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Applications of Horadam Polynomials to a Class of Close-to-Convex Functions

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Abstract. In this study, we introduce and investigate a family of close-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, \Psi(x))$ that associated with Horadam polynomials, functions in this family are defined with respect to symmetric conjugate points. The coefficient estimates of functions belonging to this family are derived. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to this family.

1. Introduction

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclarin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where } z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. Let the functions f and g be analytic in $\mathbb{D}$, we say the function f is subordinate by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwartz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$ then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D}$. For more information about the Subordination Principle we refer the readers to to the monographs [9], [10], [21] and [22].

As known univalent functions are injective (one-to-one) functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, according to Koebe one-quarter Theorem [9], the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of

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center 0 and radius 1/4. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \;\; r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{1.2}$$

For this reason, we define the class Σ as follows. A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{A}$ which are given by equation (1.1). For more information about univalent and bi-univalent functions we refer the readers to the articles [19], [20], [23], [24] the monograph [9], [12] and the references therein.

The research in the geometric function theory has been very active in recent years, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class.

Coefficient related investigations of functions belong to the class Σ began around the 1970. It is worth mentioning that, in the year 1967, Lewin [19] studied the class of bi-univalent functions and derived the bound for $|a_2|$. Later on, in the year 1969, Netanyahu [23] showed that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions belong to the class Σ. In addition, in the year 1979, Brannan and Clunie [6] proved that $|a_2| \leq \sqrt{2}$ for functions in the class Σ. Since then, many researchers investigated the coefficient bounds for various subclasses of the bi-univalent function class Σ. However, not much is known about the bounds of the general coefficients $|a_2|$ for $n \geq 4$. In fact, the coefficient estimate problem for the general coefficient $|a_n|$ is still an open problem.

In the year 1933, Fekete and Szegö [18] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{A}$ with any complex λ is called the Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [2], [3], [7], [8], [16], [17], [18], [20], [27] and the references therein.

2. Preliminaries

In this section we present some information that are curial for the main results of this paper. In the year 1985, Horadam and Mahon [13] defined the Horadam polynomials $\psi_n(x) = \psi_n(\alpha, \beta; a, b)$ by the following recurrence relation:

$$\psi_n(x) = ax\psi_{n-1}(x) + b\psi_{n-2}(x), \;\; \text{for} \;\; n \geq 3, \tag{2.1}$$

with initial values,

$$\psi_1(x) = \alpha, \ \ \psi_2(x) = \beta x, \ \ \text{and} \ \ \psi_3(x) = a\beta x^2 + b\alpha. \tag{2.2}$$

Moreover, the generating function of Horadam ploynomials is given by

$$\Psi(x,z) = \sum_{n=1}^{\infty} \psi_n(x) z^{n-1} = \frac{\alpha + (\beta - \alpha a)xz}{1 - axz - bz^2}.$$

In this paper, the argument of $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is $x \neq \mathcal{R}(z)$. By selecting particular values of α, β, a and b the Horadam polynomials leads to several known polynomials. Some of these special cases are listed below.

- If $\alpha = \beta = a = b = 1$, we get the Fibonacci polynomials $F_n(x)$.
- If $\alpha = 2$ and $\beta = a = b = 1$, we get the Lucas Polynomials $L_n(x)$.
- IF $\alpha = b = 1$ and $\beta = a = 2$, we get the Pell Polynomials $P_n(x)$.
- If $\alpha = \beta = a = 2$ and $b = 1$, we get Pell-Lucas Polynomials $Q_n(x)$.
- If $\alpha = \beta = -b = 1$ and $a = 2$, we get the Chebyshev Polynomials $T_n(x)$ of the first kind.
- If $\alpha = -b = 1$ and $\beta = a = 2$, we get the Chebyshev Polynomials $U_n(x)$ of the second kind.

For more information about Horadam polynomials and its special interesting cases, we refer the readers to the articles [1], [4], [5], [13], [14], [25], [27], [28], the monograph [15], [26] and the references therein.

