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Real-Life Applications of New Type Spherical Fuzzy Sets and Its Extension Using Aggregation Operators

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Abstract. The purpose of this article is to present a novel approach to the multiple attribute decision-making problem (MADM) based on (l_1, l_2) spherical fuzzy sets (SFS). This is an extension of the SFS. As a result of this article, we will discuss the concept of spherical fuzzy weighted averaging (SFWA), spherical fuzzy weighted geometric (SFWG), generalized spherical fuzzy weighted averaging (GSFWA) and generalized spherical fuzzy weighted geometric (GSFWG). Here is a flowchart that shows how these operators are used in the algorithm we discussed. With the help of a numerical example, we illustrate the extended Euclidean and Hamming distance measures. Additionally, the SFN approach is characterized by idempotency, boundedness, commutativity, and monotonicity. These tools help you find the best option faster, simpler, and more conveniently. The result is a more precise conclusion and a more intimate relationship between (l_1, l_2). We compare some of the current models with those that have been proposed in order to demonstrate the dependability and utility of the models under investigation. The study also revealed fascinating and intriguing findings.

1. Introduction

The complexity of systems increases every day, making it more difficult for decision-makers to make the right choice. It is difficult to accomplish a single goal, but you can do it. Motivating, setting goals, and forming opinions were difficult for many organizations. Therefore, whether individual or committee decisions are made, a number of objectives must be considered simultaneously. In light of this reflection, it would seem that the criteria are solved flexibly, which makes

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it difficult for each decision-maker to achieve an optimal solution in each of the criteria involved. In order to determine the best alternative, decision-makers should develop methods that are reliable and appropriate. Traditionally, crisp methods are ineffective when dealing with ambiguity and uncertainty in decision-making. To deal with the ambiguities, many uncertain theories exist, including fuzzy set (FS) [1], intuitionistic FS (IFS) [2], Pythagorean FS (PFS) [3] and spherical FS (SFS) [4]. An FS is a set of elements with a membership grade (MD) in the given set of values from zero to one. Later, Atanassov proposed the concept of an IFS that is divided into categories using non-membership grade (NMD), which cannot exceed one [2]. It is possible to convey a single problem to decision-making (DM) when MD and NMD scores are greater than one. PFS is characterized by a square sum of MD and NMD less than one for an IFS that has a value less than one, as determined by Yager [3]. Positive MD, neutral MD and negative MD are the three-pointers that make up the picture FS concept developed by Cuong and Kreinovich [5]. As a result, it has a number of advantages over PFS and IFS as well. A generalization of picture FS was examined by Liu et al. [6] using AOs. A generalized PFS based on AO and its applications has been proposed by Liu et al. [7]. AO features based on PFS and interval values [8]. An AO-based picture FS was presented by Liu et al. [6]. In the DM approach challenge, the sum of the positive, neutral and negative MD values rarely exceeds one. The concept of SFS is presented by Ashraf et al. [4], which makes sure the square sum of positive, neutral and negative grades does not exceed 1. The notion of SFS was examined by Fatmaa and Cengiza [9].

MADM issues cannot be solved without aggregation operators (AOs). Xu and Li [32] discussed IFS data can be averaged using some IFS averaging operators. Additionally, Xu and Yager [33] defined geometric operators derived from IFSs, such as weighted, ordered weighted, and hybrid operators. In 2002, Li [26] proposed generalized ordered weighted averaging operators (GOWAs). Zeng and Sua [34] discussed that ordered weighted distances are calculated using distance measures and AOs. Yager [3] developed some averaging and geometric AOs that can be found under PFS weighted, ordered weighted, and weighted power conditions. A fundamental PFS based on AOs properties was studied by Peng and Yuan [28]. A generalised PFS under AOs was developed by Liu et al. [7]. Ashraf et al. [25] developed fuzzy spherical Dombi AOs. Further information about SFSs and T-SFSs can be found at [30, 31]. In 2022, Temel et al. [29] discussed its application to MADM based on Muirhead power normal SFS. Peng and Dai [10] explore the neutrosophic set with MADM using MABAC and TOPSIS approaches. A generalization of PFS using TOPSIS was presented by Lhang and Xu [11]. Recently, Palanikumar et al. discussed many algebraic structures and aggregation operators with applications [14–19]. Many researchers [20–24] discussed the concept of Pythagorean fuzzy sets with its extension based on DM.

I will follow the following structure throughout the remainder of this paper. An introduction is found in section 1. PFS and SFS were discussed in section 2. Section 3 discusses some of the operations on (l_1, l_2) SFNs. Section 4 discusses the MADM model based on distance measures between (l_1, l_2) SFNs. In section 5 we discuss the interaction between MADM and some AOs for

(l_1, l_2) SFNs. A numerical example and an algorithm are discussed in section 6. The conclusion is provided in Section 7. Therefore, the following are the main outcomes of the paper:

(1) To introduce an ED, HD and score values for SFSs.
(2) By using (l_1, l_2) SFN, (l_1, l_2) SFWA, (l_1, l_2) SFWG, G (l_1, l_2) SFWA and G (l_1, l_2) SFWG operators are developed.
(3) We explore the MADM technique using (l_1, l_2) SFSs to define AOs.
(4) It is demonstrated that the proposed approaches work through a few numerical examples.
(5) To compute the different ideal values for (l_1, l_2) SFWA, (l_1, l_2) SFWG, G (l_1, l_2) SFWA and G (l_1, l_2) SFWG.
(6) We discuss the proposed and early studies in the context of comparative analysis.
(7) DM results based on the positive integer (l_1, l_2).

2. Background

This section contains a number of important definitions that we must review for further learning.

Definition 2.1. *[8] Let $\mathcal{X}$ be a universal. The PIVFS $\varsigma = \left\{\tau, \left\langle \widetilde{\Psi_{\varsigma}^{\mathcal{T}}}(\tau), \widetilde{\Psi_{\varsigma}^{\mathcal{F}}}(\tau) \right\rangle \middle| \tau \in \mathcal{X}\right\}$, where $\widetilde{\Psi_{\varsigma}^{T}}, \widetilde{\Psi_{\varsigma}^{F}} : \mathcal{X} \to Int([0,1])$ denote the MD and NMD of $\tau \in \mathcal{X}$ to the set ς, respectively, and $0 \leq (\Psi_{\varsigma}^{\mathcal{T}}(\tau))^2 + (\Psi_{\varsigma}^{\mathcal{F}}(\tau))^2 \leq 1$. For convenience, $\varsigma = \left\langle \left[\Psi_{\varsigma}^{\mathcal{T}}, \Psi_{\varsigma}^{\mathcal{T}}\right], \left[\Psi_{\varsigma}^{\mathcal{F}}, \Psi_{\varsigma}^{\mathcal{F}}\right] \right\rangle$ is called a Pythagorean interval-valued fuzzy number (PyIVFN).*

Definition 2.2. *The NS $\varsigma = \left\{x, \left\langle \Psi_{\varsigma}^{\mathcal{T}}(\tau), \Psi_{\varsigma}^{\mathcal{I}}(\tau), \Psi_{\varsigma}^{\mathcal{F}}(\tau) \right\rangle \middle| \tau \in \mathcal{X}\right\}$, where $\Psi_{\varsigma}^{\mathcal{T}}, \Psi_{\varsigma}^{\mathcal{I}}, \Psi_{\varsigma}^{\mathcal{F}} : \mathcal{X} \to [0,1]$ denotes the positive MD, neutral MD and negative MD of $\tau \in \mathcal{X}$, respectively and $0 \leq (\Psi_{\varsigma}^{\mathcal{T}}(\tau)) + (\Psi_{\varsigma}^{\mathcal{I}}(\tau)) + (\Psi_{\varsigma}^{\mathcal{F}}(\tau)) \leq 2$. For $M = \left\langle \Psi_{\varsigma}^{\mathcal{T}}, \Psi_{\varsigma}^{\mathcal{I}}, \Psi_{\varsigma}^{\mathcal{F}} \right\rangle$ is called a neutrosophic number (SFN).*

Definition 2.3. *The Pythagorean NS $\varsigma = \left\{\tau, \left\langle \Psi_{\varsigma}^{\mathcal{T}}(\tau), \Psi_{\varsigma}^{\mathcal{I}}(\tau), \Psi_{\varsigma}^{\mathcal{F}}(\tau) \right\rangle \middle| \tau \in \mathcal{X}\right\}$, where $\Psi_{\varsigma}^{\mathcal{T}}, \Psi_{\varsigma}^{\mathcal{I}}, \Psi_{\varsigma}^{\mathcal{F}} : \mathcal{X} \to [0,1]$ is denoted as the positive MD, neutral MD and negative MD of $\tau \in \mathcal{X}$, respectively and $0 \leq (\Psi_{\varsigma}^{\mathcal{T}}(\tau))^2 + (\Psi_{\varsigma}^{\mathcal{I}}(\tau))^2 + (\Psi_{\varsigma}^{\mathcal{F}}(\tau))^2 \leq 2$. For $M = \left\langle \Psi_{\varsigma}^{\mathcal{T}}, \Psi_{\varsigma}^{\mathcal{I}}, \Psi_{\varsigma}^{\mathcal{F}} \right\rangle$ is called a Pythagorean neutrosophic number (PyNSN).*

