + d_B(\delta, \varrho) \).

Then \( d_B \) is called a cone metric on \( \Delta \), and \( (\Delta, d_B) \) is called a CMS over \( \mathcal{B} \).

**Example 2.1.** Assume that \( \mathcal{B} = \mathbb{R}^2 \), \( \mathcal{P} = \{(\omega, \varrho) \in \mathcal{B} : 0 \leq \omega, \varrho \} \subset \mathbb{R}^2 \), \( \Delta = \mathbb{R} \) and \( d_B : \Delta \times \Delta \to \mathcal{B} \) such that \( d_B(\omega, \varrho) = (|\omega - \varrho|, \alpha|\omega - \varrho|) \), where \( \alpha \geq 0 \) is a constant. Then \( (\Delta, d_B) \) is a CMS over \( \mathcal{B} \).

**Definition 2.4.** [38] Let \( (\Delta, d_B) \) be a CMS over \( \mathcal{B} \), \( \omega \in \Delta \) and \( \{\omega_n\} \) be a sequence in \( \Delta \). Consequently,

(i) for each \( \theta < c \), there exists \( N \in \mathbb{N} \) such that \( d_B(\omega_n, \omega) < c \) for all \( n \geq N \).

This means that \( \{\omega_n\} \) converges to \( \omega \) and it can be written briefly as

\[
\lim_{n \to \infty} \omega_n = \omega \quad \text{or} \quad \omega_n \to \infty \quad (n \to \infty);
\]

(ii) for each \( \theta < c \), there exists \( N \in \mathbb{N} \) such that \( d_B(\omega_m, \omega_n) < c \) for all \( m, n \geq N \).

Then, \( \omega_n \) is a Cauchy sequence (CS);

(iii) if every CS in \( \Delta \) is convergent then, \( (\Delta, d_B) \) is complete.

**Remark 2.1.** [50] The spectral radius (SR) \( \rho(\omega) \) of \( \omega \) verifies \( \rho(\omega) \leq \|\omega\| \) for all \( \omega \in \mathcal{B} \), where \( \mathcal{B} \) is a BA with a unit \( I \).

**Remark 2.2.** [50] The SR \( \rho(\omega) \) of \( \omega \) satisfies

\[
\rho(\omega) \leq \|\omega\|,
\]

for all \( \omega \in \mathcal{B} \), where \( \mathcal{B} \) is a BA with a unit \( I \).

**Remark 2.3.** [53] If \( \rho(\omega) < 1 \) then \( \|\omega^n\| \to 0 \) as \( n \to \infty \).

**Lemma 2.1.** If \( \mathcal{B} \) is a unital BA with unit \( I \), \( \psi : \mathcal{B}_+ \to \mathcal{B}_+ \) and \( \rho(\psi(\omega)) < 1 \). Then \( (I_B - \psi) \) is invertible and

\[
(I_B - \psi)^{-1}(\omega) = \sum_{n=0}^{\infty} \psi^n(\omega).
\]

**Lemma 2.2.** [50] Let \( \mathcal{B} \) be a BA with a unit \( I \) and \( \omega, \varrho \in \mathcal{B} \). If \( \omega \) commutes with \( \varrho \), then

\[
\rho(\omega + \varrho) \leq \rho(\omega) + \rho(\varrho), \quad \rho(\omega \varrho) \leq \rho(\omega) \rho(\varrho).
\]

**Definition 2.4.** [54] Let \( \Psi_B \) be the set of all positive functions \( \psi_B : \mathcal{B}_+ \to \mathcal{B}_+ \) satisfying the following conditions:

(i) \( \psi(\omega + \varrho) = \psi(\omega) + \psi(\varrho) \),
(ii) \( \psi(\omega \varrho) = \psi(\omega) \psi(\varrho) \),
(iii) \( \lim_{n \to +\infty} \psi^n(\omega) = \theta \) for all \( \omega > 0 \),
(iv) \( \psi(\omega) = \theta \) iff \( \omega = 0 \).
Let us display some major definitions, constructions and basic concepts of FC (see [55, 56]). If \( h : [0, \infty) \rightarrow \mathbb{R} \) is a continuous function, then the Caputo fractional derivative (CFD) of order \( \alpha \) is defined by

\[
\mathcal{D}^\alpha h(t) = \int_0^t \frac{h^{(n)}(s)}{\Gamma(n-\alpha)(t-s)^{1+\alpha-n}} \, ds,
\]

\((n-1 < \alpha < n, n = \lfloor \alpha \rfloor + 1)\), where \( \Gamma \) is a gamma function and \( \lfloor \alpha \rfloor \) stands for the integer part of the positive real number \( \alpha \). Also, \( \mathcal{R-LJ}^\alpha \) represents the Riemann-Liouville fractional integral (R-LFI) of order \( \alpha \), and defined as follows:

\[
\mathcal{R-LJ}^\alpha h(t) = \int_0^t \frac{h(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}} \, ds.
\]

3. FP results in CMS over BA

We begin this part with proving the following lemmas:

**Lemma 3.1.** If \( \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is any monotone non-decreasing function with \( \psi(\omega) < \omega \) for all \( \omega > 0 \), \( \rho(\omega) < 1 \) and \( \|\psi^n(\omega)\| \leq \|\psi\|^n \cdot \|\omega\|^n \) then \( \rho(\psi^n(\omega)) < 1 \) where \( \rho \) is the spectral radius.

**Proof.**

\[
\rho(\psi^n(\omega)) = \rho(\psi(\omega) \cdot \psi(\omega) \ldots \psi(\omega)) = \rho(\psi(\omega)) \cdot \rho(\psi(\omega)) \ldots \rho(\psi(\omega)) = \left( \rho(\psi(\omega)) \right)^n < \left( \rho(\omega) \right)^n < 1.
\]

Therefore, \( \rho(\psi^n(\omega)) < 1 \).

**Lemma 3.2.** If \( \psi < I_B \), then \( (I_B - \psi) \) has an inverse where \( (I_B - \psi)^{-1} = \sum_{n=0}^{\infty} \psi^n \).

**Proof.**

\[
(I_B - \psi) (I_B - \psi)^{-1} = \left( I_B - \psi \left( \sum_{n=0}^{\infty} \psi^n \right) \right) = (I_B - \psi) (I_B + \psi + \psi^2 + \ldots) = I_B.
\]

Thus, \( (I_B - \psi) \) has an inverse.

**Theorem 3.1.** Suppose that \( (\Delta, \mathcal{B}, d_B) \) be a complete CMS over BA \( \mathcal{B} \). Let \( S : \Delta \rightarrow \Delta \) and \( d_B : \Delta \times \Delta \rightarrow \mathbb{B}_+ \) be mappings satisfy

\[
d_B(S\omega, S\varrho) \leq \psi(d_B(\omega, \varrho)), \quad \text{for all } \omega, \varrho \in \Delta,
\]

where \( \psi : \mathcal{B}_+ \rightarrow \mathcal{B}_+ \). Then, \( S \) has a unique FP in \( \Delta \).
Proof. Let an arbitrary element $\omega_0$ in $\Delta$. Define a sequence $\{\omega_n\}$ by $S^n\omega_0 = S\omega_{n-1} = \omega_n, \ n \geq 1$. From our contraction condition, we get

\[ d_B(\omega_{n+1}, \omega_n) = d_B(S\omega_n, S\omega_{n-1}) \leq \psi(d_B(\omega_n, \omega_{n-1})) = \psi(d_B(S\omega_{n-1}, S\omega_{n-2})) \leq \psi^2(d_B(\omega_{n-1}, \omega_{n-2})) \leq \psi^n(d_B(\omega_1, \omega_0)). \]