In the year 1987, El-Ashwah and Thomas [11] introduced and investigated the class of starlike functions with respect to symmetric conjugate points, denoted by $\mathcal{S}_{sc}^*$. The function f belong to the class $\mathcal{S}_{sc}^*$ if and only if for all $z \in \mathbb{D}$, $f(z) \in \mathcal{S}$ and satisfying the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z) - \overline{f(-\bar{z})}}\right\} > 0.$$

Moreover, a function $f \in \mathcal{S}$ is called convex with respect to symmetric conjugate points if for all $z \in \mathbb{D}$ the following condition hold:

$$\Re\left\{\frac{(zf'(z))'}{\left(f(z) - \overline{f(-\bar{z})}\right)'}\right\} > 0.$$

The main purpose of this article is to make use of Horadam polynomials in order to introduce a new subclass of the bi-univalent function class Σ. Next we define our class of close-to-convex functions which we denote by $\mathcal{S}_{sc}^c(\lambda, \Psi(x))$ where $0 \leq \lambda \leq 1$ and $x \in \mathbb{R}$.

Definition 2.1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{sc}^c(\lambda, \Psi(x))$ if it satisfies the following subordinations:*

$$\begin{aligned}&\frac{2\lambda z^3 f'''(z) + 2(1+\lambda)z^2 f''(z) + 2zf'(z)}{\lambda\left\{z^2\left(f(z) - \overline{f(-\bar{z})}\right)'' + \left(f(z) - \overline{f(-\bar{z})}\right)\right\} + (1-\lambda)\left(f(z) - \overline{f(-\bar{z})}\right)'} \\ &\prec \Psi(x,z) + 1 - \alpha,\end{aligned} \tag{2.3}$$

and

$$\frac{2\lambda w^3 g'''(w)+2(1+\lambda)w^2 g''(w)+2wg'(w)}{\lambda\left\{z^2\left(g(w)-\overline{g(-\bar{w})}\right)''+\left(g(w)-\overline{g(-\bar{w})}\right)\right\}+(1-\lambda)\left(g(w)-\overline{g(-\bar{w})}\right)'} \prec \Psi(x,w)+1-\alpha, \tag{2.4}$$

where the function $g(w)=f^{-1}(w)$ is given by the equation (1.2).

The following lemma (see, for details [17]) is a well-known fact, but it is crucial for our presented work.

Lemma 2.1. *Let $k, l \in \mathbb{R}$ and $x, y \in \mathbb{C}$. If $|x| < r$ and $|y| < r$,*

$$\left|(k+l)x+(k-l)y\right| \le \begin{cases} 2|k|r, & if\ \ |k| \ge || \\ 2|l|r, & if\ \ |k| \le |l| \end{cases}$$

The work in this paper is motivated by the research done in the articles [29], [4] and [5]. The primary goal of this study is to determine the estimates for the initial Taylor-Maclarin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda,\Psi(x))$ that subordinate to Horadam polynomials and some of their special cases. Furthermore, we examine the corresponding Fekete-Szegö functional problem for functions belong to the presenting class. We also provide relevant connections of our main results with those considered in earlier investigations.

3. Initial Coefficient Bounds for the Class $\mathcal{S}_{sc}^{c}(\lambda,\Psi(x))$

In this section, we provide estimates for the initial Taylor-Maclaurin coefficients for the functions belong to the class $\mathcal{S}_{sc}^{c}(\lambda,\Psi(x))$ which are given by equation (1.1).