Definition 2.4. *[9] The SFS ς in $\mathcal{X}$ is given by $\varsigma = \left\{\tau, \left\langle \Psi_{\varsigma}^{\mathcal{T}}(\tau), \Psi_{\varsigma}^{\mathcal{I}}(\tau), \Psi_{\varsigma}^{\mathcal{F}}(\tau) \right\rangle \middle| \tau \in \mathcal{X}\right\}$, where $\Psi_{\varsigma}^{\mathcal{T}}, \Psi_{\varsigma}^{\mathcal{I}}, \Psi_{\varsigma}^{\mathcal{F}} : \mathcal{X} \to [0,1]$ denote the truth, indeterminacy and falsity membership grade of $\tau \in \mathcal{X}$ to ς, respectively and $0 \leq (\Psi_{\varsigma}^{\mathcal{T}}(\tau))^2 + (\Psi_{\varsigma}^{\mathcal{I}}(\tau))^2 + (\Psi_{\varsigma}^{\mathcal{F}}(\tau))^2 \leq 1$.*
For all $\tau \in \mathcal{X}$, $\sqrt{1 - \left((\Psi_{\varsigma}^{\mathcal{T}}(\tau))^2 + (\Psi_{\varsigma}^{\mathcal{I}}(\tau))^2 + (\Psi_{\varsigma}^{\mathcal{F}}(\tau))^2\right)}$ is called the grade of refusal of membership of τ in ς. For convenience, $\varsigma = \left\langle \Psi_{\varsigma}^{\mathcal{T}}, \Psi_{\varsigma}^{\mathcal{I}}, \Psi_{\varsigma}^{\mathcal{F}} \right\rangle$ is called a spherical fuzzy number (SFN).

Definition 2.5. *[32] Let $\varsigma_1 = (a_1, b_1)$ and $\varsigma_2 = (a_2, b_2)$, where $a_1, b_1, a_2, b_2 \in \mathbb{N}$. Then the distance between ς_1 and ς_2 is defined as $\mathcal{D}(\varsigma_1, \varsigma_2) = \sqrt{(a_1 - a_2)^2 + \frac{1}{2}(b_1 - b_2)^2}$, where $\mathbb{N}$ is the set of all natural number.*

3. Operations for (l_1, l_2) SFN

We discuss the concept of (l_1, l_2) spherical fuzzy number (SFN). As a result, the (l_1, l_2) SFN and its operations were defined.

Definition 3.1. *The (l_1, l_2) SFS $\varsigma = \left\{\tau, \left\langle \left(\Psi^{\mathcal{T}}_{\varsigma}(\tau), \Psi^{\mathcal{I}}_{\varsigma}(\tau), \Psi^{\mathcal{F}}_{\varsigma}(\tau)\right), (l_1, l_2)\right\rangle \middle| \tau \in \mathcal{X}\right\}$, where $\Psi^{T}_{\varsigma}, \Psi^{I}_{\varsigma}, \Psi^{F}_{\varsigma} : \mathcal{X} \to [0,1]$ denote the PMD, neutral MD and NMD of $\tau \in \mathcal{X}$ to ς, respectively and $0 \leq (\Psi^{\mathcal{T}}_{\varsigma}(\tau))^{l_1} + (\Psi^{\mathcal{I}}_{\varsigma}(\tau))^{lcm(l_1,l_2)} + (\Psi^{\mathcal{F}}_{\varsigma}(\tau))^{l_2} \leq 1$. For convenience, $\varsigma = \left\langle \left(\Psi^{\mathcal{T}}_{\varsigma}, \Psi^{\mathcal{I}}_{\varsigma}, \Psi^{\mathcal{F}}_{\varsigma}\right), (l_1, l_2)\right\rangle$ is represent a (l_1, l_2) SFN.*

Definition 3.2. *Let $\varsigma = \langle(\Psi^{\mathcal{T}}, \Psi^{\mathcal{I}}, \Psi^{\mathcal{F}}), (l_1, l_2)\rangle$, $\varsigma_1 = \langle(\Psi^{\mathcal{T}}_1, \Psi^{\mathcal{I}}_1, \Psi^{\mathcal{F}}_1), (l_1, l_2)\rangle$ and $\varsigma_2 = \langle(\Psi^{\mathcal{T}}_2, \Psi^{\mathcal{I}}_2, \Psi^{\mathcal{F}}_2), (l_1, l_2)\rangle$ be any three (l_1, l_2) SFNs, and $\aleph > 0$. Then*

(1) $$\varsigma_1 \uplus \varsigma_2 = \begin{bmatrix} \sqrt[l_1]{(\Psi^{\mathcal{T}}_1)^{l_1} + (\Psi^{\mathcal{T}}_2)^{l_1} - (\Psi^{\mathcal{T}}_1)^{l_1} \cdot (\Psi^{\mathcal{T}}_2)^{l_1}}, \\ \sqrt[lcm(l_1,l_2)]{(\Psi^{\mathcal{I}}_1)^{lcm(l_1,l_2)} + (\Psi^{\mathcal{I}}_2)^{lcm(l_1,l_2)} - (\Psi^{\mathcal{I}}_1)^{lcm(l_1,l_2)} \cdot (\Psi^{\mathcal{I}}_2)^{lcm(l_1,l_2)}}, \\ (\Psi^{\mathcal{F}}_1)^{1_3} \cdot (\Psi^{\mathcal{F}}_2)^{1_3} \end{bmatrix},$$

(2) $$\varsigma_1 \oslash \varsigma_2 = \begin{bmatrix} (\Psi^{\mathcal{T}}_1)^{1_1} \cdot (\Psi^{\mathcal{T}}_2)^{1_1}, \\ \sqrt[lcm(l_1,l_2)]{(\Psi^{\mathcal{I}}_1)^{lcm(l_1,l_2)} + (\Psi^{\mathcal{I}}_2)^{lcm(l_1,l_2)} - (\Psi^{\mathcal{I}}_1)^{lcm(l_1,l_2)} \cdot (\Psi^{\mathcal{I}}_2)^{lcm(l_1,l_2)}}, \\ \sqrt[l_2]{(\Psi^{\mathcal{F}}_1)^{l_2} + (\Psi^{\mathcal{F}}_2)^{l_2} - (\Psi^{\mathcal{F}}_1)^{l_2} \cdot (\Psi^{\mathcal{F}}_2)^{l_2}} \end{bmatrix}$$

(3) $$\aleph \cdot \varsigma = \left[\sqrt[l_1]{1 - \left(1 - (\Psi^{\mathcal{T}})^{l_1}\right)^{\aleph}}, \sqrt[lcm(l_1,l_2)]{1 - \left(1 - (\Psi^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\aleph}}, ((\Psi^{\mathcal{F}})^{l_2})^{\aleph}\right],$$

(4) $$\varsigma^{\aleph} = \left[((\Psi^{\mathcal{T}})^{l_1})^{\aleph}, \sqrt[lcm(l_1,l_2)]{1 - \left(1 - (\Psi^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\aleph}}, \sqrt[l_2]{1 - \left(1 - (\Psi^{\mathcal{F}})^{l_2}\right)^{\aleph}}\right].$$

4. Find (l_1, l_2) SFN distance measure

We introduce ED and HD measures for (l_1, l_2) SFNs and study mathematical properties.