Then, for $n > m$, we find

\[ d_B(\omega_n, \omega_m) \leq d_B(\omega_n, \omega_{n-1}) + d_B(\omega_{n-1}, \omega_{n-2}) + \ldots + d_B(\omega_{m+1}, \omega_m) \leq \psi^{n-1}(d_B(\omega_1, \omega_0)) + \psi^{n-2}(d_B(\omega_1, \omega_0)) + \ldots + \psi^m(d_B(\omega_1, \omega_0)) \leq \sum_{k=m}^{n-1} \psi^k(d_B(\omega_1, \omega_0)) \rightarrow 0_B, \text{ for all } d_B(\omega_1, \omega_0) > 0_B, \ n, m \rightarrow \infty. \]

Hence $\{\omega_n\}$ is a CS in $\Delta$. Based on the completeness of $\Delta$, there exists $\omega^* \in \Delta$ such that $\lim_{n \to \infty} \omega_n = \omega^*$. Therefore, one has

\[ 0_B \leq d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, S\omega_n) + d_B(S\omega_n, S\omega^*) \leq d_B(\omega^*, \omega_{n+1}) + \psi(d_B(\omega_n, \omega^*)). \]

It follows that

\[ 0_B \leq d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, \omega^*) + \psi(d_B(\omega^*, \omega^*)) \leq 0_B + \psi(0_B). \]

Thus, $d_B(\omega^*, S\omega^*) \leq 0_B$, which is a contradiction. Then, $d_B(\omega^*, S\omega^*) = 0_B$, i.e., $\omega^* = S\omega^*$ is a FP of $S$.

Now, if $(\varphi^* \neq 0) \neq \omega^*$ is another FP of the mapping $S$, then

\[ 0_B \leq d_B(\omega^*, \varphi^*) = d_B(S\omega^*, S\varphi^*) \leq \psi(d_B(\omega^*, \varphi^*)), \]

that is,
\[(I_B - \psi)\left(d_B(\omega^*, \varrho^*)\right) \leq 0_B.\]

Since \((I_B - \psi)\) is invertible, then
\[(I_B - \psi)^{-1} (I_B - \psi)\left(d_B(\omega^*, \varrho^*)\right) \leq (I_B - \psi)^{-1}(0_B)\]
\[\iff I_B\left(d_B(\omega^*, \varrho^*)\right) \leq \sum_{n=0}^{\infty} \psi^n(0_B).\]
Again, we get a contradiction. Therefore,
\[d_B(\omega^*, \varrho^*) = 0_B \implies \omega^* = \varrho^*.\]

This implies that the FP is unique.

The example below support Theorem 3.1:

**Example 3.1.** Let \(B_+ = \mathbb{R}^2\) and \(|(\omega_1, \omega_2)| = |\omega_1| + |\omega_2|\) for all \((\omega_1, \omega_2) \in B\). Define the operation of multiplication by
\[
\omega \varrho = (\omega_1, \omega_2). (\varrho_1, \varrho_2) = (\omega_1 \varrho_1, \omega_1 \varrho_2 + \omega_2 \varrho_1).
\]

Then \(B\) is a BA with unit \(I = (1, 0)\).

Let \(P = \{(\omega_1, \omega_2) \in \mathbb{R}^2 \mid \omega_1, \omega_2 \geq 0\}\).

Let \(\Delta = \mathbb{R}\) and the metric \(d_B : \Delta \times \Delta \to \mathbb{R}_+^{2}\) be defined by
\[
d_B(\omega, \varrho) = \left(\left|\omega - \varrho\right|, \beta \left|\omega - \varrho\right|\right) \in P, \quad \text{with} \quad \beta \geq 0.
\]

Then, \((\Delta, B, d)\) is a complete CMS over \(BA\).

Now, define the mapping \(S : \Delta \to \Delta\) by
\[
S\omega = \frac{\omega}{4},
\]
and \(\psi : B_+ \to B_+\) by
\[
\psi(t) = \frac{t}{3}.
\]
where \(B_+ = \mathbb{R}_+^2\). Then, we obtain
\[
d_B(S\omega, S\varrho) = \left(\left|S\omega - S\varrho\right|, \beta \left|S\omega - S\varrho\right|\right)
= \frac{1}{4} \left(\left|\omega - \varrho\right|, \beta \left|\omega - \varrho\right|\right)
= \frac{1}{4} d_B(\omega, \varrho)
= \frac{3}{4} \left(\frac{d_B(\omega, \varrho)}{3}\right).
\]
\[
\frac{d_B(\omega, \varrho)}{3} = \psi\left(d_B(\omega, \varrho)\right).
\]

Therefore all hypotheses of Theorem 3.1 are satisfied and 0 is a unique FP of \( S. \)

**Theorem 3.2.** Let \((\Delta, B, d_B)\) be a complete CMS with \( BA \ B. \) Let \( S : \Delta \rightarrow \Delta \) and \( d_B : \Delta \times \Delta \rightarrow B_+ \) be mappings satisfy

\[
d_B(S\omega, S\varrho) \leq \psi\left(d_B(S\omega, \omega) + d_B(S\varrho, \varrho)\right), \quad \text{for all } \omega, \varrho \in \Delta,
\]

where \( \psi : B_+ \rightarrow B_+ \) and \( \rho(\psi(\omega)) < \frac{1}{3}. \) Then, \( S \) has a unique FP in \( \Delta. \)

**Proof.** Let an arbitrary element \( \omega_0 \) in \( \Delta. \) Define a sequence \( \{\omega_n\} \) by \( S^n\omega_0 = S\omega_{n-1} = \omega_n, \ n \geq 1. \) From our contraction condition, we obtain

\[
d_B(\omega_{n+1}, \omega_n) = d_B(S(\omega_n, S(\omega_{n-1})
\leq \psi\left(d_B(S\omega_n, \omega_n) + d_B(S\omega_{n-1}, \omega_{n-1})\right)
= \psi\left(d_B(\omega_{n+1}, \omega_n) + d_B(\omega_n, \omega_{n-1})\right)
\leq \psi\left(d_B(\omega_{n+1}, \omega_n)\right) + \psi\left(d_B(\omega_n, \omega_{n-1})\right).
\]

Then

\[
(I_B - \psi) d_B(\omega_{n+1}, \omega_n) \leq \psi\left(d_B(\omega_n, \omega_{n-1})\right).
\]

By multiplying \((I - \psi)^{-1}\)

\[
d_B(\omega_{n+1}, \omega_n) \leq (I_B - \psi)^{-1} \psi\left(d_B(\omega_n, \omega_{n-1})\right)
= \left[(I_B - \psi)^{-1} \psi\right]^n(d_B(\omega_1, \omega_0).
\]

Now, we shall prove that

\[
\rho\left(\left[(I_B - \psi)^{-1} \psi\right](\omega)\right) < 1, \quad \text{if } \rho(\psi(\omega)) < \frac{1}{3}.
\]

Therefore,

\[
\rho\left(\left[(I_B - \psi)^{-1} \psi\right](\omega)\right) \leq \rho\left((I_B - \psi)^{-1}(\omega)\right). \rho(\psi(\omega))
= \rho\left(\sum_{n=0}^{\infty} \psi^n(\omega)\right). \rho(\psi(\omega))
\]
\[
\leq \left( \sum_{n=0}^{\infty} \rho(\psi^n(\omega)) \right) \cdot \rho(\psi(\omega)) \\
= \left( \sum_{n=0}^{\infty} [\rho(\psi(\omega))]^n \right) \cdot \rho(\psi(\omega)) \\
\leq \frac{\rho(I_B)}{\rho(I_B) - \rho(\psi(\omega))} \cdot \rho(\psi(\omega)) \\
= \frac{1}{1 - \frac{1}{3}} \cdot \frac{1}{2} < 1.
\]

