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Different Partial Prime Bi-Ideals and Its Extension of Partial Ternary Semirings

M. Palanikumar[1], V. Nagarani[1], M.H.A. Aysha Chithukka[1], Aiyared Iampan[2,*], M.S. Malchijah Raj[3]

[1]*Department of Mathematics, SRM Valliammai Engineering College, Kattankulathur 603203, Tamilnadu, India*

[2]*Department of Mathematics, School of Science, University of Phayao, 19 Moo 2, Tambon Mae Ka, Amphur Mueang, Phayao 56000, Thailand*

[3]*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, India*

[*]*Corresponding author:* aiyared.ia@up.ac.th

Abstract. We discuss the partial bi-ideal of partial ternary semirings $\mathcal{S}$. The partial bi-ideal is a new generalization of the ideal. To determine the relationships between the three types of partial prime bi-ideals and their examples. We constructed the partial right ideal, partial lateral ideal, partial left ideal, partial ideal, and partial bi-ideal generated by a single element. We interact with the relationships between H^{Q}, L^{Q} and R^{Q}, where Q is bi-ideal. Consequently, we defined three distinct partial m-systems. The partial bi-ideal P of $\mathcal{S}$ is a partial 2-prime if and only if $Z_1Z_2Z_3 \subseteq P$, where Z_1 is a partial right ideal, Z_2 is a partial lateral ideal and Z_3 is a partial left ideal of $\mathcal{S}$ implies either one of $Z_1 \subseteq P$, $Z_2 \subseteq P$ and $Z_3 \subseteq P$. Also, we discuss H^{Q} is a unique biggest two-sided partial prime ideal of $\mathcal{S}$ contained Q. Suppose that $\mathscr{M}$ is a partial m_3-system and partial bi-ideal Q of $\mathcal{S}$ with $Q \cap \mathscr{M}$ is empty, there exists a partial 3-prime P of $\mathcal{S}$ containing Q which includes $P \cap \mathscr{M} = \emptyset$. Finally, examples were provided to illustrate the results.

1. Introduction

Partially additive semantics is used in computer programming languages. In these cases, linear algebra cannot be used because partial functions under disjoint-domain sums and functional compositions do not fall under the field definition. As algebraic structures, they can be interpreted as partial ternary semirings that can process both natural and partial ternary semirings, along with infinite partial additions and ternary multiplications. Mathematical structures such as semirings have been discussed as several types of ideals [8], ternary semirings, and rings. Introducing ideals

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for algebraic numbers and then extending them to associative rings is a concept developed by Dedekind. Bi-ideals for semigroups were first introduced by Good and Hughes. Furthermore, this is a special case of the (m, n)-ideal proposed by Lajos. With the help of bi-ideals, Lajos provides both regular and intra-regular semigroups. In addition, Lajos could analyze semigroups both regularly and intra-regularly, using quasi-ideals and generalized bi-ideals. Bi-ideals have often been used in different types of semigroups. Lajos discussed the bi-ideals of the associative rings. Quasi-ideals are generalizations of the left ideals and right ideals, which are special cases of bi-ideals. The concept of semirings is a generalization of rings. Lehmer introduced a triplex as a ternary algebraic system. He studied a class of ternary algebraic structures known as triplexes, which are commutative ternary groups. Vandiver proposed the concept of a semiring. Hestenes examined ternary algebra, using matrices and linear transformations as examples. Lister introduced the ternary ring as an algebraic structure whose triple ring product two additive subgroups of rings. Van der Walt discussed the prime bi-ideal and semiprime bi-ideal of rings. Van der Walt stated that $x_1\mathcal{S}x_2 \subseteq P$ implies either one of $x_1 \in P$ or $x_2 \in P$, for prime bi-ideal P. Roux discussed the prime bi-ideal and semiprime bi-ideal of rings. The prime bi-ideal and semiprime bi-ideal of Γ-so-rings were examined by Srinivasa et al. [19].