Definition 4.1. *For any two (l_1, l_2) SFNs $\varsigma_1 = \langle(\Psi^{\mathcal{T}}_1, \Psi^{\mathcal{I}}_1, \Psi^{\mathcal{F}}_1), (l_1, l_2)\rangle$ and $\varsigma_2 = \langle(\Psi^{\mathcal{T}}_2, \Psi^{\mathcal{I}}_2, \Psi^{\mathcal{F}}_2), (l_1, l_2)\rangle$. Then*

$$\mathcal{D}_E(\varsigma_1, \varsigma_2) = \frac{1}{2}\sqrt{\begin{bmatrix} \frac{1+(\Psi^{\mathcal{T}}_1)^2-(\Psi^{\mathcal{I}}_1)^2-(\Psi^{\mathcal{F}}_1)^2}{2} \\ -\frac{1+(\Psi^{\mathcal{T}}_2)^2-(\Psi^{\mathcal{I}}_2)^2-(\Psi^{\mathcal{F}}_2)^2}{2} \end{bmatrix}^2 + \frac{1}{2}\begin{bmatrix} \frac{1+(\Psi^{\mathcal{T}}_1)^2-(\Psi^{\mathcal{I}}_1)^2-(\Psi^{\mathcal{F}}_1)^2}{2} \\ -\frac{1+(\Psi^{\mathcal{T}}_2)^2-(\Psi^{\mathcal{I}}_2)^2-(\Psi^{\mathcal{F}}_2)^2}{2} \end{bmatrix}^2}$$

where $\mathcal{D}_E(\varsigma_1, \varsigma_2)$ is called the ED between ς_1 and ς_2.

$$\mathcal{D}_H(\varsigma_1, \varsigma_2) = \frac{1}{2}\begin{bmatrix} \left| \begin{matrix} \frac{1 + (\Psi^{\mathcal{T}}_1)^2 - (\Psi^{\mathcal{I}}_1)^2 - (\Psi^{\mathcal{F}}_1)^2}{2} \\ -\frac{1 + (\Psi^{\mathcal{T}}_2)^2 - (\Psi^{\mathcal{I}}_2)^2 - (\Psi^{\mathcal{F}}_2)^2}{2} \end{matrix} \right| \\ +\frac{1}{2}\left| \begin{matrix} \frac{1 + (\Psi^{\mathcal{T}}_1)^2 - (\Psi^{\mathcal{I}}_1)^2 - (\Psi^{\mathcal{F}}_1)^2}{2} \\ -\frac{1 + (\Psi^{\mathcal{T}}_2)^2 - (\Psi^{\mathcal{I}}_2)^2 - (\Psi^{\mathcal{F}}_2)^2}{2} \end{matrix} \right| \end{bmatrix}$$

where $\mathcal{D}_H(\varsigma_1, \varsigma_2)$ is called the HD between ς_1 and ς_2.

Theorem 4.1. *If any three (l_1, l_2) SFNs $\varsigma_1 = \langle(\Psi_1^{\mathcal{T}}, \Psi_1^{\mathcal{I}}, \Psi_1^{\mathcal{F}}), (l_1, l_2)\rangle$ and $\varsigma_2 = \langle(\Psi_2^{\mathcal{T}}, \Psi_2^{\mathcal{I}}, \Psi_2^{\mathcal{F}}), (l_1, l_2)\rangle$ and $\varsigma_3 = \langle(\Psi_3^{\mathcal{T}}, \Psi_3^{\mathcal{I}}, \Psi_3^{\mathcal{F}}), (l_1, l_2)\rangle$, then $\mathcal{D}_E(\varsigma_1, \varsigma_2)$ satisfies the following valid statements.*

(1) $\mathcal{D}_E(\varsigma_1, \varsigma_2) = 0$ *if and only if* $\varsigma_1 = \varsigma_2$.

(2) $\mathcal{D}_E(\varsigma_1, \varsigma_2) = \mathcal{D}_E(\varsigma_2, \varsigma_1)$.

(3) $\mathcal{D}_E(\varsigma_1, \varsigma_3) \leq \mathcal{D}_E(\varsigma_1, \varsigma_2) + \mathcal{D}_E(\varsigma_2, \varsigma_3)$.

Proof. It is straightforward to prove (1) and (2). Now, $\Big(\mathcal{D}_E(\varsigma_1, \varsigma_2) + \mathcal{D}_E(\varsigma_2, \varsigma_3)\Big)^2$ implies

$$\frac{1}{4}\Big((O-P)^2 + \frac{1}{2}(O-P)^2\Big) + \frac{1}{4}\Big((P-Q)^2 + \frac{1}{2}(P-Q)^2\Big) + \frac{1}{2}\left(\sqrt{(O-P)^2 + \frac{1}{2}(O-P)^2} \times \sqrt{(P-Q)^2 + \frac{1}{2}(P-Q)^2}\right),$$

where

$$O = \frac{1 + (\Psi_1^{\mathcal{T}})^2 - (\Psi_1^{\mathcal{I}})^2 - (\Psi_1^{\mathcal{F}})^2}{2},$$
$$P = \frac{1 + (\Psi_2^{\mathcal{T}})^2 - (\Psi_2^{\mathcal{I}})^2 - (\Psi_2^{\mathcal{F}})^2}{2},$$
$$Q = \frac{1 + (\Psi_3^{\mathcal{T}})^2 - (\Psi_3^{\mathcal{I}})^2 - (\Psi_3^{\mathcal{F}})^2}{2}.$$

Hence, $\Big(\mathcal{D}_E(\varsigma_1, \varsigma_2) + \mathcal{D}_E(\varsigma_2, \varsigma_3)\Big)^2$

$$\begin{aligned}
&\geq \frac{1}{4}\Big((O-P)^2 + \frac{1}{2}(O-P)^2\Big) + \frac{1}{4}\Big((P-Q)^2 + \frac{1}{2}(P-Q)^2\Big) \\
&\quad + \frac{1}{2}\Big((O-P) \times (P-Q) + \frac{1}{2}(O-P) \times (P-Q)\Big) \\
&= \frac{1}{4}\Big((O-P)^2 + (P-Q)^2 + 2(O-P) \times (P-Q)\Big) \\
&\quad + \frac{1}{4}\Big(\frac{1}{2}(O-P)^2 + \frac{1}{2}(P-Q)^2 + (O-P) \times (P-Q)\Big) \\
&= \frac{1}{4}(O-P+P-Q)^2 + \frac{1}{8}(O-P+P-Q)^2 \\
&= \frac{1}{4}\Big[(O-Q)^2 + \frac{1}{2}(O-Q)^2\Big] \\
&= \mathcal{D}_E(\varsigma_1, \varsigma_3)^2.
\end{aligned}$$

□

Corollary 4.1. *Let $\varsigma_1 = \langle(\Psi_1^{\mathcal{T}}, \Psi_1^{\mathcal{I}}, \Psi_1^{\mathcal{F}}), (l_1, l_2)\rangle$ and $\varsigma_2 = \langle(\Psi_2^{\mathcal{T}}, \Psi_2^{\mathcal{I}}, \Psi_2^{\mathcal{F}}), (l_1, l_2)\rangle$ and $\varsigma_3 = \langle(\Psi_3^{\mathcal{T}}, \Psi_3^{\mathcal{I}}, \Psi_3^{\mathcal{F}}), (l_1, l_2)\rangle$ be the any three (l_1, l_2) SFNs. Then*

(1) $\mathcal{D}_H(\varsigma_1, \varsigma_2) = 0$ *if and only if* $\varsigma_1 = \varsigma_2$.

(2) $\mathcal{D}_H(\varsigma_1, \varsigma_2) = \mathcal{D}_H(\varsigma_2, \varsigma_1)$.

(3) $\mathcal{D}_H(\varsigma_1,\varsigma_3) \leq \mathcal{D}_H(\varsigma_1,\varsigma_2) + \mathcal{D}_H(\varsigma_2,\varsigma_3)$.

Proof. As a result of Theorem 4.1, the proof follows. □

5. AOs based on (l_1,l_2) SFN

Here we describe the AOs using (l_1,l_2) SFWA, (l_1,l_2) SFWG, G (l_1,l_2) SFWA, and G (l_1,l_2) SFWG.

5.1. (l_1,l_2) **SFWA.**

Definition 5.1. *Let $\varsigma_i = \langle(\Psi_i^{\mathcal{T}},\Psi_i^{\mathcal{I}},\Psi_i^{\mathcal{F}}),(l_1,l_2)\rangle$ be the (l_1,l_2) SFNs, $W=(\omega_1,\omega_2,...,\omega_n)$ be the weight of ς_i, $\omega_i \geq 0$ and $\uplus_{i=1}^n \omega_i = 1$. Then (l_1,l_2) SFWA $(\varsigma_1,\varsigma_2,...,\varsigma_n) = \uplus_{i=1}^n \omega_i\varsigma_i$.*

Theorem 5.1. *Let $\varsigma_i = \left\langle\left(\Psi_i^{\mathcal{T}},\Psi_i^{\mathcal{I}},\Psi_i^{\mathcal{F}}\right),(l_1,l_2)\right\rangle$ be the (l_1,l_2) SFNs. Then $(l_1,l_2)SFWA(\varsigma_1,\varsigma_2,...,\varsigma_n)$*

$$= \begin{bmatrix} \sqrt[l_1]{1-\odot_{i=1}^n\left(1-(\Psi_i^{\mathcal{T}})^{l_1}\right)^{\omega_i}}, \\ \sqrt[lcm(l_1,l_2)]{1-\odot_{i=1}^n\left(1-(\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}}, \odot_{i=1}^n((\Psi_i^{\mathcal{F}})^{l_2})^{\omega_i} \end{bmatrix}.$$