Thus, for \( n > m \), we get
\[
d_B(\omega_n, \omega_m) \leq d_B(\omega_n, \omega_{n-1}) + d_B(\omega_{n-1}, \omega_{n-2}) + \ldots + d_B(\omega_{m+1}, \omega_m) \\
\leq \left[ (I_B - \psi)^{-1} \right]^{n-1} \left( d_B(\omega_1, \omega_0) \right) + \left[ (I_B - \psi)^{-1} \right]^{n-2} \left( d_B(\omega_1, \omega_0) \right) \\
+ \ldots + \left[ (I_B - \psi)^{-1} \right]^{m} (d_B(\omega_1, \omega_0)) \\
\leq \sum_{l=m}^{n-1} \left[ (I_B - \psi)^{-1} \right]^{l} (d_B(\omega_1, \omega_0)) \rightarrow 0_B, \text{ for all } d_B(\omega_1, \omega_0) > 0_B, \quad n, m \rightarrow \infty.
\]

Hence \( \{\omega_n\} \) is a CS in \( \Delta \). Based on the completeness of \( \Delta \), there exists \( \omega^* \in \Delta \) such that \( \lim_{n \rightarrow \infty} \omega_n = \omega^* \). Therefore, one has
\[
0_B \leq d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, S\omega_n) + d_B(S\omega_n, S\omega^*) \\
\leq d_B(\omega^*, \omega_{n+1}) + \psi \left( d_B(S\omega_n, \omega_n) + d_B(S\omega^*, \omega^*) \right) \\
\leq d_B(\omega^*, \omega_{n+1}) + \psi \left( d_B(\omega_{n+1}, \omega_n) \right) + \psi \left( d_B(S\omega^*, \omega^*) \right).
\]

Taking \( \lim_{n \rightarrow \infty} \), we get
\[
d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, \omega^*) + \psi \left( d_B(\omega^*, \omega^*) + d_B(\omega^*, S\omega^*) \right) \\
\leq 0_B + 0_B + \psi \left( d_B(\omega^*, S\omega^*) \right),
\]
that is,
\[
(I_B - \psi) \left( d_B(\omega^*, S\omega^*) \right) \leq 0_B.
\]
Since \((I_B - \psi)\) is invertible, then
\[
(I_B - \psi)^{-1} (I_B - \psi) \left( d_B(\omega^*, S\omega^*) \right) \leq (I_B - \psi)^{-1} (0_B).
\]
Thus, \(d_B(\omega^*, S\omega^*) \leq 0_B\), which is a contradiction. Then, \(d_B(\omega^*, S\omega^*) = 0_B\), i.e., \(\omega^* = S\omega^*\) is a FP of \(S\).

Now, if \((\varphi^* \neq 0) \neq \omega^*\) is another FP of the mapping \(S\), then
\[
0_B \leq d_B(\omega^*, \varphi^*) = d_B(S\omega^*, S\varphi^*)
\]
\[
\leq \psi \left( d_B(S\omega^*, \omega^*) + d_B(S\varphi^*, \varphi^*) \right)
\]
\[
= \psi \left( d_B(\omega^*, \omega^*) \right) + \psi \left( d_B(\varphi^*, \varphi^*) \right),
\]
that is,
\[
d_B(\omega^*, \varphi^*) \leq 0_B,
\]
which is a contradiction. Therefore,
\[
d_B(\omega^*, \varphi^*) = 0_B \implies \omega^* = \varphi^*.
\]

This implies that the FP is unique.

The example below support Theorem 3.2:

**Example 3.2.** Assume the same hypotheses of Example 3.1 then, we get
\[
d_B(S\omega, \omega) + d_B(S\varphi, \varphi) = \left( |S\omega - \omega| , \beta |S\omega - \omega| \right) + \left( |S\varphi - \varphi| , \beta |S\varphi - \varphi| \right)
\]
\[
= \frac{3}{4} \left( |\omega| , \beta |\omega| \right) + \frac{3}{4} \left( |\varphi| , \beta |\varphi| \right)
\]
\[
= \frac{3}{4} \left( |\omega| + |\varphi| , \beta \left( |\omega| + |\varphi| \right) \right)
\]
\[
\geq \frac{3}{4} \left( |\omega - \varphi| , \beta |\omega - \varphi| \right)
\]
\[
= 3 \left( \left| \frac{\omega}{4} - \frac{\varphi}{4} \right| , \beta \left| \frac{\omega}{4} - \frac{\varphi}{4} \right| \right)
\]
\[
= 3 \left( |S\omega - S\varphi| , \beta |S\omega - S\varphi| \right)
\]
\[
= 3 d_B(S\omega, S\varphi),
\]
i.e.,
\[
d_B(S\omega, S\varphi) \leq \frac{1}{3} \left( d_B(S\omega, \omega) + d_B(S\varphi, \varphi) \right).
\]
It follows that
\[ d_B(S\omega, S\varrho) \leq \psi \left( d_B(S\omega, \omega) + d_B(S\varrho, \varrho) \right). \]

Therefore all conditions of Theorem 3.2 are verified and \(0\) is a unique FP of \(S\).

**Theorem 3.3.** Let \((\Delta, \mathcal{B}, d_B)\) be a complete CMS over \(\mathcal{B} \mathcal{A} \mathcal{B}\). Let \(S : \Delta \rightarrow \Delta\) and \(d_B : \Delta \times \Delta \rightarrow \mathcal{B}_+\) be mappings satisfy
\[ d_B(S\omega, S\varrho) \leq \psi \left( d_B(S\omega, \omega) + d_B(S\varrho, \varrho) \right), \]
for all \(\omega, \varrho \in \Delta\),

where \(\psi : \mathcal{B}_+ \rightarrow \mathcal{B}_+\) and \(\rho(\psi(\omega)) < \frac{1}{3}\). Then, \(S\) has a unique FP in \(\Delta\).

**Proof.** Let \(\omega_0\) be an arbitrary element in \(\Delta\). Define a sequence \(\{\omega_n\}\) by \(S\omega_0 = S\omega_n = \omega_n, \; n \geq 1\). From our contraction condition, we get
\[ d_B(\omega_{n+1}, \omega_n) = d_B(S\omega_n, S\omega_{n-1}) \]
\[ \leq \psi \left( d_B(S\omega_n, \omega_{n-1}) + d_B(S\omega_{n-1}, \omega_n) \right) \]
\[ = \psi \left( d_B(\omega_{n+1}, \omega_{n-1}) + d_B(\omega_n, \omega_{n-1}) \right) \]
\[ \leq \psi \left( d_B(\omega_{n+1}, \omega_n) + d_B(\omega_n, \omega_{n-1}) \right) \]
\[ \leq \psi \left( d_B(\omega_{n+1}, \omega_n) \right) + \psi \left( d_B(\omega_n, \omega_{n-1}) \right). \]

It follows that
\[ (I_B - \psi) d_B(\omega_{n+1}, \omega_n) \leq \psi(d_B(\omega_n, \omega_{n-1})). \]

By multiplying \((I - \psi)^{-1}\)
\[ d_B(\omega_{n+1}, \omega_n) \leq (I_B - \psi)^{-1} \psi(d_B(\omega_n, \omega_{n-1})) \]
\[ \vdots \]
\[ \leq \left[ (I_B - \psi)^{-1} \psi \right]^n(d_B(\omega_1, \omega_0)). \]

Now, we show that
\[ \rho \left[ \left( (I_B - \psi)^{-1} \psi \right) (\omega) \right] < 1, \quad \text{if} \quad \rho(\psi(\omega)) < \frac{1}{3}. \]

Therefore,
\[ \rho \left[ \left( (I_B - \psi)^{-1} \psi \right) (\omega) \right] \leq \rho \left( (I_B - \psi)^{-1}(\omega) \right) \rho(\psi(\omega)) \]
\[ = \rho \left( \sum_{n=0}^{\infty} \psi^n(\omega) \right) \rho(\psi(\omega)) \]
\[ \begin{align*}
&\leq \left( \sum_{n=0}^{\infty} \rho \left( \psi^n(\omega) \right) \right) \cdot \rho \left( \psi(\omega) \right) \\
&= \left( \sum_{n=0}^{\infty} \left[ \rho \left( \psi(\omega) \right) \right]^n \right) \cdot \rho \left( \psi(\omega) \right) \\
&\leq \frac{\rho \left( I_{B} \right)}{\rho \left( I_{B} \right) - \rho \left( \psi(\omega) \right)} \cdot \rho \left( \psi(\omega) \right) \\
&= \frac{1}{1 - \frac{1}{3}} \cdot \frac{1}{3} < 1.
\end{align*} \]