Recently, Badmaev et al. [1–5] discussed various applications for Boolean functions generated by maximal partial ultraclones. Prime ideals for rings and semirings can be found in [6]. Partial addition and ternary product-based so-semiring is discussed by Bhagyalakshmi et al. [9]. Palanikumar et al. [15] discussed the concept of a novel method for generating the M-tri-basis of an ordered Γ-semigroup. The theory of partial semirings of continuous valued functions is explained by Shalaginova et al. [18]. Various ideals of partial semirings and gamma partial semirings are discussed by Rao et al. [**?**, 17]. Dutta et al. examined the prime ideals and radicals of ternary semirings [7]. Palanikumar et al. [11] covered the rings' various prime and semiprime bi-ideals. Palanikumar et al. [10, 12–14, 16] discussed the various ideals of semigroups, semirings and ternary semirings. Research on partially additive semantics was conducted. Flowchart untying axiom is the reason for the emphasis on "$\sum$" in computer science. The semantics of programming languages and integration theory are two examples of partially defined infinite operations. Using computer science, we can improve our understanding of computer programs without changing their functions. A positive partial monoid satisfies "positivity" property: if $\sum(\varsigma_i | i \in \mathscr{X})$ is zero, then each ς_i is zero. Considering an abelian monoid that meets the positivity condition of $\varsigma_1 + \varsigma_2 = 0$ implies $\varsigma_1 = 0 = \varsigma_2$ is a partial monoid, where the partition associativity of summable families makes abelian necessary, where the families have finite support and usual sum.

Let M be a fixed set. If $\mathscr{X}$ is a set, then the function $z : \mathscr{X} \to M$ is a $\mathscr{X}$-indexed family. Here, ς_i instead of $\varsigma(i)$. The co-domain is suppressed in the family notation instead of being explicitly indicated in $\varsigma : \mathscr{X} \to M$. Semantics describes "meaning" and computer language semantics, among other technical terms. As a function, semantics uses a syntactically correct program as input and produces a description of the function that the program has calculated. Certain $\mathscr{X}$-indexed families in M will receive an element "$\sum_i(\varsigma_i|i \in \mathscr{X})$" from the partial addition that will be axiomatizing. We shall only deal with countable families because the semantic concepts we want to represent do not involve uncountable sums. The failure to subdivide a sum can be appropriately explained by one axiom. For an example $\varsigma_1 + \varsigma_2 + \varsigma_3 + \varsigma_4 + \varsigma_5 + \varsigma_6 + \varsigma_7 + \varsigma_8 = \varsigma_3 + (\varsigma_1 + \varsigma_6 + \varsigma_2) + (\varsigma_4 + \varsigma_5) + (\varsigma_7 + \varsigma_8)$. If $\mathscr{X} = 1, 2, ..., 8$, $\mathscr{X}_{y_1} = \{3\}$, $\mathscr{X}_{y_2} = \{1, 6, 2\}$, $\mathscr{X}_{y_3} = \{4, 5\}$, $\mathscr{X}_{y_4} = \{7, 8\}$ and $\mathscr{Y} = \{y_1, y_2, y_3, y_4\}$. Hence, $\sum(\varsigma_i|i \in \mathscr{X}) = \sum(\sum(\varsigma_i|i \in \mathscr{X}_j)|j \in \mathscr{Y})$. Here $(\mathscr{X}_j|j \in \mathscr{Y})$ is a partition of $\mathscr{X}$. If $j \neq k$, then $\mathscr{X}_j \cap \mathscr{X}_k = \emptyset$ and $\mathscr{X} = \cup(\mathscr{X}_j|j \in \mathscr{Y})$. We make it clear that any number of j (including an infinite number of j) is acceptable in our definition of a partition $\mathscr{X}_j = \emptyset$ as long as the previously mentioned partition qualities hold. Based on the results of this study, we hope to:

(1) We discuss that partial 1-prime implies that partial 2-prime implies partial 3-prime, and its reverse implications do not hold.
(2) Constructing a partial m_1-system implies that a partial m_2-system implies a partial m_3-system and its reverse implications do not hold.
(3) We discuss the notion of R^Q, L^Q and H^Q and its relation with examples.