Proof. If $n=2$, then (l_1,l_2) SFWA$(\varsigma_1,\varsigma_2) = \omega_1\varsigma_1 \uplus \omega_2\varsigma_2$, where

$$\omega_1\varsigma_1 = \begin{bmatrix} \sqrt[l_1]{1-\left(1-(\Psi_1^{\mathcal{T}})^{l_1}\right)^{\omega_1}}, \\ \sqrt[lcm(l_1,l_2)]{1-\left(1-(\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_1}}; ((\Psi_1^{\mathcal{F}})^{l_2})^{\omega_1} \end{bmatrix}$$

$$\omega_2\varsigma_2 = \begin{bmatrix} \sqrt[l_1]{1-\left(1-(\Psi_2^{\mathcal{T}})^{l_1}\right)^{\omega_2}}, \\ \sqrt[lcm(l_1,l_2)]{1-\left(1-(\Psi_2^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_2}}; ((\Psi_2^{\mathcal{F}})^{l_2})^{\omega_2} \end{bmatrix}.$$

Now,

$$\omega_1\varsigma_1 \uplus \omega_2\varsigma_2 = \begin{bmatrix} \sqrt[l_1]{\begin{array}{c}\left(1-\left(1-(\Psi_1^{\mathcal{T}})^{l_1}\right)^{\omega_1}\right)+\left(1-\left(1-(\Psi_2^{\mathcal{T}})^{l_1}\right)^{\omega_2}\right) \\ -\left(1-\left(1-(\Psi_1^{\mathcal{T}})^{l_1}\right)^{\omega_1}\right)\cdot\left(1-\left(1-(\Psi_2^{\mathcal{T}})^{l_1}\right)^{\omega_2}\right)\end{array}}, \\ \\ \sqrt[lcm(l_1,l_2)]{\begin{array}{c}\left(1-\left(1-(\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_1}\right)+\left(1-\left(1-(\Psi_2^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_2}\right) \\ -\left(1-\left(1-(\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_1}\right)\cdot\left(1-\left(1-(\Psi_2^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_2}\right)\end{array}}, \\ \\ ((\Psi_1^{\mathcal{F}})^{l_2})^{\omega_1}\cdot((\Psi_2^{\mathcal{F}})^{l_2})^{\omega_2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[l_1]{1-\left(1-(\Psi_1^{\mathcal{T}})^{l_1}\right)^{\omega_1}\left(1-(\Psi_2^{\mathcal{T}})^{l_1}\right)^{\omega_2}}, \\ \sqrt[l_1]{1-\left(1-(\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_1}\left(1-(\Psi_2^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_2}}, ((\Psi_1^{\mathcal{F}})^{l_2})^{\omega_1}\cdot((\Psi_2^{\mathcal{F}})^{l_2})^{\omega_2} \end{bmatrix}$$

Hence, $(l_1, l_2)SFWA(\varsigma_1, \varsigma_2)$

$$= \left[\begin{array}{c} \sqrt[l_1]{1 - \odot_{i=1}^{2}\left(1 - (\Psi_i^{\mathcal{T}})^{l_1}\right)^{\omega_i}}, \\ \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{2}\left(1 - (\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}}; \odot_{i=1}^{2}((\Psi_i^{\mathcal{F}})^{l_2})^{\omega_i} \end{array} \right].$$

It valid for $n \geq 3$,

Thus, $(l_1, l_2)SFWA(\varsigma_1, \varsigma_2, ..., \varsigma_l)$

$$= \left[\begin{array}{c} \sqrt[l_1]{1 - \odot_{i=1}^{l}\left(1 - (\Psi_i^{\mathcal{T}})^{l_1}\right)^{\omega_i}}, \\ \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{l}\left(1 - (\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}}; \odot_{i=1}^{l}((\Psi_i^{\mathcal{F}})^{l_2})^{\omega_i} \end{array} \right].$$

If $n = l + 1$, then (l_1, l_2) SFWA $(\varsigma_1, \varsigma_2, ..., \varsigma_l, \varsigma_{l+1})$

$$= \left[\begin{array}{c} \sqrt[l_1]{\begin{array}{c} \uplus_{i=1}^{l}\left(1 - \left(1 - (\Psi_i^{\mathcal{T}})^{l_1}\right)^{\omega_i}\right) + \left(1 - \left(1 - (\Psi_{l+1}^{\mathcal{T}})^{l_1}\right)^{\omega_{l+1}}\right) \\ - \odot_{i=1}^{l}\left(1 - \left(1 - (\Psi_i^{\mathcal{T}})^{l_1}\right)^{\omega_i}\right) \cdot \left(1 - \left(1 - (\Psi_{l+1}^{\mathcal{T}})^{l_1}\right)^{\omega_{l+1}}\right), \end{array}} \\ \\ \sqrt[lcm(l_1,l_2)]{\begin{array}{c} \uplus_{i=1}^{l}\left(1 - \left(1 - (\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}\right) + \left(1 - \left(1 - (\Psi_{l+1}^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_{l+1}}\right) \\ - \odot_{i=1}^{l}\left(1 - \left(1 - (\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}\right) \cdot \left(1 - \left(1 - (\Psi_{l+1}^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_{l+1}}\right), \end{array}} \\ \odot_{i=1}^{l}((\Psi_i^{\mathcal{F}})^{l_2})^{\omega_i} \cdot ((\Psi_{l+1}^{\mathcal{F}})^{l_2})^{\omega_{l+1}}, \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[l_1]{1 - \odot_{i=1}^{l+1}\left(1 - (\Psi_i^{\mathcal{T}})^{l_1}\right)^{\omega_i}}, \\ \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{l+1}\left(1 - (\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}}, \odot_{i=1}^{l+1}((\Psi_i^{\mathcal{F}})^{l_2})^{\omega_i} \end{array} \right].$$

□

Theorem 5.2. *Let* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2) \right\rangle$ *be the* (l_1, l_2) *SFNs. Then* (l_1, l_2) *SFWA* $(\varsigma_1, \varsigma_2, ..., \varsigma_n) =$ *L (idempotency property).*

Proof. Since $\Psi_i^{\mathcal{T}} = \Psi^{\mathcal{T}}$, $\Psi_i^{\mathcal{I}} = \Psi^{\mathcal{I}}$ and $\Psi_i^{\mathcal{F}} = \Psi^{\mathcal{F}}$ and $\uplus_{i=1}^{n}\omega_i = 1$. Now, $(l_1, l_2)SFWA(\varsigma_1, \varsigma_2, ..., \varsigma_n)$

$$= \left[\begin{array}{c} \sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - (\Psi_i^{\mathcal{T}})^{l_1}\right)^{\omega_i}}, \\ \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - (\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}}; \odot_{i=1}^{n}((\Psi_i^{\mathcal{F}})^{l_2})^{\omega_i} \end{array} \right]$$

$$= \left[\begin{array}{c} \sqrt[l_1]{1 - \left(1 - (\Psi^{\mathcal{T}})^{l_1}\right)^{\uplus_{i=1}^{n}\omega_i}}, \\ \sqrt[lcm(l_1,l_2)]{1 - \left(1 - (\Psi^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\uplus_{i=1}^{n}\omega_i}}; ((\Psi^{\mathcal{F}})^{l_2})^{\uplus_{i=1}^{n}\omega_i}, \end{array} \right]$$

$$= \left[\sqrt[l_1]{1 - \left(1 - (\Psi^{\mathcal{T}})^{l_1}\right)}, \ \sqrt[lcm(l_1,l_2)]{1 - \left(1 - (\Psi^{\mathcal{I}})^{lcm(l_1,l_2)}\right)}, \ (\Psi^{\mathcal{F}})^{l_2} \right]$$

$$= \quad L.$$

□

Theorem 5.3. *Let* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2)\right\rangle$ *be the* (l_1, l_2) *SFNs. Then* (l_1, l_2) *SFWA*$(\varsigma_1, \varsigma_2, ..., \varsigma_n)$, *where* $\overleftarrow{\Psi^{\mathcal{T}}} = \min \Psi_{ij}^{\mathcal{T}}$, $\overrightarrow{\Psi^{\mathcal{T}}} = \max \Psi_{ij}^{\mathcal{T}}$, $\overleftarrow{\Psi^{\mathcal{I}}} = \min \Psi_{ij}^{\mathcal{I}}$, $\overrightarrow{\Psi^{\mathcal{I}}} = \max \Psi_{ij}^{\mathcal{I}}$, $\overleftarrow{\Psi^{\mathcal{F}}} = \min \Psi_{ij}^{\mathcal{F}}$, $\overrightarrow{\Psi^{\mathcal{F}}} = \max \Psi_{ij}^{\mathcal{F}}$ *and where* $1 \le i \le n$, $j = 1, 2, ..., i_j$. *Then,* $\left\langle \overleftarrow{\Psi^{\mathcal{T}}}, \overleftarrow{\Psi^{\mathcal{I}}}, \overrightarrow{\Psi^{\mathcal{F}}}\right\rangle \le (l_1, l_2)SFWA(\varsigma_1, \varsigma_2, ..., \varsigma_n) \le \left\langle \overrightarrow{\Psi^{\mathcal{T}}}, \overrightarrow{\Psi^{\mathcal{I}}}, \overleftarrow{\Psi^{\mathcal{F}}}\right\rangle$. *(boundedness property).*