Theorem 3.1. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda,\Psi(x))$. Then*

$$|a_2| \le \frac{|\beta x|\sqrt{|\beta x|}}{\sqrt{|[(4\lambda+3)\beta-2(\lambda+2)^2 a]\beta x^2-2b\alpha(\lambda+2)^2|}}, \tag{3.1}$$

and

$$|a_3| \le \frac{|\beta x|}{2(4\lambda+3)}+\frac{\beta^2 x^2}{4(\lambda+2)^2}. \tag{3.2}$$

Proof. Let f be in the class $\mathcal{S}_{sc}^{c}(\lambda,\Psi(x))$. Then, using Definition 2.1, there are two analytic functions u and v on the unit disk $\mathbb{D}$ such that

$$\frac{2\lambda z^3 f'''(z)+2(1+\lambda)z^2 f''(z)+2zf'(z)}{\lambda\left\{z^2\left(f(z)-\overline{f(-\bar{z})}\right)''+\left(f(z)-\overline{f(-\bar{z})}\right)\right\}+(1-\lambda)\left(f(z)-\overline{f(-\bar{z})}\right)'} \prec \Psi(x,u(z))+1-\alpha, \tag{3.3}$$

and

$$\frac{2\lambda w^3 g'''(w) + 2(1+\lambda)w^2 g''(w) + 2wg'(w)}{\lambda\left\{z^2\left(g(w)-\overline{g(-\bar{w})}\right)'' + \left(g(w)-\overline{g(-\bar{w})}\right)\right\} + (1-\lambda)\left(g(w)-\overline{g(-\bar{w})}\right)'} \prec \Psi(x, v(w)) + 1 - \alpha, \tag{3.4}$$

where for all $z, w \in \mathbb{D}$ the analytic functions $u(z)$ and $v(w)$ are given by

$$u(z) = \sum_{n=1}^{\infty} u_n z^n, \quad v(w) = \sum_{n=1}^{\infty} v_n w^n,$$

$u(0) = 0 = v(0)$, $|u(z)| < 1$ and $|v(w)| < 1$. Moreover, it is well-known that (see, for details [9]) for all $j \in \mathbb{N}$ we have $|u_j| \le 1$ and $|v_j| \le 1$.

Thus, by comparing coefficients in both sides of equation (3.3) and equation (3.4), we get the following equations:

$$2(\lambda+2)a_2 = \psi_2(x)u_1, \tag{3.5}$$

$$2(4\lambda+3)a_3 = \psi_2(x)u_2 + \psi_3(x)u_1^2, \tag{3.6}$$

$$-2(\lambda+2)a_2 = \psi_2(x)v_1, \tag{3.7}$$

and

$$2(4\lambda+3)(2a_2^2 - a_3) = \psi_2(x)v_2 + \psi_3(x)v_1^2. \tag{3.8}$$

From equation (3.5) and equation (3.7), we find that

$$u_1 = -v_1, \tag{3.9}$$

and

$$8(\lambda+2)^2 a_2^2 = \psi_2^2(x)(u_1^2 + v_1^2) \tag{3.10}$$

Equation (3.10) gives us the following

$$a_2^2 = \frac{\psi_2^2(x)(u_1^2+v_1^2)}{8(\lambda+2)^2}, \tag{3.11}$$

and

$$u_1^2 + v_1^2 = \frac{8(\lambda+2)^2 a_2^2}{\psi_2^2(x)}. \tag{3.12}$$

If we add equation (3.6) and equation (3.8), then make use of equation (3.9) and equation (3.12) we obtain

$$4(4\lambda+3)a_2^2 = \psi_2(x)(u_2+v_2) + \psi_3(x)(u_1^2+v_1^2),$$

which gives

$$4(4\lambda+3)a_2^2 = \psi_2(x)(u_2+v_2) + \frac{8\psi_3(x)(\lambda+2)^2 a_2^2}{\psi_2^2(x)}.$$

Therefore, we get the following

$$a_2^2 = \frac{\psi_2^3(x)(u_2+v_2)}{4(4\lambda+3)\psi_2^2(x) - 8\psi_3(x)(\lambda+2)^2}. \tag{3.13}$$

Using the facts $|u_2| \leq 1$ and $|v_2| \leq 1$, and using the initial values (2.2), we get the desired bound for the modulus of a_2.