Thus, for \( n > m \), we get
\[
d_B(\omega_n, \omega_m) \leq d_B(\omega_n, \omega_{n-1}) + d_B(\omega_{n-1}, \omega_{n-2}) + \ldots + d_B(\omega_{m+1}, \omega_m)
\]
\[
\leq \left[ (I_B - \psi)^{-1} \psi \right]^{n-1} \left( d_B(\omega_1, \omega_0) \right) + \left[ (I_B - \psi)^{-1} \psi \right]^{n-2} \left( d_B(\omega_1, \omega_0) \right)
\]
\[
+ \ldots + \left[ (I_B - \psi)^{-1} \psi \right]^{m} \left( d_B(\omega_1, \omega_0) \right)
\]
\[
\leq \sum_{l=m}^{n-1} \left[ (I_B - \psi)^{-1} \psi \right]^{l} \left( d_B(\omega_1, \omega_0) \right) \rightarrow 0_B, \quad \text{for all } d_B(\omega_1, \omega_0) > 0_B, \quad n, m \rightarrow \infty.
\]

Hence \( \{\omega_n\} \) is a CS in \( \Delta \). Based on the completeness of \( \Delta \), there exists \( \omega^* \in \Delta \) such that \( \lim_{n \rightarrow \infty} \omega_n = \omega^* \). Therefore, one has
\[
0_B \leq d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, S\omega_n) + d_B(S\omega_n, S\omega^*)
\]
\[
\leq d_B(\omega^*, \omega_{n+1}) + \psi \left( d_B(S\omega_n, \omega^*) + d_B(S\omega^*, \omega_n) \right)
\]
\[
\leq d_B(\omega^*, \omega_{n+1}) + \psi \left( d_B(\omega_{n+1}, \omega^*) \right) + \psi \left( d_B(S\omega^*, \omega_n) \right).
\]

Taking \( \lim_{n \rightarrow \infty} \), we get
\[
d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, \omega^*) + \psi \left( d_B(\omega^*, \omega^*) \right) + \psi \left( d_B(S\omega^*, \omega^*) \right)
\]
\[
\leq 0_B + 0_B + \psi \left( d_B(\omega^*, S\omega^*) \right),
\]
that is,
\[
(I_B - \psi) \left( d_B(\omega^*, S\omega^*) \right) \leq 0_B.
\]

Since \( (I_B - \psi) \) is invertible, then
\[
(I_B - \psi)^{-1} (I_B - \psi) \left( d_B(\omega^*, S\omega^*) \right) \leq (I_B - \psi)^{-1} (0_B).
\]
Thus, $d_B(\omega^*, S\omega^*) \leq 0_B$, which is a contradiction. Then, $d_B(\omega^*, S\omega^*) = 0_B$, i.e., $\omega^* = S\omega^*$ is a FP of $S$.

Now, if $(\varphi^* \neq 0) \neq \omega^*$ is another FP of the mapping $S$, then

$$0_B \leq d_B(\omega^*, \varphi^*) = d_B(S\omega^*, S\varphi^*) \leq \psi\left(d_B(S\omega^*, \varphi^*) + d_B(S\varphi^*, \omega^*)\right) \leq \psi\left(d_B(\omega^*, \varphi^*)\right) + \psi\left(d_B(\varphi^*, \omega^*)\right),$$

that is,

$$(I_B - 2\psi)(d_B(\omega^*, \varphi^*)) \leq 0_B.$$ 

Since $\psi(\omega) = 0$ iff $a = 0$ and $I_B(\omega) \neq 2\psi(\omega)$. Then,

$$d_B(\omega^*, \varphi^*) = 0_B \implies \omega^* = \varphi^*.$$

This implies that the FP is unique.

The example below support Theorem 3.3:

**Example 3.3.** Assume the same assumptions of Example 3.1 then, we have

$$d_B(S\omega, S\varphi) = \left(\|S\omega - S\varphi\|, \beta \|S\omega - S\varphi\|\right)$$

$$= \frac{1}{4} \left(\|\omega - \varphi\|, \beta \|\omega - \varphi\|\right)$$

$$= \frac{1}{4} \left(\|\omega - S\omega + S\omega - \varphi\|, \beta \|\omega - S\omega + S\omega - \varphi\|\right)$$

$$\leq \frac{1}{4} \left(\|\omega - S\omega\| + \|S\omega - \varphi\|, \beta \left(\|\omega - S\omega\| + \|S\omega - \varphi\|\right)\right)$$

$$= \frac{1}{4} \left[(\|\omega - S\omega\|, \beta \|\omega - S\omega\|)\right] + \frac{1}{4} \left[(\|S\omega - \varphi\|, \beta \|S\omega - \varphi\|)\right]$$

$$= \frac{1}{4} \left[(\|\omega - S\varphi + S\varphi - S\omega\|, \beta \|\omega - S\varphi + S\varphi - S\omega\|)\right]$$

$$+ \frac{1}{4} \left[(\|S\omega - \varphi\|, \beta \|S\omega - \varphi\|)\right]$$

$$\leq \frac{1}{4} \left[(\|\omega - S\varphi\|, \beta \|\omega - S\varphi\|)\right] + \frac{1}{4} \left[(\|S\varphi - S\omega\|, \beta \|S\varphi - S\omega\|)\right]$$

$$+ \frac{1}{4} \left[(\|S\omega - \varphi\|, \beta \|S\omega - \varphi\|)\right]$$

$$= \frac{1}{4} \left[(\|S\varphi - \omega\|, \beta \|S\varphi - \omega\|)\right] + \frac{1}{4} \left[(\|S\omega - S\varphi\|, \beta \|S\omega - S\varphi\|)\right]$$
Theorem 3.4. Let \( \Delta, \mathcal{B}, d_\mathcal{B} \) be a complete CMS with BA \( \mathcal{B} \). Let \( S : \Delta \to \Delta \) and \( d_\mathcal{B} : \Delta \times \Delta \to \mathcal{B}_+ \) be mappings satisfy
\[
d_\mathcal{B}(S\omega, S\varphi) \leq \frac{1}{3} \left( d_\mathcal{B}(S\omega, \varphi) + d_\mathcal{B}(S\varphi, \omega) \right),
\]
for all \( \omega, \varphi \in \Delta \),

where \( \psi : \mathcal{B}_+ \to \mathcal{B}_+ \) and \( \rho(\psi(\omega)) < \frac{1}{5} \). Then, \( S \) has a unique FP in \( \Delta \).