This study expands the concept of prime bi-ideals of ternary semiring into prime partial bi-ideals of partial ternary semiring. Section 1 provides an introduction to this study. In Section 2, we briefly describe ternary and partial ternary semirings. In Section 3, the concept of partial prime bi-ideals is examined using numerical examples. The semiprime partial bi-ideals are discussed in Section 4 along with an illustration. The conclusions are provided in Section 5.

List of abbreviations

The following abbreviations are used in this manuscript:

Right ideal	RI
Lateral ideal	LATI
Left ideal	LI
Ideal	ID
Bi-ideal	BI
Partial RI	$\mathscr{P}RI$
Partial lateral ideal	$\mathscr{P}LATI$
Partial LATI	$\mathscr{P}LI$
Partial ID	$\mathscr{P}ID$
Partial BI	$\mathscr{P}BI$
Partial prime BI	$\mathscr{P}PBI$
Partial 1-prime	$\mathscr{P}1P$
Partial 2-prime	$\mathscr{P}2P$
Partial 3-prime	$\mathscr{P}3P$
Partial m_1-system	$\mathscr{P}$-m_1-system
Partial m_2-system	$\mathscr{P}$-m_2-system
Partial m_3-system	$\mathscr{P}$-m_3-system

2. Basic concepts

This study provides a short overview of the fundamental terms used in ternary semirings and partial ternary semirings which will be useful for future research.

Definition 2.1. *A partial monoid $(M, \sum)$, $\sum$ is a partial addition defined on some families $(\varsigma_i|i \in \mathscr{X})$ in M, but not necessarily all of them.*

(i) $\mathscr{X} = \{j\}$, *and* $(\varsigma_i|i \in \mathscr{X})$ *are one-element families in M,* $\sum(\varsigma_i|i \in \mathscr{X}) = \varsigma_j$.
(ii) If a family in M is $(\varsigma_i|i \in \mathscr{X})$ *and a partition of* $\mathscr{X}$ *is* $(\mathscr{X}_j|j \in \mathscr{Y})$, *then* $(\varsigma_i|i \in \mathscr{X})$ *is summable if and only if* $(\varsigma_i|i \in \mathscr{X}_j)$ *is summable,* $(\sum(\varsigma_i|i \in \mathscr{X}_j)|j \in \mathscr{Y})$ *and* $\sum(\varsigma_i|i \in \mathscr{X}) = \sum(\sum(\varsigma_i|i \in \mathscr{X}_j)|j \in \mathscr{Y})$.

Definition 2.2. *Suppose that* $(\mathcal{S}, \sum)$ *is a partial monoid. The function* $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ *is called a partial semiring if*
(i) $l(mn) = (lm)n$,
(ii) The summability of $(\varsigma_i|i \in \mathscr{X})$ *in* $\mathcal{S}$ *denotes* $(l\,\varsigma_i|i \in \mathscr{X})$ *in* $\mathcal{S}$ *and* $l[\sum(\varsigma_i|i \in \mathscr{X})] = \sum(l\varsigma_i|i \in \mathscr{X})$.
(iii) If a family $(\varsigma_i|i \in \mathscr{X})$ *is summable, then* $(\varsigma_i\, l|i \in \mathscr{X})$ *is also summable and* $[\sum(\varsigma_i|i \in \mathscr{X}]l = \sum(\varsigma_i l|i \in \mathscr{X})$.

Definition 2.3. *[19] Let* $\mathscr{A} \subseteq \mathcal{S}$. *If* $\mathscr{A}$ *is said to be a* $\mathscr{P}$*LI(*$\mathscr{P}$*RI) of* $\mathcal{S}$. *Then*
(i) $(\varsigma_i|i \in \mathscr{X})$ *is a summable in* $\mathcal{S}$ *and* $\varsigma_i \in \mathscr{A}$ *for every* $i \in \mathscr{X}$, *hence conclude that* $\sum_i \varsigma_i \in \mathscr{A}$.
(ii) For each $x \in \mathcal{S}$ *and* $y \in \mathscr{A}$ *imply* $xy \in \mathscr{A}$ $(yx \in \mathscr{A})$.