Now, we look for the bound on $|a_3|$. In order to do this, we subtract equation (3.8) from equation (3.6) which gives

$$4(4\lambda+3)(a_3 - a_2^2) = \psi_2(x)(u_2 - v_2) + \psi_3(x)(u_1^2 - v_1^2).$$

In view of equation (3.9), we obtain

$$a_3 = \frac{\psi_2(x)(u_2 - v_2)}{4(4\lambda+3)} + a_2^2 \tag{3.14}$$

If follows from equation (3.11) that

$$a_3 = \frac{\psi_2(x)(u_2 - v_2)}{4(4\lambda+3)} + \frac{\psi_2^2(x)(u_1^2 + v_1^2)}{8(\lambda+2)^2}.$$

Therefore, using the initial values (2.2) and the facts $|u_2| \leq 1$, $|v_2| \leq 1$, we get the desired bound for the modulus of a_3. This completes the proof of Theorem 3.1. □

The following are just corollaries related to the special cases of Horadam polynomials. The following corollary gives initial coefficient estimates for functions belonging to the class of closed-to-convex functions $\mathcal{S}_{sc}^c(\lambda, F_n(x))$ that associated with Fibonacci polynomials.

Corollary 3.1. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^c(\lambda, F_n(x))$. Then*

$$|a_2| \leq \frac{|x|\sqrt{|x|}}{\sqrt{(2\lambda^2+4\lambda+5)x^2 + 2(\lambda+2)^2}},$$

and

$$|a_3| \leq \frac{|x|}{2(4\lambda+3)} + \frac{x^2}{4(\lambda+2)^2}.$$

The following corollary gives initial coefficient estimates for functions belonging to the class of closed-to-convex functions $\mathcal{S}_{sc}^c(\lambda, L_n(x))$ that associated with Lucas Polynomials.

Corollary 3.2. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^c(\lambda, L_n(x))$. Then*

$$|a_2| \leq \frac{|x|\sqrt{|x|}}{\sqrt{(2\lambda^2+4\lambda+5)x^2 + 4(\lambda+2)^2}},$$

and

$$|a_3| \leq \frac{|x|}{2(4\lambda+3)} + \frac{x^2}{4(\lambda+2)^2}.$$

The following corollary gives initial coefficient estimates for functions belonging to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, P_n(x))$ that associated with Pell Polynomials.

Corollary 3.3. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, P_n(x))$. Then*

$$|a_2| \leq \frac{|x|\sqrt{|2x|}}{\sqrt{(4\lambda^2+8\lambda+10)x^2+(\lambda+2)^2}},$$

and

$$|a_3| \leq \frac{|x|}{4\lambda+3}+\frac{x^2}{(\lambda+2)^2}.$$

The following corollary gives initial coefficient estimates for functions belonging to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, Q_n(x))$ that associated with Pell-Lucas Polynomials.

Corollary 3.4. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, Q_n(x))$. Then*

$$|a_2| \leq \frac{|x|\sqrt{|2x|}}{\sqrt{(4\lambda^2+8\lambda+10)x^2+2(\lambda+2)^2}},$$

and

$$|a_3| \leq \frac{|x|}{4\lambda+3}+\frac{x^2}{(\lambda+2)^2}.$$

The following corollary gives initial coefficient estimates for functions belonging to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, T_n(x))$ that associated with Chebyshev Polynomials of the first kind.

Corollary 3.5. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, T_n(x))$. Then*

$$|a_2| \leq \frac{|x|\sqrt{|x|}}{\sqrt{|2(\lambda+2)^2-(4\lambda^2+10\lambda+13)x^2|}},$$

and

$$|a_3| \leq \frac{|x|}{2(4\lambda+3)}+\frac{x^2}{4(\lambda+2)^2}.$$

The following corollary gives initial coefficient estimates for functions belonging to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, U_n(x))$ that associated with Chebyshev Polynomials of the second kind

Corollary 3.6. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, U_n(x))$. Then*

$$|a_2| \leq \frac{|x|\sqrt{|2x|}}{\sqrt{|(\lambda+2)^2-(4\lambda^2+8\lambda+10)x^2|}},$$

and

$$|a_3| \leq \frac{|x|}{4\lambda+3}+\frac{x^2}{(\lambda+2)^2}.$$

4. Fekete-Szegö Inequality for the Class $\mathcal{S}_{sc}^{c}(\lambda, \Psi(x))$

In this section, we maximize the modulus of the functional $\Psi_{\zeta}(f) = a_3 - \zeta a_2^2$ for real numbers ζ and for functions f belong to our class $\mathcal{S}_{sc}^{c}(\lambda, \Psi(x))$.