Proof. Let \( \omega_0 \) be an arbitrary element in \( \Delta \). Define a sequence \( \{\omega_n\} \) by \( S^n\omega_0 = S\omega_{n-1} = \omega_n, \ n \geq 1 \). From our contraction condition, we have
\[
d_\mathcal{B}(\omega_{n+1}, \omega_n) = d_\mathcal{B}(S\omega_n, S\omega_{n-1}) \]

\[
\leq \psi\left( d_\mathcal{B}(\omega_n, \omega_{n-1}) + \frac{d_\mathcal{B}(S\omega_n, \omega_n) \cdot d_\mathcal{B}(S\omega_{n-1}, \omega_{n-1})}{1 + d_\mathcal{B}(\omega_n, \omega_{n-1})} \right) \]

\[
= \psi\left( d_\mathcal{B}(\omega_n, \omega_{n-1}) + \frac{d_\mathcal{B}(\omega_{n+1}, \omega_n) \cdot d_\mathcal{B}(\omega_{n+1}, \omega_{n-1})}{1 + d_\mathcal{B}(\omega_n, \omega_{n-1})} \right) \]

\[
\leq \psi\left( d_\mathcal{B}(\omega_{n+1}, \omega_n) + d_\mathcal{B}(\omega_n, \omega_{n-1}) \right) \]

\[
\leq \psi\left( d_\mathcal{B}(\omega_{n+1}, \omega_n) \right) + \psi\left( d_\mathcal{B}(\omega_n, \omega_{n-1}) \right) \]

Then
\[
(\mathcal{I} - \psi) d_\mathcal{B}(\omega_{n+1}, \omega_n) \leq \psi\left( d_\mathcal{B}(\omega_{n+1}, \omega_n) \right).
\]

By multiplying \( (\mathcal{I} - \psi)^{-1} \)
\[
d_\mathcal{B}(\omega_{n+1}, \omega_n) \leq \left[ (\mathcal{I} - \psi)^{-1} \psi \right]\left( d_\mathcal{B}(\omega_n, \omega_{n-1}) \right)\]

\[ \vdots \]
\[\leq \left((I_B - \psi)^{-1}\psi\right)^n d_B(\omega_1, \omega_0).\]

Now, we verify that
\[\rho \left(\left((I_B - \psi)^{-1}\psi\right)(\omega)\right) < 1, \quad \text{if } \rho(\psi(\omega)) < \frac{1}{5}.\]

Therefore,
\[\rho \left(\left((I_B - \psi)^{-1}\psi\right)(\omega)\right) \leq \rho \left(\left((I_B - \psi)^{-1}\psi\right)(\omega)\right) \cdot \rho(\psi(\omega))\]
\[= \rho \left(\sum_{n=0}^{\infty} \psi^n(\omega)\right) \cdot \rho(\psi(\omega))\]
\[\leq \left(\sum_{n=0}^{\infty} \rho(\psi^n(\omega))\right) \cdot \rho(\psi(\omega))\]
\[= \left(\sum_{n=0}^{\infty} \rho(\psi(\omega))^n\right) \cdot \rho(\psi(\omega))\]
\[\leq \frac{\rho(I_B)}{\rho(I_B) - \rho(\psi(\omega))} \cdot \rho(\psi(\omega))\]
\[= \frac{1}{1 - \frac{1}{5}} = \frac{1}{4} < 1.\]

Thus, for \(n > m\), we get
\[d_B(\omega_n, \omega_m)\]
\[\leq d_B(\omega_n, \omega_{n-1}) + d_B(\omega_{n-1}, \omega_{n-2}) + \ldots + d_B(\omega_{m+1}, \omega_m)\]
\[\leq \left((I_B - \psi)^{-1}\psi\right)^{n-1} d_B(\omega_1, \omega_0) + \left((I_B - \psi)^{-1}\psi\right)^{n-2} d_B(\omega_1, \omega_0)\]
\[+ \ldots + \left((I_B - \psi)^{-1}\psi\right)^{m} d_B(\omega_1, \omega_0)\]
\[\leq \sum_{l=m}^{n-1} \left((I_B - \psi)^{-1}\psi\right)^l d_B(\omega_1, \omega_0) \rightarrow 0_B, \quad \text{for all } d_B(\omega_1, \omega_0) > 0_B, \quad n, m \rightarrow \infty.\]

Hence \(\{\omega_n\}\) is a CS in \(\Delta\). Based on the completeness of \(\Delta\), there exists \(\omega^* \in \Delta\) such that \(\lim_{n \to \infty} \omega_n = \omega^*\). Therefore, one has
\[0_B \leq d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, S\omega_n) + d_B(S\omega_n, S\omega^*)\]
\[\leq d_B(\omega^*, \omega_{n+1}) + \psi \left(d_B(\omega_n, \omega^*) + \frac{S\omega_n, \omega_n \cdot d_B(S\omega^*, \omega^*)}{1 + d_B(\omega_n, \omega^*)}\right)\]
\[
= d_G(\omega^*, \omega_{n+1}) + \psi \left( d_G(\omega_n, \omega^*) + \frac{d_G(\omega_{n+1}, \omega_n) \cdot d_G(\omega^*, \omega^*)}{1 + d_G(\omega_n, \omega^*)} \right)
\]
\[
\leq d_G(\omega^*, \omega_{n+1}) + \psi \left( d_G(\omega_n, \omega^*) \right).
\]
Taking \( \lim_{n \to \infty} \), we get
\[
d_G(\omega^*, S\omega^*) \leq d_G(\omega^*, \omega^*) + \psi \left( d_G(\omega^*, \omega^*) \right) \leq 0_B + \psi(0_B) = 0_B.
\]
Thus, \( d_G(\omega^*, S\omega^*) \leq 0_B \), which is a contradiction. Then, \( d_G(\omega^*, S\omega^*) = 0_B \), i.e., \( \omega^* = S\omega^* \) is a FP of \( S \).

Now, if \( (\varrho^* \neq 0) \neq \omega^* \) is another FP of the mapping \( S \), then
\[
0_B \leq d_G(\omega^*, \varrho^*) = d_G(S\omega^*, S\varrho^*) \leq \psi \left( d_G(\omega^*, \varrho^*) + \frac{d_G(S\omega^*, \omega^*) \cdot d_G(S\varrho^*, \varrho^*)}{1 + d_G(\omega^*, \varrho^*)} \right) = \psi \left( d_G(\omega^*, \varrho^*) \right),
\]
that is,
\[
(\mathcal{I}_B - \psi) \left( \psi \left( d_G(\omega^*, \varrho^*) \right) \right) \leq 0_B.
\]
Since \( (\mathcal{I}_B - \psi) \) is invertible, then
\[
(\mathcal{I}_B - \psi)^{-1} (\mathcal{I}_B - \psi) \left( \psi \left( d_G(\omega^*, \varrho^*) \right) \right) \leq (\mathcal{I}_B - \psi)^{-1} (0_B).
\]
Then,
\[
d_G(\omega^*, \varrho^*) = 0_B \implies \omega^* = \varrho^*.
\]
This implies that the FP is unique.

In Theorem 3.4, if \( \psi = \lambda I \) then we obtain the result below:

**Corollary 3.1.** Let \((\Delta, \mathcal{B}, d_B)\) be a complete CMS with BA \( \mathcal{B} \). Let \( S : \Delta \to \Delta \) and \( d_B : \Delta \times \Delta \to \mathcal{B}_+ \) be mappings satisfy
\[
d_B(S\omega, S\varrho) \leq \lambda \left( d_B(\omega, \varrho) + \frac{d_B(S\omega, \omega) \cdot d_B(S\varrho, \varrho)}{1 + d_B(\omega, \varrho)} \right), \quad \text{for all } \omega, \varrho \in \Delta,
\]
where \( 0 \leq \lambda < 1 \) is a constant and \( \rho(\omega) < \frac{1}{5} \). Then, \( S \) has a unique FP in \( \Delta \).
Theorem 3.5. Let \((\Delta, \mathcal{B}, d_{\mathcal{B}})\) be a complete CMS over BA \(\mathcal{B}\). Let \(S : \Delta \rightarrow \Delta\) and \(d_{\mathcal{B}} : \Delta \times \Delta \rightarrow \mathcal{B}_+\) be mappings satisfy
\[
d_{\mathcal{B}}(S\omega, S\varrho) \leq \psi \left( d_{\mathcal{B}}(\omega, \varrho) + \frac{d_{\mathcal{B}}(S\omega, \varrho) \cdot d_{\mathcal{B}}(S\varrho, \omega)}{1 + d_{\mathcal{B}}(\omega, \varrho)} \right),
\]
for all \(\omega, \varrho \in \Delta\), where \(\psi : \mathcal{B}_+ \rightarrow \mathcal{B}_+\). Then, \(S\) has a unique FP in \(\Delta\).

