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Subclasses of Janowski Associated with q-Derivative

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Abstract. Herglotz representation and Bieberbach type properties are discussed for the q-analog classes of Janowski starlike and convex functions.

1. Introduction

The unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ is normally treated as a standard domain because simply connected proper subsets of the complex plane are conformally equavlent to $\mathcal{U}$ in view of Riemann mapping theorem. The functions whose ranges describe certain geometries like star, close-to-star, covex, close-to-convex, spiral, some in certain directions, some uniformly, some with respect to conjugate symmetric points and so on are known as geometric functions and these geometries can be described in succinct mathematical terms along with establishing closed links between certain prescribed properties of analytic functions and the geometries of their ranges. A subset D of the complex plane is called q-geometric if $qz \in D$ for fixed $q \in \mathbb{R}$, whenever $z \in D$. This paper is devoted to the study of q analog of Janowski starlike and convex functions and explore certain results like inclusion properties, Herglotz representation and Bieberbach cojecture. Let $\mathcal{P}$ denote the class of functions with positive real part. The class of functions f analytic on the open unit disk $\mathcal{U}$ normalized by $f(0) = f'(0) - 1 = 0$ will be denoted by $\mathcal{A}$. We denote by S^* and C the class of starlike and convex functions in $\mathcal{A}$ respectively.

Let $P(A, B)$, $-1 \leq A < B \leq 1$, consists of functions $p(z) = 1 + p_1 z + \dots$, is analytic in the unit disk such that $p(z) = \dfrac{1 + A\omega(z)}{1 + B\omega(z)}$, $z \in \mathcal{U}$,

where $\omega(z) \in \Omega = \{\omega \textit{ is analytic in } \mathcal{U} : \omega(0) = 0, |\omega(z)| < 1, z \in \mathcal{U}\}$.

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Now $S^*(A, B)$ consists of functions $f \in \mathcal{A}$ such that $\dfrac{zf'(z)}{f(z)} \in P(A, B)$ and $C(A, B)$ consists of functions $f \in \mathcal{A}$ such that $1 + \dfrac{zf''(z)}{f'(z)} \in P(A, B)$. These classes introduced by Janowski [6].

Let f be a function, real or complex valued on a q-geometric set B, $|q| \neq 1$. The q-differnce operator which was introduced by Jackson [5] is defined as

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \; for\, z \in B - \{0\}. \tag{1.1}$$

In addition, the q-derivative at zero defined for $|q| > 1$, $D_q f(0) = D_{q^{-1}} f(0)$.

In some literature, the q-derivative at 0 is defined to be $f'(0)$ if it exists.

The concept of q-integral is useful in this setting. Thomae and Jackson [5] introduced the q-integral

$$\int_0^1 f(t) d_q t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n$$

and Jackson gave the more general definition

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t$$

where, $I_q(f(x)) = \displaystyle\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n$.

Now we generlize $S^*(A, B)$ and $C(A, B)$ replacing the derivative and the domain by thier corresponding q-analogs.

Definition 1.1. A functions $f \in \mathcal{A}$ is said to belong to the class $S_q^*(A, B)$, $-1 \leq A < B \leq 1$, if

$$\left| \frac{(1-B)\dfrac{zD_q f(z)}{f(z)} - (1-A)}{A-B} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

Equivalently we have, $f \in S_q^*(A, B)$ if and only if

$$\left| \frac{zD_q f(z)}{f(z)} - \frac{((1-q) + (Aq - B))}{1-q} \right| \leq \frac{(A-B)}{(1-q)(1-B)}.$$

Definition 1.2. A functions $f \in \mathcal{A}$ is said to belong to the class $C_q(A, B)$, $-1 \leq A < B \leq 1$, if $z(D_q f)(z) \in S_q^*(A, B)$.

As $q \to 1^-$ the closed disk $|\omega - \dfrac{1}{1-q}| \leq \dfrac{1}{1-q}$ becomes the right half plane and the class $S_q^*(A, B)$ reduces to $S^*(A, B)$ and $C_q(A, B)$ reduces to C(A,B). In particular, when $A = 1$, and $B = -1$, the class $S_q^*(A, B)$ reduces to the class introduced by Ismail [4] and for $A = 1 - 2\alpha, B = -1$, we arrive at the classes $S_q^*(\alpha)$ and $C_q(\alpha)$ introduced by Agrawal and Sahoo [2] and Agrawal [1] respectively.