Proof. Since, $\overleftarrow{\Psi^{\mathcal{T}}} = \min \Psi_{ij}^{\mathcal{T}}, \overrightarrow{\Psi^{\mathcal{T}}} = \max \Psi_{ij}^{\mathcal{T}}$ and $\overleftarrow{\Psi^{\mathcal{T}}} \le \Psi_{ij}^{\mathcal{T}} \le \overrightarrow{\Psi^{\mathcal{T}}}$. Now,
$\overleftarrow{\Psi^{\mathcal{T}}} = \sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - (\overleftarrow{\Psi^{\mathcal{T}}})^{l_1}\right)^{\omega_i}} \le \sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - (\Psi_{ij}^{\mathcal{T}})^{l_1}\right)^{\omega_i}} \le \sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - (\overrightarrow{\Psi^{\mathcal{T}}})^{l_1}\right)^{\omega_i}} = \overrightarrow{\Psi^{\mathcal{T}}}$.
Since, $\overleftarrow{\Psi^{\mathcal{I}}} = \min \Psi_{ij}^{\mathcal{I}}$, $\overrightarrow{\Psi^{\mathcal{I}}} = \max \Psi_{ij}^{\mathcal{I}}$ and $\overleftarrow{\Psi^{\mathcal{I}}} \le \Psi_{ij}^{\mathcal{I}} \le \overrightarrow{\Psi^{\mathcal{I}}}$. Now,

$$\begin{aligned}\overleftarrow{\Psi^{\mathcal{I}}} &= \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - (\overleftarrow{\Psi^{\mathcal{I}}})^{lcm(l_1,l_2)}\right)^{\omega_i}} \le \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - (\Psi_{ij}^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}} \\ &\le \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - (\overrightarrow{\Psi^{\mathcal{I}}})^{lcm(l_1,l_2)}\right)^{\omega_i}} = \overrightarrow{\Psi^{\mathcal{I}}}.\end{aligned}$$

Since, $\overleftarrow{(\Psi^{\mathcal{F}})^{l_2}} = \min(\Psi_{ij}^{\mathcal{F}})^{l_2}$, $\overrightarrow{(\Psi^{\mathcal{F}})^{l_2}} = \max(\Psi_{ij}^{\mathcal{F}})^{l_2}$ and $\overleftarrow{(\Psi^{\mathcal{F}})^{l_2}} \le (\Psi_{ij}^{\mathcal{F}})^{l_2} \le \overrightarrow{(\Psi^{\mathcal{F}})^{l_2}}$. We have, $\overleftarrow{(\Psi^{\mathcal{F}})^{l_2}} = \odot_{i=1}^{n}((\overleftarrow{(\Psi^{\mathcal{F}})^{l_2}})^{\omega_i} \le \odot_{i=1}^{n}((\Psi_{ij}^{\mathcal{F}})^{l_2})^{\omega_i} \le \odot_{i=1}^{n}((\overrightarrow{(\Psi^{\mathcal{F}})^{l_2}})^{\omega_i} = \overrightarrow{(\Psi^{\mathcal{F}})^{l_2}}$.
Therefore,

$$\begin{aligned} &\frac{1}{2} \times \left[\frac{\left(\sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - (\overleftarrow{\Psi^{\mathcal{T}}})^{l_1}\right)^{\omega_i}}\right)^2}{2} - \frac{\left(\sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - (\overleftarrow{\Psi^{\mathcal{I}}})^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^2}{2} + 1 - \frac{\left(\odot_{i=1}^{n}((\overrightarrow{\Psi^{\mathcal{F}}})^{l_2})^{\omega_i}\right)^2}{2}\right] \\ \le\ &\frac{1}{2} \times \left[\frac{\left(\sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - (\Psi_{ij}^{\mathcal{T}})^{l_1}\right)^{\omega_i}}\right)^2}{2} - \frac{\left(\sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - (\Psi_{ij}^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^2 +}{2} + 1 - \frac{\left(\odot_{i=1}^{n}((\Psi_{ij}^{\mathcal{F}})^{l_2})^{\omega_i}\right)^2}{2}\right] \\ \le\ &\frac{1}{2} \times \left[\frac{\left(\sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - (\overrightarrow{\Psi^{\mathcal{T}}})^{l_1}\right)^{\omega_i}}\right)^2}{2} - \frac{\left(\sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - (\overrightarrow{\Psi^{\mathcal{I}}})^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^2 +}{2} + 1 - \frac{\left(\odot_{i=1}^{n}((\overleftarrow{\Psi^{\mathcal{F}}})^{l_2})^{\omega_i}\right)^2}{2}\right]. \end{aligned}$$

Hence, $\left\langle \overleftarrow{\Psi^{\mathcal{T}}}, \overleftarrow{\Psi^{\mathcal{I}}}, \overrightarrow{\Psi^{\mathcal{F}}}\right\rangle \le (l_1, l_2)SFWA(\varsigma_1, \varsigma_2, ..., \varsigma_n) \le \langle \overrightarrow{\Psi^{\mathcal{T}}}, \overrightarrow{\Psi^{\mathcal{I}}}, \overleftarrow{\Psi^{\mathcal{F}}}\rangle$. □

Theorem 5.4. *Let* $\varsigma_i = \langle(\Psi^{\mathcal{T}}_{t_{ij}}, \Psi^{\mathcal{I}}_{t_{ij}}, \Psi^{\mathcal{F}}_{t_{ij}}), (l_1, l_2)\rangle$ *and* $W_i = \langle(\Psi^{\mathcal{T}}_{h_{ij}}, \Psi^{\mathcal{I}}_{h_{ij}}, \Psi^{\mathcal{F}}_{h_{ij}}), (l_1, l_2)\rangle$*, be the* (l_1, l_2) *SFWAs. For any* i*, if there is* $(\Psi^{\mathcal{T}}_{t_{ij}})^2 \leq (\Psi^{\mathcal{T}}_{h_{ij}})^2$ *and* $(\Psi^{\mathcal{I}}_{t_{ij}})^2 \leq (\Psi^{\mathcal{I}}_{h_{ij}})^2$ *and* $(\Psi^{\mathcal{F}}_{t_{ij}})^2 \geq (\Psi^{\mathcal{F}}_{h_{ij}})^2$ *or* $\varsigma_i \leq W_i$*. Prove that* $(l_1, l_2)SFWA(\varsigma_1, \varsigma_2, ..., \varsigma_n) \leq (l_1, l_2)SFWA(W_1, W_2, ..., W_n)$*, where* $(i = 1, 2, ..., n); (j = 1, 2, ..., i_j)$ *(monotonicity property).*

Proof. For any i, $(\Psi^{\mathcal{T}}_{t_{ij}})^2 \leq \left(\Psi^{\mathcal{T}}_{h_{ij}}\right)^2$.
Therefore, $1 - (\Psi^{\mathcal{T}}_{t_i})^2 \geq 1 - (\Psi^{\mathcal{T}}_{h_i})^2$.
Hence, $\odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{T}}_{t_i}\right)^2\right)^{\omega_i} \geq \odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{T}}_{h_i}\right)^2\right)^{\omega_i}$
and $\sqrt[l_1]{1 - \odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{T}}_{t_i}\right)^{l_1}\right)^{\omega_i}} \leq \sqrt[l_1]{1 - \odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{T}}_{h_i}\right)^{l_1}\right)^{\omega_i}}$.
For any i, $(\Psi^{\mathcal{I}}_{t_{ij}})^{lcm(l_1,l_2)} \leq \left(\Psi^{\mathcal{I}}_{h_{ij}}\right)^{lcm(l_1,l_2)}$.
Therefore, $1 - \left(\Psi^{\mathcal{I}}_{t_i}\right)^{lcm(l_1,l_2)} \geq 1 - \left(\Psi^{\mathcal{I}}_{h_i}\right)^{lcm(l_1,l_2)}$.
Hence, $\odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{I}}_{t_i}\right)^{lcm(l_1,l_2)}\right)^{\omega_i} \geq \odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{I}}_{h_i}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}$.
This implies that $\sqrt[lcm(l_1,l_2)]{1 - \odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{I}}_{t_i}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}} \leq \sqrt[lcm(l_1,l_2)]{1 - \odot^n_{i=1}\left(1 - \left(\Psi^{\mathcal{I}}_{h_i}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}$.
For any i, $\left(\Psi^{\mathcal{F}}_{t_{ij}}\right)^2 \geq \left(\Psi^{\mathcal{F}}_{h_{ij}}\right)^2$ and $\left(\Psi^{\mathcal{F}}_{t_{ij}}\right)^{l_2} \geq \left(\Psi^{\mathcal{F}}_{h_{ij}}\right)^{l_2}$.
Therefore, $1 - \frac{\left(\odot^n_{i=1}\Psi^{\mathcal{F}}_{t_{ij}}\right)^{l_2}}{2} \leq 1 - \frac{\left(\odot^n_{i=1}\Psi^{\mathcal{F}}_{h_{ij}}\right)^{l_2}}{2}$.