Theorem 4.1. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, \Psi(x))$. Then for some $\zeta \in \mathbb{R}$ and for $x\beta \neq 0$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x\beta|}{2(4\lambda+3)}, & \text{if } |1-\zeta| \leq \frac{|A|}{\beta^2 x^2 (4\lambda+3)} \\ \frac{|x\beta|^3 |1-\zeta|}{2|A|}, & \text{if } |1-\zeta| \geq \frac{|A|}{\beta^2 x^2 (4\lambda+3)}, \end{cases} \tag{4.1}$$

where

$$A = (4\lambda+3)(x\beta)^2 - 2(\lambda+2)^2(a\beta x^2 + b\alpha).$$

Proof. For some real number ζ, using equation (3.14), we obtain

$$a_3 - \zeta a_2^2 = \frac{\psi_2(x)(u_2 - v_2)}{4(4\beta+3)} + (1-\zeta)a_2^2.$$

Using equation (3.13), we obtain

$$\begin{aligned} a_3 - \zeta a_2^2 &= \frac{\psi_2(x)(u_2 - v_2)}{4(4\lambda+3)} + \frac{(1-\zeta)\psi_2^3(x)(u_2+v_2)}{4(4\lambda+3)\psi_2^2(x) - 8(\lambda+2)^2\psi_3(x)} \\ &= \psi_2(x)\left\{\left(\Delta(\zeta,\lambda) + \frac{1}{4(4\lambda+3)}\right)u_2 + \left(\Delta(\zeta,\lambda) - \frac{1}{4(4\lambda+3)}\right)v_2\right\}, \end{aligned}$$

where

$$\Delta(\zeta,\lambda) = \frac{(1-\zeta)\psi_2^2(x)}{4\psi_2^2(x)(4\lambda+3) - 8\psi_3(x)(\lambda+2)^2}.$$

Thus, using Lemma 2.1, we get

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x\beta|}{2(4\lambda+3)}, & \text{if } |\Delta(\zeta,\lambda)| \leq \frac{1}{4(4\lambda+3)} \\ 2|x\beta||\Delta(\zeta,\lambda)|, & \text{if } |\Delta(\zeta,\lambda)| \geq \frac{1}{4(4\lambda+3)}. \end{cases} \tag{4.2}$$

Therefore, using the initial values (2.2) then simplifying (4.2) we get the desired result (4.1). This completes the proof of Theorem 4.1. □

The following are just corollaries related to the special cases of Horadam polynomials. The following corollary gives Fekete-Szegö inequality for functions belong to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, F_n(x))$ that associated with Fibonacci polynomials.

Corollary 4.1. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, F_n(x))$. Then for some $\zeta \in \mathbb{R}$ and for $x \neq 0$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x|}{2(4\lambda+3)}, & \text{if } |1-\zeta| \leq \frac{(2\lambda^2+4\lambda+5)x^2+2(\lambda+2)^2}{x^2(4\lambda+3)} \\ \frac{|x|^3|1-\zeta|}{(4\lambda^2+8\lambda+10)x^2+4(\lambda+2)^2}, & \text{if } |1-\zeta| \geq \frac{(2\lambda^2+4\lambda+5)x^2+2(\lambda+2)^2}{x^2(4\lambda+3)}. \end{cases}$$

The following corollary gives Fekete-Szegö inequality for functions belong to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, L_n(x))$ that associated with Lucas Polynomials.