Now we proceed to prove some basic interesting properties which are used in proving our main results namely the Herglotz representation for functions belonging to the classes $S_q^*(A, B)$ and $C_q(A, B)$ in the form of a Poisson-Stietjes integral and the Bieberbach conjecture problem.

2. Some Basic Lemmas

We need the following Lemmas to prove our core results. The proofs of the Lemma 2.1 to Lemma 2.4 is straight forward from the definition of the classes $S_q^*(A,B)$ and $C_q(A,B)$.

Lemma 2.1. *If $f \in S_q^*(A,B)$, then there exists a unique function $g \in S_q^*$ such that*

$$\frac{(1-B)\dfrac{zD_qf(z)}{f(z)}-(1-A)}{A-B}=\frac{z(D_qg)(z)}{g(z)} \tag{2.1}$$

holds. Similarly, for a given function $g \in S_q$ there exists a unique function $f \in S_q^*(A,B)$ satisfying (2.1).

Lemma 2.2. *If $f \in C_q(A,B)$, $-1 \le A < B \le 1$, then there exists a unique function $g \in S_q^*(A,B)$, $-1 \le A < B \le 1$, such that*

$$g(z)=z(D_qf)(z) \tag{2.2}$$

holds. Similarly, for a given function $g \in S_q^*(A,B)$ there exists a unique function $f \in C_q(A,B)$ satisfying (2.2).

Lemma 2.3. *Let $f \in \mathcal{A}$ then $f \in S_q^*(A,B)$ if and only if*

$$\left|\frac{f(qz)}{f(z)}-\frac{(1-A)}{1-B}q\right| \le \frac{A-B}{1-B}, \ \ z \in \mathcal{U}.$$

Lemma 2.4. *Let $f \in \mathcal{A}$ then $f \in C_q(A,B)$ if and only if*

$$\left|q\frac{(D_qf)(qz)}{(D_qf)(z)}-\frac{(1-A)}{1-B}q\right| \le \frac{A-B}{1-B}, \ \ z \in \mathcal{U}.$$

Lemma 2.5. *The class $S_q^*(A,B)$ satisfies the inclusion relation*

$$\bigcap_{q<p<1} S_p^*(A,B) \subset S_q^*(A,B) \text{ and } \bigcap_{0<q<1} S_p^*(A,B) = S^*(A,B).$$

Proof. The inclusions
$\bigcap_{q<p<1} S_p^*(A,B) \subset S_p^*(A,B)$ and $\bigcap_{0<q<1} S_p^*(A,B) \subset S^*(A,B)$ clearly hold.

We need only to show that $S^*(A,B) \subset \bigcap_{0<q<1} S_p^*(A,B)$.

Consider $f \in S^*(A,B)$, there exists a unique $g \in S^*$ satisfying

$$\frac{(1-B)\dfrac{zf'(z)}{f(z)}-(1-A)}{A-B}=\frac{zg'(z)}{g(z)}.$$

Since $S^* = \bigcap_{0<q<1} S_p^*$, it follows that $g \in S_p^*$ for all $q \in (0,1)$. Thus by Lemma 2.1 there exists a unique $h \in S_q^*(A,B)$ satisfying the identity (2.1) with $h(z)=f(z)$. □

Analogously we have

Lemma 2.6. *Let $f \in \mathcal{A}$ then $f \in C_q(A, B)$ if and only if*

$$\left| q\frac{(D_q f)(qz)}{(D_q f)(z)} - \frac{(1-A)}{1-B} q \right| \leq \frac{A-B}{1-B}, \;\; z \in \mathcal{U}.$$

Lemma 2.7. *The class $C_q(A, B)$ satisfies the inclusion relation*

$$\bigcap_{q<p<1} C_p(A, B) \subset C_q(A, B) \text{ and } \bigcap_{0<q<1} C_p(A, B) = C(A, B).$$

We proceed to introduce two sets which are being used to prove our main results. $B_q = \{g : g \in \mathcal{A},\, g(0) = q \text{ and } g : \mathcal{U} \to \mathcal{U}\}$ and $B_q^0 = \{g : g \in B_q \text{ and } 0 \notin g(\mathcal{U})\}$.

Lemma 2.8. *If $h \in B_q$ then the infinite product $\prod_{n=0}^{\infty} \left\{ \frac{(A-B)h(zq^n) + (1-A)q}{(1-B)q} \right\}$ converges uniformly on compact subsets of $\mathcal{U}$*

Proof. Set $g(z) = \frac{(A-B)h(z) + (1-A)q}{(1-B)}$.