Proof. Let \(\omega_0\) be an arbitrary element in \(\Delta\). Define a sequence \(\{\omega_n\}\) by \(S^n\omega_0 = S\omega_{n-1} = \omega_n, \ n \geq 1\). From our contraction condition, we get
\[
d_{\mathcal{B}}(\omega_{n+1}, \omega_n) = d_{\mathcal{B}}(S\omega_n, S\omega_{n-1})
\leq \psi \left( d_{\mathcal{B}}(\omega_n, \omega_{n-1}) + \frac{d_{\mathcal{B}}(S\omega_n, \omega_{n-1}) \cdot d_{\mathcal{B}}(S\omega_{n-1}, \omega_n)}{1 + d_{\mathcal{B}}(\omega_n, \omega_{n-1})} \right)
= \psi \left( d_{\mathcal{B}}(\omega_n, \omega_{n-1}) + \frac{d_{\mathcal{B}}(\omega_{n+1}, \omega_{n-1}) \cdot d_{\mathcal{B}}(\omega_{n-1}, \omega_n)}{1 + d_{\mathcal{B}}(\omega_n, \omega_{n-1})} \right)
\leq \psi \left( d_{\mathcal{B}}(\omega_n, \omega_{n-1}) \right).
\]

Then
\[
d_{\mathcal{B}}(\omega_{n+1}, \omega_n) \leq \psi \left( d_{\mathcal{B}}(\omega_n, \omega_{n-1}) \right)
\vdots
\leq \psi^n \left( d_{\mathcal{B}}(\omega_1, \omega_0) \right).
\]

Then, for \(n > m\), we find
\[
d_{\mathcal{B}}(\omega_n, \omega_m) \leq d_{\mathcal{B}}(\omega_n, \omega_{n-1}) + d_{\mathcal{B}}(\omega_{n-1}, \omega_{n-2}) + \ldots + d_{\mathcal{B}}(\omega_m+1, \omega_m)
\leq \psi^{n-1} \left( d_{\mathcal{B}}(\omega_1, \omega_0) \right) + \psi^{n-2} \left( d_{\mathcal{B}}(\omega_1, \omega_0) \right) + \ldots + \psi^m \left( d_{\mathcal{B}}(\omega_1, \omega_0) \right)
\leq \sum_{k=m}^{n-1} \psi^k \left( d_{\mathcal{B}}(\omega_1, \omega_0) \right) \rightarrow 0_{\mathcal{B}}, \quad \text{for all } d_{\mathcal{B}}(\omega_1, \omega_0) > 0_{\mathcal{B}}, \ n, m \rightarrow \infty.
\]

Hence \(\omega_n\) is a CS in \(\Delta\). Based on the completeness of \(\Delta\), there exists \(\omega^* \in \Delta\) such that \(\lim_{n \rightarrow \infty} \omega_n = \omega^*\). Therefore, one has
\[
0_{\mathcal{B}} \leq d_{\mathcal{B}}(\omega^*, S\omega) \leq d_{\mathcal{B}}(\omega^*, S\omega_n) + d_{\mathcal{B}}(S\omega_n, S\omega^*)
\leq d_{\mathcal{B}}(\omega^*, \omega_{n+1}) + \psi \left( d_{\mathcal{B}}(\omega_n, \omega^*) + \frac{d_{\mathcal{B}}(S\omega_n, \omega^*) \cdot d_{\mathcal{B}}(S\omega^*, \omega_n)}{1 + d_{\mathcal{B}}(\omega_n, \omega^*)} \right)
\]
Now, if \(0 \leq d\) is \(\in\), then \(\psi\) satisfies

\[
\begin{align*}
\Delta \trianglerighteq d_B(\omega^*, \omega_n) + \psi\left(d_B(\omega_n, \omega^*) + \frac{d_B(\Omega_{n+1}, \omega^*) \cdot d_B(\Omega_{n}, \omega^*)}{1 + d_B(\Omega_{n}, \omega^*)}\right) \\
\leq d_B(\omega^*, \omega_{n+1}) + \psi\left(d_B(\omega_n, \omega^*) + d_B(\omega_{n+1}, \omega^*)\right) \\
\leq d_B(\omega^*, \omega_{n+1}) + \psi\left(d_B(\omega_n, \omega^*)\right) + \psi\left(d_B(\omega_{n+1}, \omega^*)\right).
\end{align*}
\]

Taking \(\lim_{n \to \infty}\), we get

\[
d_B(\omega^*, S\omega^*) \leq d_B(\omega^*, \omega^*) + \psi\left(d_B(\omega^*, \omega^*)\right) + \psi\left(d_B(\omega^*, \omega^*)\right) = 0_B + \psi(0_B) + \psi(0_B) = 0_B.
\]

Thus, \(d_B(\omega^*, S\omega^*) \leq 0_B\), that is a contradiction. Then, \(d_B(\omega^*, S\omega^*) = 0_B\), i.e., \(\omega^* = S\omega^*\) is a FP of \(S\).

Now, if \((\omega^* \neq 0) \neq \omega^*\) is another FP of the mapping \(S\), then

\[
0_B \leq d_B(\omega^*, \omega^*) = d_B(S\omega^*, S\omega^*)
\]

\[
\leq \psi\left(d_B(\omega^*, \omega^*) + \frac{d_B(S\omega^*, \omega^*) \cdot d_B(S\omega^*, \omega^*)}{1 + d_B(\omega^*, \omega^*)}\right)
\]

\[
\leq \psi\left(d_B(\omega^*, \omega^*) + \frac{d_B(S\omega^*, \omega^*) \cdot d_B(\omega^*, \omega^*)}{1 + d_B(\omega^*, \omega^*)}\right)
\]

\[
= 2 \psi\left(d_B(\omega^*, \omega^*)\right),
\]

that is,

\[
(I_B - 2 \psi)\left(d_B(\omega^*, \omega^*)\right) \leq 0_B.
\]

Since \(\psi(\omega) = 0_B\) if \(\omega = 0\) and \(I_B(\omega) \neq 2 \psi(\omega)\). Then,

\[
d_B(\omega^*, \omega^*) = 0_B \implies \omega^* = \omega^*.
\]

This implies that the FP is unique.

In Theorem 3.5, if \(\psi = \lambda I\) then we have the result below:

**Corollary 3.2.** Let \((\Delta, B, d_B)\) be a complete CMS over \(BA\) \(B\). Let \(S : \Delta \to \Delta\) and \(d_B : \Delta \times \Delta \to B_+\) be mappings satisfy

\[
d_B(S\omega, S\varphi) \leq \lambda \left(d_B(\omega, \varphi) + \frac{d_B(S\omega, \varphi) \cdot d_B(S\varphi, \omega)}{1 + d_B(\omega, \varphi)}\right), \quad \text{for all } \omega, \varphi \in \Delta,
\]

where \(0 \leq \lambda < 1\) is a constant. Then, \(S\) has a unique FP in \(\Delta\).
4. Solve Urysohn integral equations

Here, we apply Theorem 3.2 to discuss the existence and uniqueness of solution to the following Urysohn IEs:

\[
\begin{align*}
\omega(t) &= h(t) + \int_0^1 K_1(t, s, \omega(s)) \, ds, \\
\varrho(t) &= h(t) + \int_0^1 K_2(t, s, \varrho(s)) \, ds,
\end{align*}
\]

where \(K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}\) and \(h : [0, 1] \to \mathbb{R}\).