Corollary 4.2. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, L_n(x))$. Then for some $\zeta \in \mathbb{R}$ and for $x \neq 0$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x|}{2(4\lambda+3)}, & if \ |1-\zeta| \leq \frac{(2\lambda^2+4\lambda+5)x^2+4(\lambda+2)^2}{x^2(4\lambda+3)} \\ \frac{|x|^3|1-\zeta|}{(4\lambda^2+8\lambda+10)x^2+8(\lambda+2)^2}, & if \ |1-\zeta| \geq \frac{(2\lambda^2+4\lambda+5)x^2+4(\lambda+2)^2}{x^2(4\lambda+3)}. \end{cases}$$

The following corollary gives Fekete-Szegö inequality for functions belong to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, P_n(x))$ that associated with Pell Polynomials.

Corollary 4.3. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, P_n(x))$. Then for some $\zeta \in \mathbb{R}$ and for $x \neq 0$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x|}{4\lambda+3}, & if \ |1-\zeta| \leq \frac{(4\lambda^2+8\lambda+10)x^2+(\lambda+2)^2}{2x^2(4\lambda+3)} \\ \frac{4|x|^3|1-\zeta|}{(4\lambda^2+8\lambda+10)x^2+(\lambda+2)^2}, & if \ |1-\zeta| \geq \frac{(4\lambda^2+8\lambda+10)x^2+(\lambda+2)^2}{2x^2(4\lambda+3)}. \end{cases}$$

The following corollary gives Fekete-Szegö inequality for functions belong to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, Q_n(x))$ that associated with Pell-Lucas Polynomials.

Corollary 4.4. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, Q_n(x))$. Then for some $\zeta \in \mathbb{R}$ and for $x \neq 0$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x|}{4\lambda+3}, & if \ |1-\zeta| \leq \frac{(2\lambda^2+4\lambda+5)x^2+(\lambda+2)^2}{x^2(4\lambda+3)} \\ \frac{|x|^3|1-\zeta|}{(2\lambda^2+4\lambda+5)x^2+(\lambda+2)^2}, & if \ |1-\zeta| \geq \frac{(2\lambda^2+4\lambda+5)x^2+(\lambda+2)^2}{x^2(4\lambda+3)}. \end{cases}$$

The following corollary gives Fekete-Szegö inequality for functions belong to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, T_n(x))$ that associated with Chebyshev Polynomials of the first kind.

Corollary 4.5. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, T_n(x))$. Then for some $\zeta \in \mathbb{R}$ and for $x \neq 0$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x|}{2(4\lambda+3)}, & if \ |1-\zeta| \leq \frac{|2(\lambda+2)^2-(4\lambda^2+10\lambda+13)x^2|}{x^2(4\lambda+3)} \\ \frac{|x|^3|1-\zeta|}{|4(\lambda+2)^2-2(4\lambda^2+10\lambda+13)x^2|}, & if \ |1-\zeta| \geq \frac{|2(\lambda+2)^2-(4\lambda^2+10\lambda+13)x^2|}{x^2(4\lambda+3)}. \end{cases}$$

The following corollary gives Fekete-Szegö inequality for functions belong to the class of closed-to-convex functions $\mathcal{S}_{sc}^{c}(\lambda, U_n(x))$ that associated with Chebyshev Polynomials of the second kind

Corollary 4.6. *Let the function f given by (1.1) be in the class $\mathcal{S}_{sc}^{c}(\lambda, U_n(x))$. Then for some $\zeta \in \mathbb{R}$ and for $x \neq 0$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|x|}{4\lambda+3}, & if \ |1-\zeta| \leq \frac{|(\lambda+2)^2-(4\lambda^2+8\lambda+10)x^2|}{2x^2(4\lambda+3)} \\ \frac{2|x|^3|1-\zeta|}{|(\lambda+2)^2-(4\lambda^2+8\lambda+10)x^2|}, & if \ |1-\zeta| \geq \frac{|(\lambda+2)^2-(4\lambda^2+8\lambda+10)x^2|}{2x^2(4\lambda+3)}. \end{cases}$$

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