Definition 2.4. *[19] Complete rings can be summable if all the families in a partial ring can be summable.*

Remark 2.1. *[19]* $\mathcal{S}$ *is a complete partial ring. Then* $\mathscr{P}$*RI (*$\mathscr{P}$*LI,* $\mathscr{P}$*ID,* $\mathscr{P}$*BI) generated by "*ς*" are defined as*
(i) $<\varsigma>_r = \{z \in \mathcal{S}|z = n\varsigma + \sum_i \varsigma r_i,\ \ r_i \in \mathcal{S}, n \in \mathbb{N}\}$.
(ii) $<\varsigma>_l = \{z \in \mathcal{S}|z = n\varsigma + \sum_i r_i\varsigma,\ \ r_i \in \mathcal{S}, n \in \mathbb{N}\}$.
(iii) $<\varsigma> = \{z \in \mathcal{S}|z = n\varsigma + \sum_i \varsigma r_i + \sum_j r_j\varsigma + \sum_k \varsigma r_k\varsigma,\ \ r_i, r_j, r_k \in \mathcal{S}, n \in \mathbb{N}\}$.
(iv) $<\varsigma>_b = \{z \in \mathcal{S}|z = n\varsigma + m\varsigma^2 + \sum_i \varsigma r_i\varsigma,\ \ r_i \in \mathcal{S}, n, m \in \mathbb{N}\}$.

Definition 2.5. *[7]* $\mathcal{S}$ *is a ternary semiring if*
(i) $(\mathcal{S}, +)$ *is a commutative semigroup.*
(iia) $(\varsigma_1\varsigma_2\varsigma_3)\varsigma_4\varsigma_5 = \varsigma_1(\varsigma_2\varsigma_3\varsigma_4)\varsigma_5 = \varsigma_1\varsigma_2(\varsigma_3\varsigma_4\varsigma_5)$.
(iib) $(\varsigma_1 + \varsigma_2)\varsigma_3\varsigma_4 = \varsigma_1\varsigma_3\varsigma_4 + \varsigma_2\varsigma_3\varsigma_4$.
(iic) $\varsigma_1(\varsigma_2 + \varsigma_3)\varsigma_4 = \varsigma_1\varsigma_2\varsigma_4 + \varsigma_1\varsigma_3\varsigma_4$.
(iid) $\varsigma_1\varsigma_2(\varsigma_3 + \varsigma_4) = \varsigma_1\varsigma_2\varsigma_3 + \varsigma_1\varsigma_2\varsigma_4\ \forall\ \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5 \in \mathcal{S}$.

Definition 2.6. *[7] Let* $\mathscr{X} \subseteq \mathcal{S}$ *is represent a*
(i) ternary subsemiring when $\mathscr{X}$ *is a additive subsemigroup and* $\varsigma_1\varsigma_2\varsigma_3 \in \mathscr{X}\ \forall\ \varsigma_1, \varsigma_2, \varsigma_3 \in \mathscr{X}$.
(ii) RI(LATI,LI) if $\varsigma r_1 r_2 \in \mathscr{X}\ (r_1\varsigma r_2 \in \mathscr{X}, r_1 r_2\varsigma \in \mathscr{X})\ \forall\ r_1, r_2 \in \mathcal{S}$ *and* $\varsigma \in \mathscr{X}$.

3. Different $\mathscr{P}$PBIs

In the following, $\mathcal{S}$ refers to a partial ternary semiring unless otherwise specified. If we change the ID P by BI P by Theorem 3.1 [7], all three conditions are different. In this section, we introduce three different $\mathscr{P}$PBIs for $\mathcal{S}$.