$$\frac{1}{2} \times \left[\begin{array}{c} \frac{\left(\sqrt[l_1]{1 - \odot^n_{i=1}\left(1 - (\Psi^{\mathcal{T}}_{ti})^{l_1}\right)^{\omega_i}}\right)^2}{2} - \frac{\left(\sqrt[lcm(l_1,l_2)]{1 - \odot^n_{i=1}\left(1 - (\Psi^{\mathcal{I}}_{ti})^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^2}{2} \\ +1 - \frac{\left(\odot^n_{i=1}(\Psi^{\mathcal{F}}_{tij})^{l_2}\right)^2}{2} \end{array}\right]$$

$$\leq \frac{1}{2} \times \left[\begin{array}{c} \frac{\left(\sqrt[l_1]{1 - \odot^n_{i=1}\left(1 - (\Psi^{\mathcal{T}}_{hi})^{l_1}\right)^{\omega_i}}\right)^2}{2} - \frac{\left(\sqrt[lcm(l_1,l_2)]{1 - \odot^n_{i=1}\left(1 - (\Psi^{\mathcal{I}\varsigma}_{hi})^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^2}{2} \\ +1 - \frac{\left(\odot^n_{i=1}(\Psi^{\mathcal{F}}_{hij})^{l_2}\right)^2}{2} \end{array}\right].$$

Hence, $(l_1, l_2)SFWA\,(\varsigma_1, \varsigma_2, ..., \varsigma_n) \leq (l_1, l_2)SFWA\,(W_1, W_2, ..., W_n)$. □

5.2. (l_1, l_2) **SFWG.**

Definition 5.2. *Let* $\varsigma_i = \left\langle\left(\Psi^{\mathcal{T}}_i, \Psi^{\mathcal{I}}_i, \Psi^{\mathcal{F}}_i\right)\right\rangle$ *be the* (l_1, l_2) *SFNs. Then* (l_1, l_2) *SFWG* $(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \odot^n_{i=1}\varsigma_i^{\omega_i}$.

Theorem 5.5. *Let* $\varsigma_i = \left\langle\left(\Psi^{\mathcal{T}}_i, \Psi^{\mathcal{I}}_i, \Psi^{\mathcal{F}}_i\right), (l_1, l_2)\right\rangle$ *be the* (l_1, l_2) *SFNs. Then* (l_1, l_2) *SFWG* $(\varsigma_1, \varsigma_2, ..., \varsigma_n) =$

$$\left[\begin{array}{c} \odot^n_{i=1}((\Psi^{\mathcal{T}}_i)^{l_1})^{\omega_i}; \\ \sqrt[lcm(l_1,l_2)]{1 - \odot^n_{i=1}\left(1 - (\Psi^{\mathcal{I}}_i)^{lcm(l_1,l_2)}\right)^{\omega_i}},\ \sqrt[l_2]{1 - \odot^n_{i=1}\left(1 - (\Psi^{\mathcal{F}}_i)^{l_2}\right)^{\omega_i}} \end{array}\right].$$

Proof. Based on Theorem 5.1, the following results follow. □

Theorem 5.6. *Let* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2)\right\rangle$ *be the* (l_1, l_2) *SFNs and all are equal. Then* (l_1, l_2) *SFWG*$(\varsigma_1, \varsigma_2, ..., \varsigma_n) = L$.

Proof. Based on Theorem 5.2, the following results follow. □

Remark 5.1. *It has other properties, including boundedness and monotonicity, as well as having* (l_1, l_2) *SFWG.*

Proof. Based on Theorems 5.3 and 5.4, the following results follow. □

5.3. Generalized (l_1, l_2) SFWA (G (l_1, l_2) SFWA).

Definition 5.3. *Let* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2)\right\rangle$ *be the* (l_1, l_2) *SFN. Then G* (l_1, l_2) *SFWA* $(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \left(\uplus_{i=1}^{n} \omega_i \varsigma_i^{\aleph}\right)^{1/\aleph}$.

Theorem 5.7. *Let* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2)\right\rangle$ *be the* (l_1, l_2) *SFNs. Then G* (l_1, l_2) *SFWA* $(\varsigma_1, \varsigma_2, ..., \varsigma_n)$

$$= \left[\begin{array}{c} \left(\sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - \left(\left(\Psi_i^{\mathcal{T}}\right)^{l_1}\right)^{l_1}\right)^{\omega_i}}\right)^{1/l_1}, \\ \left(\sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - \left(\left(\Psi_i^{\mathcal{I}}\right)^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^{1/lcm(l_1,l_2)}, \\ \sqrt[l_2]{1 - \left(1 - \left(\odot_{i=1}^{n}\left(\sqrt[l_2]{1 - \left(1 - \left(\Psi_i^{\mathcal{F}}\right)^{l_2}\right)^{l_2}}\right)^{\omega_i}\right)^{l_2}\right)^{1/l_2}} \end{array}\right].$$

Proof. We can prove this first by demonstrating that,

$$\uplus_{i=1}^{n} \omega_i \varsigma_i^{l_1} = \left[\begin{array}{c} \sqrt[l_1]{1 - \odot_{i=1}^{n}\left(1 - \left(\left(\Psi_i^{\mathcal{T}}\right)^{l_1}\right)^{l_1}\right)^{\omega_i}}; \\ \sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - \left(\left(\Psi_i^{\mathcal{I}}\right)^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}; \\ \odot_{i=1}^{n}\left(\sqrt[l_2]{1 - \left(1 - \left(\Psi_i^{\mathcal{F}}\right)^{l_2}\right)^{l_2}}\right)^{\omega_i}, \end{array}\right].$$

Put $n=2$, $\omega_1\varsigma_1 \uplus \omega_2\varsigma_2$

$$
= \left[\begin{array}{c}
\sqrt[l_1]{\begin{array}{l}\left(\sqrt[l_1]{1-\left(1-\left((\Psi_1^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_1}}\right)^{l_1}+\left(\sqrt[l_1]{1-\left(1-\left((\Psi_2^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_1}}\right)^{l_1}, \\ -\left(\sqrt[l_1]{1-\left(1-\left((\Psi_1^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_1}}\right)^{l_1}\cdot\left(\sqrt[l_1]{1-\left(1-\left((\Psi_2^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_1}}\right)^{l_1}\end{array}} \\
, \\
\sqrt[lcm(l_1,l_2)]{\begin{array}{l}\left(\sqrt[lcm(l_1,l_2)]{1-\left(1-\left((\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_1}}\right)^{lcm(l_1,l_2)}+ \\ \left(\sqrt[lcm(l_1,l_2)]{1-\left(1-\left((\Psi_2^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_1}}\right)^{lcm(l_1,l_2)}, \\ -\left(\sqrt[lcm(l_1,l_2)]{1-\left(1-\left((\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_1}}\right)^{lcm(l_1,l_2)}\cdot \\ \left(\sqrt[lcm(l_1,l_2)]{1-\left(1-\left((\Psi_2^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_1}}\right)^{lcm(l_1,l_2)}\end{array}} \\
\left(\sqrt[l_2]{1-\left(1-(\Psi_1^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_1}\cdot\left(\sqrt[l_2]{1-\left(1-(\Psi_2^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_1},
\end{array}\right]
$$

$$
= \left[\begin{array}{c}
\sqrt[l_1]{1-\odot_{i=1}^{2}\left(1-\left((\Psi_1^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_i}}; \\
\sqrt[lcm(l_1,l_2)]{1-\odot_{i=1}^{2}\left(1-\left((\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}; \\
\odot_{i=1}^{2}\left(\sqrt[l_2]{1-\left(1-(\Psi_i^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_i};
\end{array}\right].
$$