Since $h \in B_q$, it follows that $g \in B_q$. □

Lemma 2.9. *If $h \in B_q^0$ then the infinite product $\prod_{n=0}^{\infty} \left\{ \frac{(A-B)h(zq^n) + (1-A)q}{(1-B)q} \right\}$ converges uniformly on compact subsets of $\mathcal{U}$ to a nonzero function in $\mathcal{U}$ with no zeros. Furthermore, the function*

$$f(z) = \frac{z}{\prod_{n=0}^{\infty} \left\{ \frac{(A-B)h(zq^n) + (1-A)q}{(1-B)q} \right\}} \tag{2.3}$$

belongs to $S_q^*(A, B)$ and $h(z) = \frac{(1-B)\frac{f(qz)}{f(z)} - (1-A)q}{(A-B}$.

Proof. The convergence of the infinite product is proved in Lemma 2.8. Since $h \in B_q^0$ we have $h(z) \neq 0$ in $\mathcal{U}$ and the infinite product does not vanish in $\mathcal{U}$. Thus the function $f \in \mathcal{A}$ and we have the relation $\frac{f(qz)}{f(z)} = \frac{(A-B)h(z) + (1-A)q}{1-B}$.

Equivalently $h(z) = \frac{(1-B)\frac{f(qz)}{f(z)} - (1-A)q}{A-B}$.

Since $h \in B_q^0$, we get $f \in S_q^*(A, B)$. □

On similar lines we have

Lemma 2.10. *If $h \in B_q^0$ then the infinite product $\prod_{n=0}^{\infty} \left\{ \frac{(A-B)h(zq^n) + (1-A)q}{(1-B)q} \right\}$ converges uniformly on compact subsets of $\mathcal{U}$ to a nonzero function in $\mathcal{U}$ with no zeros. Furthermore, the function*

$$z(D_q f)(z) = \frac{z}{\prod_{n=0}^{\infty} \left\{ \frac{(A-B)h(zq^n) + (1-A)q}{(1-B)q} \right\}}$$

belongs to $C_q(A, B)$ *and* $h(z) = \dfrac{(1-B)q\dfrac{(D_qf)(qz)}{(D_qf)(z)} - (1-A)q}{(A-B}$.

Now we define two classes $B_q(A, B)$ and $B_q^0(A, B)$ as follows.

$$B_q(A, B) = \{g:\ g \in \mathcal{A},\ g(0) = \frac{(1-B)q}{(A-B)+q(1-A)}\ \textit{and}\ g: \mathcal{U} \to \mathcal{U}\}$$

and

$$B_q^0(A, B) = \{g:\ g \in B_q(A, B),\ 0 \notin g(\mathcal{U})\}.$$

Lemma 2.11. *A function* $g \in B_q^0(A, B)$ *if and only if it has the representation*

$$g(z) = exp\Big\{\Big(\log \frac{q(1-B)}{(A-B)+q(1-A)}\Big)p(z)\Big\} \tag{2.4}$$

where $p(z)$ *belongs to the class* $\mathcal{P} = \{p:\ p(0) = 1,\ \textit{and}\ \Re\{p(z)\} > 0,\ \textit{for}\ z \in \mathcal{U}\}$.

Proof. For $g \in B_q^0(A, B)$, define $L(z) = \log g(z)$. We can easily show that $p(z) = \dfrac{L(z)}{\log\Big(\dfrac{(1-B)q}{(A-B)+q(1-A)}\Big)} \in \mathcal{P}$ and satisfies (2.4).

Conversely, if g is given by (2.4) then it is obvious that $g \in B_q^0(A, B)$. □

Lemma 2.12. *The mapping* $\rho: S_q^*(A, B) \to B_q^0$ *defined by*

$$\rho(f(z)) = \frac{(1-B)\dfrac{f(qz)}{f(z)} - (1-A)q}{A-B}$$

is a bijection.