Let \(\mathcal{B}\) and \(\mathcal{P}\) are defined as Example 3.1. Let \(\Delta = C([0, 1], \mathbb{R})\) where \(C[0, 1]\) denotes the set of all real continuous functions (RCFs), \(\mathcal{P}\) is a cone and \(d_{\mathcal{B}} : \Delta \times \Delta \to \mathcal{B}\) is defined as follows:

\[
\begin{align*}
d_{\mathcal{B}}(\omega, \varrho) &= \left( \|\omega - \varrho\|_{\infty}, \beta \|\omega - \varrho\|_{\infty} \right) \\
&= \left( \sup_{t \in [0, 1]} |\omega(t) - \varrho(t)|, \beta \sup_{t \in [0, 1]} |\omega(t) - \varrho(t)| \right) \in \mathcal{P},
\end{align*}
\]

where \(\omega, \varrho \in \Delta\). Then, \((\Delta, \mathcal{B}, d)\) is a complete CMS over \(\mathcal{B}\).

Now, we define \(\psi : \mathcal{B}_{+} \to \mathcal{B}_{+}\) as

\[\psi(r) = \mu r,\]

where \(\mu \in [0, 1)\) and \(\mathcal{B}_{+} = \mathbb{R}^2_{+}\).

**Theorem 4.1.** Define the mapping \(S : \Delta \to \Delta\) by

\[
\begin{align*}
S\omega(t) &= h(t) + \int_0^1 K_1(t, s, \omega(s)) \, ds, \\
S\varrho(t) &= h(t) + \int_0^1 K_2(t, s, \varrho(s)) \, ds,
\end{align*}
\]

for all \(t \in [0, 1]\). If the following inequality

\[|K_1(t, s, \omega(s)) - K_2(t, s, \varrho(s))| \leq \mu |\omega(t) - \varrho(t)|,\]

holds, the IEs (4.1) and (4.2) have a unique solution.

**Proof.** It is clear that finding the solution of the equations (4.1) and (4.2) is equivalent to finding the FP of the mapping \(S\).

Now, consider

\[
\begin{align*}
d_{\mathcal{B}}(Sx, S\varrho) &= \left( \|S\omega - S\varrho\|_{\infty}, \beta \|S\omega - S\varrho\|_{\infty} \right) \\
&= \left( \sup_{t \in [0, 1]} |S\omega(t) - S\varrho(t)|, \beta \sup_{t \in [0, 1]} |S\omega(t) - S\varrho(t)| \right)
\end{align*}
\]
Theorem 4.2. Assume that all conditions of Theorem 4.1 are true. Define

\[ d_B(Sx, S\varrho) = \left( \|S\omega - S\varrho\|_\infty, \beta \|S\varrho - \varrho\|_\infty \right) \]

as \( \psi(r) = \lambda r \) where \( \lambda = \frac{\mu}{2} \in [0, 1) \). Then the proposed equations have a unique solution.

Proof.

\[
\begin{align*}
&= \left( \sup_{t \in [0,1]} \left| \int_0^1 K_1(t, s, \omega(s)) \, ds - \int_0^1 K_2(t, s, \varrho(s)) \, ds \right|, \\
&\quad \beta \sup_{t \in [0,1]} \left| \int_0^1 K_1(t, s, \omega(s)) \, ds - \int_0^1 K_2(t, s, \varrho(s)) \, ds \right|) \\
&\leq \left( \sup_{t \in [0,1]} \int_0^1 |K_1(t, s, \omega(s)) - K_2(t, s, \varrho(s))| \, ds, \beta \sup_{t \in [0,1]} \int_0^1 |K_1(t, s, \omega(s)) - K_2(t, s, \varrho(s))| \, ds \right) \\
&\leq \mu \left( \sup_{t \in [0,1]} \int_0^1 |\omega(t) - \varrho(t)| \, ds, \beta \sup_{t \in [0,1]} \int_0^1 |\omega(t) - \varrho(t)| \, ds \right) \\
&\leq \mu \left( \|\omega(t) - \varrho(t)\|_\infty, \beta \|\omega(t) - \varrho(t)\|_\infty \right) \\
&= \mu d_B(\omega, \varrho) = \psi(d_B(\omega, \varrho)),
\end{align*}
\]

where \( \mu \in [0, 1) \). Therefore all requirements of Theorem 3.1 hold, then the problems (4.1) and (4.2) have a unique solution.

Now, we can apply the results shown in Theorem 3.3 to obtain the same results of Theorem 4.1.

**Theorem 4.2.** Assume that all conditions of Theorem 4.1 are true. Define \( \psi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2 \) as \( \psi(r) = \lambda r \) where \( \lambda = \frac{\mu}{2} \in [0, 1) \). Then the proposed equations have a unique solution.

Proof.
Hence, all stipulations of Theorem 3.3 are verified. Then the considered problems have a unique solution.

5. Solving a Caputo fractional derivative

In this part, we will implicate some theoretical results to study the existence and uniqueness of the solution to a CFD of order $\alpha$, it takes the form

$$\text{cD}^\alpha(\omega(t)) = f(t, \omega(t)) \quad (0 < t < 1, \ 1 < \alpha \leq 2),$$

via the integral boundry conditions

$$\omega(0) = 0, \quad \omega(1) = \int_0^{\eta} \omega(s) \, ds \quad (0 < \eta < 1), \quad \text{with} \ \omega \in C([0,1], \mathbb{R}),$$
where $C([0, 1], \mathbb{R})$ is the set of all RCFs from $[0, 1]$ into $\mathbb{R}$, $C^D_\alpha$ represents the CFD of order $\alpha$ and $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a CF (see [57]).

Now, we present our main theorem in this section.

**Theorem 5.1.** Problem (5.1) has a solution provided that the following hypotheses are true:

(A) there exists $\psi \in \Psi$ such that

$$|f(t, \omega) - f(t, \varphi)| \leq \frac{\Gamma(\alpha + 1)}{3} \psi(|\omega - \varphi|, \lambda |\omega - \varphi|),$$

for all $\omega, \varphi \in \mathbb{R}$, $\lambda \geq 0$ and $t \in [0, 1]$;

(B) for all $t \in [0, 1]$, there exists a mapping $S : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ such that

$$S\omega(t) = \int_0^t \frac{f(s, \omega(s))}{\Gamma(\alpha) (t-s)^{1-\alpha}} ds$$

$$+ 2t \int_0^1 \left( \int_0^s \frac{f(r, \omega(r))}{\Gamma(\alpha) (s-r)^{1-\alpha}} dr \right) ds;$$

(C) for all $t \in [0, 1]$ and $\omega, \varphi \in C([0, 1], \mathbb{R})$, let the metric $d_B : C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}^2_+$ be defined as

$$d_B(\omega, \varphi) = \left( \|\omega - \varphi\|_{\infty}, \lambda \|\omega - \varphi\|_{\infty} \right)$$

$$= \left( \sup_{t \in [0,1]} |\omega(t) - \varphi(t)|, \lambda \sup_{t \in [0,1]} |\omega(t) - \varphi(t)| \right) \in \mathcal{P};$$

(D) for all $t \in [0, 1]$, if $\{\omega_n\}$ is a sequence in $C([0, 1], \mathbb{R})$ and $\omega \in C([0, 1], \mathbb{R})$, then $\omega_n \rightarrow \omega$ in $C([0, 1], \mathbb{R})$.

Then, the problem (5.1) has at least one solution.