Hence,

$$
\uplus_{i=1}^{l}\omega_i\varsigma_i^{\aleph} = \left[\begin{array}{c}
\sqrt[l_1]{1-\odot_{i=1}^{l}\left(1-\left((\Psi_1^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_i}}; \\
\sqrt[lcm(l_1,l_2)]{1-\odot_{i=1}^{l}\left(1-\left((\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}; \\
\odot_{i=1}^{l}\left(\sqrt[l_2]{1-\left(1-(\Psi_i^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_i};
\end{array}\right].
$$

If $n = l+1$, then $\uplus_{i=1}^{l}\omega_i\varsigma_i^{\aleph} + \omega_{l+1}\varsigma_{l+1}^{\aleph} = \uplus_{i=1}^{l+1}\omega_i\varsigma_i^{\aleph}$.
Now,$\uplus_{i=1}^{l}\omega_i\varsigma_i^{\aleph} + \omega_{l+1}\varsigma_{l+1}^{\aleph} = \omega_1\varsigma_1^{\aleph} \uplus \omega_2\varsigma_2^{\aleph} \uplus ... \uplus \omega_l\varsigma_l^{\aleph} \uplus \omega_{l+1}\varsigma_{l+1}^{\aleph}$

$$= \left[\begin{array}{c} \sqrt[l_1]{\begin{array}{c}\left(\sqrt[l_1]{1-\odot_{i=1}^{l}\left(1-\left((\Psi_i^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_i}}\right)^{l_1} + \left(\sqrt[l_1]{1-\left(1-\left((\Psi_{l+1}^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_1}}\right)^{l_1}, \\ -\left(\sqrt[l_1]{1-\odot_{i=1}^{l}\left(1-\left((\Psi_i^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_i}}\right)^{l_1}\cdot\left(\sqrt[l_1]{1-\left(1-\left((\Psi_{l+1}^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_1}}\right)^{l_1}\end{array}} \\ \sqrt[lcm(l_1,l_2)]{\begin{array}{c}\left(\sqrt[lcm(l_1,l_2)]{1-\odot_{i=1}^{l}\left(1-\left((\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^{lcm(l_1,l_2)} + \\ \left(\sqrt[lcm(l_1,l_2)]{1-\left(1-\left((\Psi_{l+1}^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_1}}\right)^{lcm(l_1,l_2)}, \\ -\left(\sqrt[lcm(l_1,l_2)]{1-\odot_{i=1}^{l}\left(1-\left((\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^{lcm(l_1,l_2)}\cdot \\ \left(\sqrt[lcm(l_1,l_2)]{1-\left(1-\left((\Psi_{l+1}^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_1}}\right)^{lcm(l_1,l_2)}\end{array}} \\ \odot_{i=1}^{l}\left(\sqrt[l_2]{1-\left(1-(\Psi_i^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_i}\cdot\left(\sqrt[l_2]{1-\left(1-(\Psi_{l+1}^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_1}, \end{array}\right]$$

$$\uplus_{i=1}^{l+1}\omega_i\varsigma_i^{l_1} = \left[\begin{array}{c}\sqrt[l_1]{1-\odot_{i=1}^{l+1}\left(1-\left((\Psi_1^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_i}}; \\ \sqrt[lcm(l_1,l_2)]{1-\odot_{i=1}^{l+1}\left(1-\left((\Psi_1^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}; \\ \odot_{i=1}^{l+1}\left(\sqrt[l_2]{1-\left(1-(\Psi_i^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_i}\end{array}\right].$$

$$\left(\uplus_{i=1}^{l+1}\omega_i\varsigma_i^{\aleph}\right)^{1/\aleph} = \left[\begin{array}{c}\left(\sqrt[\aleph]{1-\odot_{i=1}^{l+1}\left(1-\left((\Psi_i^{\mathcal{T}})^{l_1}\right)^{l_1}\right)^{\omega_i}}\right)^{1/l_1}; \\ \left(\sqrt[lcm(l_1,l_2)]{1-\odot_{i=1}^{l+1}\left(1-\left((\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^{1/lcm(l_1,l_2)}; \\ \sqrt[l_2]{1-\left(1-\left(\odot_{i=1}^{l+1}\left(\sqrt[l_2]{1-\left(1-(\Psi_i^{\mathcal{F}})^{l_2}\right)^{l_2}}\right)^{\omega_i}\right)^2\right)^{1/l_2}}, \end{array}\right]$$

$\square$

Remark 5.2. *An operator modified from the G (l_1, l_2) SFWA operator to the (l_1, l_2) SFWA operator is performed if $\aleph = 1$.*

Theorem 5.8. *If all* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2) \right\rangle$ *and all are equal. Then* $G\ (l_1, l_2)\ SFWA(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \varsigma$.

Proof. There is a proof based on the Theorem 5.2. □

Remark 5.3. *In the* $G\ (l_1, l_2)$ *SFWA operator, boundedness and monotonicity are satisfied.*

Proof. There is a proof based on the Theorems 5.3 and 5.4. □

5.4. **Generalized** (l_1, l_2) **SFWG (G** (l_1, l_2) **SFWG).**

Definition 5.4. *Let* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2) \right\rangle$ *be the* (l_1, l_2) *SFNs.*
Then $G\ (l_1, l_2)\ SFWG\ (\varsigma_1, \varsigma_2, ..., \varsigma_n) = \frac{1}{\aleph}\left(\odot_{i=1}^{n} (\aleph \varsigma_i)^{\omega_i} \right)$.

Theorem 5.9. *Let* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2) \right\rangle$ *be the* (l_1, l_2) *SFNs. Then* $G\ (l_1, l_2)\ SFWG(\varsigma_1, \varsigma_2, ..., \varsigma_n)$

$$= \left[\begin{array}{c} \sqrt[l_1]{1 - \left(1 - \left(\odot_{i=1}^{n} \left(\sqrt[l_1]{1 - \left(1 - (\Psi_i^{\mathcal{T}})^{l_1}\right)^{l_1}}\right)^{\omega_i}\right)^{l_1}\right)^{1/l_1}}; \\ \left(\sqrt[lcm(l_1,l_2)]{1 - \odot_{i=1}^{n}\left(1 - \left((\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)}\right)^{lcm(l_1,l_2)}\right)^{\omega_i}}\right)^{1/lcm(l_1,l_2)}; \\ \left(\sqrt[l_2]{1 - \odot_{i=1}^{n}\left(1 - \left((\Psi_i^{\mathcal{F}})^{l_2}\right)^{l_2}\right)^{\omega_i}}\right)^{1/l_2} \end{array} \right].$$

Proof. Based on Theorem 5.7 the following results follow. □

Remark 5.4. *There is a conversion that takes place when* $\aleph = 1$*, which converts the* $G\ (l_1, l_2)$ *SFWG into the* (l_1, l_2) *SFWG.*

Remark 5.5. *Boundness and monotonicity properties that are satisfied by* $G\ (l_1, l_2)$ *SFWG operators.*

Proof. Based on Theorems 5.3 and 5.4, the following results follow. □

Theorem 5.10. *If all* $\varsigma_i = \left\langle \left(\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}}\right), (l_1, l_2) \right\rangle$ *are equal. Then* $G\ (l_1, l_2)\ SFWG(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \varsigma$.

6. MADM approach based on (l_1, l_2) SFN

Let $\varsigma = \{\varsigma_1, \varsigma_2, ..., \varsigma_n\}$ be the set of n-alternatives, $C = \{C_1, C_2, ..., C_m\}$ be the set of m-attributes, $w = \{w_1, w_2, ..., w_m\}$ be the weights of attributes, for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$ $\varsigma_{ij} = \left\langle \left(\Psi_{ij}^{\mathcal{T}}, \Psi_{ij}^{\mathcal{I}}, \Psi_{ij}^{\mathcal{F}}\right), (l_1, l_2) \right\rangle$ denote (l_1, l_2) SFN of alternative ς_i in attribute C_j. Since $\Psi_i^{\mathcal{T}}, \Psi_i^{\mathcal{I}}, \Psi_i^{\mathcal{F}} \in [0, 1]$ and $0 \le (\Psi_i^{\mathcal{T}})^{l_1} + (\Psi_i^{\mathcal{I}})^{lcm(l_1,l_2)} + (\Psi_i^{\mathcal{F}})^{l_2} \le 1$, where $l_1, lcm(l_1, l_2), l_2$ are a positive integers. In the present example, the n-alternatives and m-attributes sets give rise to a $n \times m$ decision matrix, which is symbolized by $\mathfrak{R} = (\widehat{\varsigma_{ij}})_{n \times m}$. In an MADM problem, a set of attributes with preferred weights is collected in order to come up with the best possible decision out of finite alternatives. To obtain a decision, all the alternatives to each attribute are given by (l_1, l_2) SFNs and ED, HD and score values are used. Representations are obtained by summarizing positive and negative values for each attribute for each alternative. Decisions are made based on the following algorithm.