Proof. For $h \in B_q^0$, define a mapping $\sigma: B_q^0 \to \mathcal{A}$ by

$$\sigma(h(z)) = f(z) = \frac{z}{\prod_{n=0}^{\infty}\{\dfrac{(A-B)h(zq^n)+(1-A)q}{(1-B)q}\}}.$$

From Lemma 2.9 that $\sigma(h) \in S_q^*(A, B)$ and $(\rho o \sigma)(h) = h$. Also

$$(\sigma o \rho)f(z) = \frac{z}{\prod_{n=0}^{\infty}\Big\{\dfrac{f(zq^{n+1})}{qf(zq^n)}\Big\}} = \frac{z}{\dfrac{z}{f(z)}} = f(z).$$

The map ρ is invertible because $\rho o \sigma$ and $\sigma o \rho$ are identity maps and σ is the inverse of ρ. Hence $\rho(f)$ is a bijection. □

On simiar lines a bijection from $C_q(A, B)$ to B_q^0 is defined as in the following Lemma

Lemma 2.13. *The mapping $\tau : C_q(A, B) \to B_q^0$ defined by*

$$\tau(f(z)) = \frac{(1-B)q\dfrac{(D_qf)(qz)}{(D_qf)(z)} - (1-A)q}{A-B}.$$

is a bijection.

3. Main Results

Now we prove our core results using the preliminary Lemmas proved in section 2.

Theorem 3.1. *Let $f \in \mathcal{A}$ then $f \in S_q^*(A, B)$ if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\psi|=1} \psi z F_q'(A, B)(\psi z) d\mu(\psi),$$

where

$$F_q(A, B)(z) = \sum_{n=1}^{\infty} \frac{(-2)\log\left(\frac{(1-B)q}{(A-B)+q(1-A)}\right)}{1-q^n} z^n. \tag{3.1}$$

Proof. For $0 < q < 1$ and $-1 \leq B < A \leq 1$, let $F_q(A, B)(z)$ be defined by (3.1). It is obvious that f has the representation (2.3) with $h \in B_q^0$. Taking the logarithmic derivative of f we have,

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{n=0}^{\infty} \frac{(A-B)zq^n h'(zq^n)}{(A-B)h(zq^n) + (1-A)q}. \tag{3.2}$$

Let $g(z) = \dfrac{(A-B)h(z) + (1-A)q}{(A-B) + (1-A)q}$

Clearly, $g \in B_q^*(A, B)$ and using Lemma 2.9, g has the representation (2.3). Taking the logarithmic derivative, we have

$$\frac{zg'(z)}{g(z)} = \left(\log \frac{(1-B)q}{(A-B) + q(1-A)}\right) zp'(z) \tag{3.3}$$

where $p(z) \in \mathcal{P}$. Using the Herglotz representation of $p(z)$ there exists a probability measure μ supported on the unit circle $|\psi| = 1$ such that

$$zp'(z) = \int_{|\theta|=1} 2\psi z(1-\psi z)^{-2} d\mu(\psi). \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), we have

$$\begin{aligned}
\frac{zf'(z)}{f(z)} &= 1 - 2\left(\log \frac{(1-B)q}{(A-B)+q(1-A)}\right) \sum_{n=0}^{\infty} \int_{|\psi|=1} \psi z q^n (1-\psi z q^n)^{-2} d\mu(\psi) \\
&= 1 - 2\left(\log \frac{(1-B)q}{(A-B)+q(1-A)}\right) \int_{|\psi|=1} \left\{ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m\psi^m z^m q^{mn} \right\} d\mu(\psi) \\
&= 1 - 2\left(\log \frac{(1-B)q}{(A-B)+q(1-A)}\right) \int_{|\psi|=1} \left\{ \sum_{m=1}^{\infty} m\psi^m z^m \frac{1}{1-q^m} \right\} d\mu(\psi)
\end{aligned}$$

$$= 1 + \int_{|\psi|=1} \psi z F'_q(A, B)(\psi z) d\mu(\psi).$$

□

Analogous result for $C_q(A, B)$ is as follows:

Theorem 3.2. *Let $f \in \mathcal{A}$ then $f \in C_q(A, B)$ if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{z(D_q f)'(z)}{(D_q f)(z)} = \int_{|\psi|=1} \psi z F'_q(A, B)(\psi z) d\mu(\psi),$$

where, $F_q(A, B)(z)$ is defined in (3.1).

Now we prove the Bieberbach cojecture problem for q-Janowski starlike and convex functions. The extremal function obtained here is the generalization of the is generalization of the K$\ddot{o}$ebe function.

Theorem 3.3. *Let*

$$G_q(A, B)(z) = z\, exp[F_q(A, B)(z)] = z + \sum_{n=2}^{\infty} a_n z^n. \tag{3.5}$$

Then $G_q(A, B) \in S^*_q(A, B)$. If $z + \sum_{n=2}^{\infty} a_n z^n \in S^*_q(A, B)$, then $|a_n| \leq c_n$. Equality holds if and only if f is a rotation of $G_q(A, B)$.