**Proof.** Let $\Delta = C([0, 1], \mathbb{R})$ where $\Delta$ is the Banach space equipped with the supremum norm $\|\omega\|_{\infty} = \sup_{t \in [0,1]} |\omega(t)| \quad \forall \, \omega \in \Delta$. Then, $\omega \in \Delta$ is a solution of (5.1) if and only if $\omega \in \Delta$ is a solution of the following IE:

$$\omega(t) = \int_0^t \frac{f(s, \omega(s))}{\Gamma(\alpha) (t-s)^{1-\alpha}} ds$$

$$+ 2t \int_0^1 \left( \int_0^s \frac{f(r, \omega(r))}{\Gamma(\alpha) (s-r)^{1-\alpha}} dr \right) ds, \quad t \in [0, 1].$$

Define the function $\psi : \mathcal{B}_+ \rightarrow \mathcal{B}_+$ as

$$\psi(p) \leq p,$$
where $\mathcal{B}_+ = \mathbb{R}_+^2$. Now, let $\varpi, \varrho \in \Delta$ and $t \in [0, 1]$. By assumption (A), we have

$$\left| S_{\varpi}(t) - S_{\varrho}(t) \right| = \left| \int_0^t \frac{f(s, \varpi(s))}{\Gamma(\alpha)(t-s)^{1-\alpha}} \, ds - \int_0^t \frac{f(s, \varrho(s))}{\Gamma(\alpha)(t-s)^{1-\alpha}} \, ds \right|$$

$$+ \frac{2t}{\Gamma(\alpha)} \left| \int_0^1 \left( \int_0^s \frac{f(r, \varpi(r))}{\Gamma(\alpha)(s-r)^{1-\alpha}} \, dr \right) ds - \int_0^1 \left( \int_0^s \frac{f(r, \varrho(r))}{\Gamma(\alpha)(s-r)^{1-\alpha}} \, dr \right) ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left| f(s, \varpi(s)) - f(s, \varrho(s)) \right| ds$$

$$+ \frac{2t}{\Gamma(\alpha)} \int_0^1 \left| \int_0^s \frac{f(r, \varpi(r)) - f(r, \varrho(r))}{(s-r)^{1-\alpha}} \, dr \right| ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left| f(s, \varpi(s)) - f(s, \varrho(s)) \right| ds$$

$$+ \frac{2t}{\Gamma(\alpha)} \int_0^1 \left| \int_0^s \frac{f(r, \varpi(r)) - f(r, \varrho(r))}{|s-r|^{1-\alpha}} \, dr \right| ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left| f(s, \varpi(s)) - f(s, \varrho(s)) \right| ds$$

$$+ \frac{2t}{\Gamma(\alpha)} \int_0^1 \left( \int_0^s \frac{f(r, \varpi(r)) - f(r, \varrho(r))}{|s-r|^{1-\alpha}} \, dr \right) ds$$

$$\leq \frac{\Gamma(\alpha + 1)}{3 \Gamma(\alpha)} \psi \left( |\varpi(s) - \varrho(s)|, \lambda |\varpi(s) - \varrho(s)| \right) \int_0^t \left| t - s |^{\alpha-1} \right| ds$$

$$+ \frac{2t \Gamma(\alpha + 1)}{3 \Gamma(\alpha)} \psi \left( |\varpi(r) - \varrho(r)|, \lambda |\varpi(r) - \varrho(r)| \right) \int_0^1 \left( \int_0^s |s-r|^{\alpha-1} \, dr \right) ds$$

$$\leq \frac{\Gamma(\alpha + 1)}{3 \alpha \Gamma(\alpha)} \psi \left( |\varpi(s) - \varrho(s)|, \lambda |\varpi(s) - \varrho(s)| \right) \int_0^t \left| t - s |^{\alpha-1} \right| ds$$

$$+ \frac{2t \Gamma(\alpha + 1)}{3 \alpha \Gamma(\alpha)} \psi \left( |\varpi(r) - \varrho(r)|, \lambda |\varpi(r) - \varrho(r)| \right) \int_0^1 \left( \int_0^s |s-r|^{\alpha-1} \, dr \right) ds$$

$$\leq \psi \left( |\varpi(s) - \varrho(s)|, \lambda |\varpi(s) - \varrho(s)| \right) \frac{\mu^t}{3}$$

$$+ \frac{2t}{3} B(\alpha + 1, 1) \psi \left( |\varpi(r) - \varrho(r)|, \lambda |\varpi(r) - \varrho(r)| \right) \int_0^t s^{(\alpha+1)-1} (1-s)^{(1-1) \frac{1}{1-1}} ds$$
\[
\begin{align*}
&\leq \psi\left(\sup_{t \in [0,1]} |\varpi(t) - \varrho(t)|, \lambda \sup_{t \in [0,1]} |\varpi(t) - \varrho(t)|\right) \frac{t^{\alpha}}{3} \\
&\quad + \frac{2t}{3(\alpha + 1)} \psi\left(\sup_{t \in [0,1]} |\varpi(t) - \varrho(t)|, \lambda \sup_{t \in [0,1]} |\varpi(t) - \varrho(t)|\right) \\
&\quad + \frac{2t}{3(\alpha + 1)} \psi\left(\sup_{t \in [0,1]} |\varpi(t) - \varrho(t)|, \lambda \sup_{t \in [0,1]} |\varpi(t) - \varrho(t)|\right) \\
&\quad \leq \psi\left(\|\varpi(t) - \varrho(t)\|_\infty\right) \left(\frac{t^{\alpha}}{3} + \frac{2t}{3(\alpha + 1)}\right),
\end{align*}
\]
where \(B\) denotes the beta function.

Taking supremum on both sides, we get
\[
\begin{align*}
\sup_{t \in [0,1]} |S\varpi(t) - S\varrho(t)| &\leq \psi\left(\|\varpi(t) - \varrho(t)\|_\infty, \lambda \|\varpi(t) - \varrho(t)\|_\infty\right) \times \sup_{t \in [0,1]} \left(\frac{t^{\alpha}}{3} + \frac{2t}{3(\alpha + 1)}\right),
\end{align*}
\]
which implies that
\[
\|S\varpi(t) - S\varrho(t)\|_\infty \leq \psi\left(\|\varpi(t) - \varrho(t)\|_\infty, \lambda \|\varpi(t) - \varrho(t)\|_\infty\right).
\]
Consequently,
\[
\begin{align*}
d_B(S\varpi, S\varrho) &= \left(\|S\varpi - S\varrho\|_\infty, \lambda \|S\varpi - S\varrho\|_\infty\right) \\
&\leq \left(\psi\left(\|\varpi(t) - \varrho(t)\|_\infty, \lambda \|\varpi(t) - \varrho(t)\|_\infty\right), \lambda \psi\left(\|\varpi(t) - \varrho(t)\|_\infty, \lambda \|\varpi(t) - \varrho(t)\|_\infty\right)\right) \\
&= (1, \lambda) \psi\left(\|\varpi(t) - \varrho(t)\|_\infty, \lambda \|\varpi(t) - \varrho(t)\|_\infty\right) \\
&\leq \psi\left(\|\varpi(t) - \varrho(t)\|_\infty, \lambda \|\varpi(t) - \varrho(t)\|_\infty\right) \\
&= \psi\left(d_B(\varpi, \varrho)\right).
\end{align*}
\]
Then, all assertions of Theorem 3.1 are satisfied.

**Remark 5.1.** By a similar way of Theorem 5.1, if \(\psi : B_+ \to B_+\) is defined by \(\psi(p) \leq p\) where \(B_+ = \mathbb{R}^2_+\). We can show that all conditions of Theorem 3.3 are verified.
Conclusions

In this article, we introduced our findings under novel $\psi$–contraction mapping within the framework of a complete CMS over $\mathbf{BA}$. We investigated several definitions, corollaries and theorems under $\psi$-contraction mapping in such spaces. These definitions, corollaries and theorems are regarded as an extension and generalisation of the findings in the literature. On the other hand, we presented several cases of $\psi$-contraction mapping in main results and applications to the findings of our fundamental theorems using Ursohn integral equations (UIEs) and nonlinear FDEs. We concentrated on the nonlinear FDEs and UIEs as applications that mark a substantial contribution to the fields of FC and IEs. There are numerous guidelines and methods for the future work as well as investigations and discoveries in this particular discipline such as studying and analyzing variations of $\psi$-contraction mappings, such as weak $\psi$-contractions or mixed $\psi$-contractions to learn more about their properties and applications. Also, we can establish significant developments on the existence and uniqueness of solutions to certain integral and FDEs by utilizing the power of $\psi$-contraction mappings.

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