6.1. **Algorithm for** (l_1, l_2) **SFN.** **Step-1:** Enter the values for (l_1, l_2) SFN as follows:
Step-2: Calculate aggregate values for each alternative. On the basis of (l_1, l_2) SFN information aggregation operators, attribute C_j in $\widehat{\varsigma_i}$, $\varsigma_{ij} = \left\langle \left(\Psi^{\mathcal{T}}_{ij}, \Psi^{\mathcal{I}}_{ij}, \Psi^{\mathcal{F}}_{ij} \right), (l_1, l_2) \right\rangle$ is aggregated into $\varsigma_i = \left\langle \left(\Psi^{\mathcal{T}}_{i}, \Psi^{\mathcal{I}}_{i}, \Psi^{\mathcal{F}}_{i} \right), (l_1, l_2) \right\rangle$.

Step-3: Compute the both ideal values are as follows: $\overbrace{\varsigma}^{+} = \langle 1, 1, 0 \rangle$, $\overbrace{\varsigma}^{-} = \langle 0, 0, 1 \rangle$.
Step-4: In order to determine the EDs between each alternative and the ideal values, follow these steps:

$$\mathfrak{R}_i^{+} = \mathfrak{R}_E\left(\overbrace{\varsigma_i}, \overbrace{\varsigma}^{+} \right);\ \mathfrak{R}_i^{-} = \mathfrak{R}_E\left(\overbrace{\varsigma_i}, \overbrace{\varsigma}^{-} \right).$$

Step-5: The following formula is used to calculate relative closeness:

$$\mathfrak{R}_i^{*} = \frac{\mathfrak{R}_i^{-}}{\mathfrak{R}_i^{+} + \mathfrak{R}_i^{-}}.$$

Step-6: A value of max $\mathfrak{R}_i^{*}$ yields the optimal output.

6.2. **Real life example.** People can use personal computers at home, in college, or in their businesses. Monitors, like screens, are built into computers to provide users with favourable colours. The screen has a higher resolution rate, which provides a clearer view. You can add printers, speakers, desktop scanners, and hard drives with more power to your personal computer. It is convenient to take a laptop anywhere, whether on business trips, vacations, or anywhere else. Today's laptops are lightweight and portable, making transporting them easy. $\varsigma = \{\varsigma_a, \varsigma_b, \varsigma_c, \varsigma_d, \varsigma_e\}$ is the five-type laptop (alternative) that customers can choose from. Four attributes are considered: the performance of battery life (C_1), storage capacity (C_2), version of operating system (C_3), over all cost (C_4) and their corresponding weights are $w = \{0.4, 0.3, 0.2, 0.1\}$. The objective is to select the best option from a large number of alternatives by assessing experts based on these criteria. There are many reasons why one might need to purchase a laptop. Consumers go through a number of stages before making a purchase. The following information is used to make decisions:

	C_1	C_2	C_3	C_4
ς_a	$(0.7, 0.55, 0.7)$	$(0.75, 0.6, 0.7)$	$(0.65, 0.7, 0.6)$	$(0.7, 0.5, 0.65)$
ς_b	$(0.65, 0.6, 0.65)$	$(0.8, 0.55, 0.6)$	$(0.7, 0.6, 0.45)$	$(0.65, 0.7, 0.55)$
ς_c	$((0.5, 0.65, 0.5)$	$(0.85, 0.5, 0.45)$	$(0.75, 0.5, 0.7)$	$(0.55, 0.6, 0.7)$
ς_d	$(0.7, 0.55, 0.6)$	$(0.7, 0.45, 0.65)$	$(0.55, 0.65, 0.75)$	$(0.7, 0.5, 0.6)$
ς_e	$(0.6, 0.75, 0.65)$	$(0.65, 0.7, 0.55)$	$(0.75, 0.45, 0.6)$	$(0.6, 0.65, 0.7)$

Aggregate information with (l_1, l_2) SFWA operators are as follows:

	(l_1, l_2) SFWA operator $(l_1 = 1, l_2 = 1)$
$\overbrace{\varsigma_a}$	$(0.7071, 0.5952, 0.6737)$
$\overbrace{\varsigma_b}$	$(0.7131, 0.5974, 0.5798)$
$\overbrace{\varsigma_c}$	$(0.6999, 0.5760, 0.5359)$
$\overbrace{\varsigma_d}$	$(0.6747, 0.5407, 0.6426)$
$\overbrace{\varsigma_e}$	$(0.6502, 0.6803, 0.6129)$

To determine the ideal values of positive and negative alternatives, follow these steps:

$$\overbrace{\varsigma}^{+} = (1, 1, 0) \text{ and } \overbrace{\varsigma}^{-} = (0, 0, 1)$$

ED values are shown for alternative options with both ideal values as follows:

$\mathfrak{R}_1^+$	$\mathfrak{R}_2^+$	$\mathfrak{R}_3^+$	$\mathfrak{R}_4^+$	$\mathfrak{R}_5^+$
0.0944	0.0565	0.0396	0.0766	0.1273

$\mathfrak{R}_1^-$	$\mathfrak{R}_2^-$	$\mathfrak{R}_3^-$	$\mathfrak{R}_4^-$	$\mathfrak{R}_5^-$
0.2118	0.2497	0.2666	0.2296	0.1789

The closeness values are calculated as follows:

$\mathfrak{R}_1^*$	$\mathfrak{R}_2^*$	$\mathfrak{R}_3^*$	$\mathfrak{R}_4^*$	$\mathfrak{R}_5^*$
0.6918	0.8154	0.8708	0.7498	0.5843

The following alternatives are ranked:

$$\varsigma_c > \varsigma_b > \varsigma_d > \varsigma_a > \varsigma_e.$$

As a result, ς_c is the best option for the purchase.

6.3. **Analysis of existing and proposed methods.** Based on the above information, we recommend using (l_1, l_2) SFWA, SFWG, GSFWA and GSFWG approaches. There are several distances to choose from:

$(l_1 = 1, l_2 = 1)$	WA	WG	GWA	GWG
TOPSIS-Euclidean distance (proposed)	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$
TOPSIS-Hamming distance (proposed)	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$
Score-values (proposed)	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$

	WA	*WG*	*GWA*	*GWG*
TOPSIS-Euclidean distance [27]	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$
TOPSIS-Hamming distance [27]	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$
Score-values [27]	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_c > \varsigma_b > \varsigma_d$ $\varsigma_a > \varsigma_e$	$\varsigma_b > \varsigma_c > \varsigma_d$ $\varsigma_a > \varsigma_e$

Change the (l_1, l_2) values from SFWA approach. As a result, we have the following closeness values and orders:

	Relative closeness values					
(l_1, l_2)	$\mathfrak{R}_1^*$	$\mathfrak{R}_2^*$	$\mathfrak{R}_3^*$	$\mathfrak{R}_4^*$	$\mathfrak{R}_5^*$	*Order*
$(1,1)$	0.6918	0.8154	0.8708	0.7498	0.5843	$\varsigma_c > \varsigma_b > \varsigma_d > \varsigma_a > \varsigma_e$
$(2,1)$	0.6899	0.8169	0.8817	0.7484	0.5803	$\varsigma_c > \varsigma_b > \varsigma_d > \varsigma_a > \varsigma_e$
$(2,2)$	0.9377	0.9629	0.9264	0.9908	0.8148	$\varsigma_d > \varsigma_b > \varsigma_a > \varsigma_c > \varsigma_e$

Hence the optimal alternative change from ς_c into ς_d. The values of (l_1, l_2) are used to calculate alternative rankings using SFWG, GSFWA, and GSFWG operators respectively.

7. Conclusion

In this study, we established ED, HD and score values for (l_1, l_2) SFSs, which also have the advantage of being mathematically simple. The ED, HD and score values are shown to be superior by employing appropriate. For (l_1, l_2) SFWA, (l_1, l_2) SFWG, G (l_1, l_2) SFWA and G (l_1, l_2) SFWG, we have proposed improved AO rules. In order to create these operators, we have also talked about a few aspects and provided a few algebraic operations. There is no doubt that the research contained in this article, which is still at a very early stage of its development, will provide a major benefit to future researchers in this field. It is a large field that is open to future academics who have an interest in it. Therefore, the ideas presented here will be beneficial to them in the future. We will discuss in more detail the following topics:

(1) There is a relationship between cubic FS and IVPFS based on AOs.

(2) We examine the normal vague set, normal spherical set and normal vague SFS.

(3) It is possible to solve the problem using complex SFWAs, complex SFWGs, and complex GSFWAs and GSFWGs.

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