Proof. For $0 < q < 1$, $-1 \leq B < A \leq 1$, let $G_q(A, B)$ be defined by (3.5).

Let $f \in S^*_q(A, B)$ choose $p \in \mathcal{P}$ as in Lemma 2.11 and satisfy

$$\rho(f)(z) = h(z) = \frac{(1-B)\frac{f(qz)}{f(z)} - (1-A)q}{A-B} \in B^0_q.$$

Since $h \in B^0_q$, $g(z) = \dfrac{\left((1 - \frac{(1-A)}{1-B})h(z) + \frac{1-A}{1-B}q\right)}{1 - \frac{(1-A)}{(1-B)}(1-q)} \in B^0_q(A, B)$

$$= \frac{(A-B)h(z) + (1-A)q}{(1-B) - (1-A)(1-q)}.$$

By Lemma 2.11, $g(z)$ has the representation (2.4) and on solving we get

$$\frac{f(qz)}{f(z)} = \left(1 - \frac{(1-A)}{(1-B)}(1-q)\right)\exp\left\{\left(\log \frac{q(1-B)}{1 - \frac{(1-A)}{(1-B)}(1-q)}\right)p(z)\right\}.$$

Define the function $\phi(z) = \log\{\dfrac{f(z)}{z}\}$ and set

$$\phi(z) = \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \phi_n z^n. \tag{3.6}$$

On solving , we get $\log \frac{q(1-B)}{(1-B)-(1-A)(1-q)} + \phi(qz) = \phi(z) + \left(\log \frac{q(1-B)}{(1-B)-(1-A)(1-q)}\right)p(z),$

which implies that

$$\phi_n = p_n \frac{\left(\log \frac{q(1-B)}{(1-B)-(1-A)(1-q)}\right)}{(q^n-1)}.$$

Since $|p_n| \le 2$, we have

$$|\phi_n| \le (-2)\frac{\left(\log \frac{q(1-B)}{(1-B)-(1-A)(1-q)}\right)}{(1-q^n)}.$$

From this inequality, together with the expression of $G_q(A, B)$ and (3.6), we get required results. □

Theorem 3.4. *Let*

$$E_q(A, B)(z) = I_q\{z\, exp[F_q(A, B)(z)]\} = z + \sum_{n=2}^{\infty}\left(\frac{1-q}{1-q^n}\right)c_n z^n. \tag{3.7}$$

where c_n is the n th coefficient of the function $z\, exp[F_q(A, B)(z)]$. Then $E_q(A, B) \in C_q(A, B)$, $-1 \le B < A \le 1$. Also if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_q(A, B)$, then $|a_n| \le \left(\frac{1-q}{1-q^n}\right)c_n$. Equality holds if and only if f is a rotation of $E_q(A, B)(z)$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_q(A, B)$. By definition of $C_q(A, B)$,

$$z(D_q f)(z) = z + \sum_{n=0}^{\infty}\left(\frac{1-q}{1-q^n}\right)a_n z^n \in S_q^*(A, B).$$

Then by Theorem 3.3, we have

$$\left|\frac{1-q^n}{1-q}a_n\right| \le c_n.$$

Next, we show that equality holds for the function $E_q(A, B) \in C_q(A, B)$. As a special case to Theorem 3.2, when the measure has a unit mass, it is clear that $E_q(A, B)(z) \in C_q(A, B)$. Let $E_q(A, B)(z) = z + \sum_{n=0}^{\infty} b_n z^n$. From this representation of E_q and the definition of $D_q f$, we get

$$z(D_q E_q)(z) = z + \sum_{n=2}^{\infty} b_n \frac{(1-q^n)}{(1-q)} z^n. \tag{3.8}$$

Since $E_q(A, B)(z) = I_q\{z\, exp[E_q(A, B)(z)]\}$, $z(D_q E_q)(z) = z\,\{exp[F_q(A, B)]\}$ and since c_n is the n-th coefficient of the function $z\,\{exp[F_q(A, B)]\}$, we have

$$z(D_q E_q)(z) = z + \sum_{n=2}^{\infty} c_n z^n. \tag{3.9}$$

By comparing (3.8) and (3.9) we get $b_n = c_n \frac{(1-q)}{(1-q^n)}$.

$$i,\ e,\ \ E_q(A, B)(z) = z + \sum_{n=2}^{\infty} b_n \frac{(1-q^n)}{(1-q)} z^n.$$

□

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