Theorem 3.1. *[7] Let P be an ID of $\mathcal{S}$. In this case, the statements are equivalent.*
(i) P is a PID.
(ii) $\varsigma_1\mathcal{S}\varsigma_2\mathcal{S}\varsigma_3 \subseteq P, \varsigma_1\mathcal{S}\mathscr{T}\varsigma_2\mathcal{S}\mathscr{T}\varsigma_3 \subseteq P, \varsigma_1\mathcal{S}\mathscr{T}\varsigma_2\mathcal{S}\varsigma_3\mathcal{S} \subseteq P, \mathcal{S}\varsigma_1\mathcal{S}\varsigma_2\mathcal{S}\mathscr{T}\varsigma_3 \subseteq P$ implies any one of $\varsigma_1 \in P$, $\varsigma_2 \in P$ and $\varsigma_3 \in P$.
(iii) $<\varsigma_1><\varsigma_2><\varsigma_3>\subseteq P$ implies any one of $\varsigma_1 \in P$, $\varsigma_2 \in P$ and $\varsigma_3 \in P$.

Definition 3.1. *[7] $\mathscr{M} \subseteq \mathcal{S}$ is called a m-system if $\varsigma_1, \varsigma_2, \varsigma_3 \in \mathcal{S}$, there exist $r_1, r_2 \in \mathscr{M}$, that is associated with $\varsigma_1 r_1 \varsigma_2 r_2 \varsigma_3 \in \mathscr{M}$.*

Definition 3.2. *Consider the partial ternary semiring with "$\sum$" is defined as*

$$\sum_i (\varsigma_i | i \in \mathscr{X}) = \begin{cases} \sum_i \varsigma_i & \text{if } \varsigma_i \in \mathscr{X} \text{ is finite} \\ & \\ \text{undefined} & \text{elsewhere} \end{cases}$$

and "$\cdot$" is defined by the ternary multiplication.

Definition 3.3. *Let $(\mathcal{S}, \sum)$ be a partial monoid. A mapping $\mathcal{S}\times\mathcal{S}\times\mathcal{S} \to \mathcal{S}$ is called partial ternary semiring if*
(i) $(l\cdot m\cdot n)\cdot o\cdot p = l\cdot(m\cdot n\cdot o)\cdot p = l\cdot m\cdot(n\cdot o\cdot p)$,
(ii) a family $(\varsigma_j | j \in \mathscr{X})$ is summable implies $(\varsigma_j xy | j \in \mathscr{X})$ is summable and $[\sum(\varsigma_j | j \in \mathscr{X})]xy = \sum(\varsigma_j xy | j \in \mathscr{X})$.
(iii) a family $(\varsigma_j | j \in \mathscr{X})$ is summable implies $(x\varsigma_j y | j \in \mathscr{X})$ is summable and $x[\sum(\varsigma_j | j \in \mathscr{X}]y = \sum(x\varsigma_j y | j \in \mathscr{X})$.
(iv) a family $(\varsigma_j | j \in \mathscr{X})$ is summable implies $(xy\varsigma_j | j \in \mathscr{X})$ is summable and $xy[\sum(\varsigma_j | j \in \mathscr{X}] = \sum(xy\varsigma_j | j \in \mathscr{X})$.

Definition 3.4. *Let $\mathscr{A} \subseteq \mathcal{S}$, $\mathscr{A}$ is represent a $\mathscr{P}RI(\mathscr{P}LATI, \mathscr{P}LI)$ of $\mathcal{S}$ if*
(i) $(\varsigma_i | i \in \mathscr{X})$ is summable in $\mathcal{S}$ and $\varsigma_i \in \mathscr{A}$ $\forall\ i \in \mathscr{X}$ implies $\sum_i \varsigma_i \in \mathscr{A}$.
(ii) $\forall y, \varsigma \in \mathcal{S}$ and $x \in \mathscr{A}$ implies $xy\varsigma \in \mathscr{A}$ $(yx\varsigma \in \mathscr{A},\ y\varsigma x \in \mathscr{A})$.

Here, we introduce various ideals generated by a single element. Let $a \in \mathcal{S}$. The principle
(i) $\mathscr{P}RI$ generated by "ς" is $<\varsigma>_r = \{x \in \mathcal{S} | x = \sum_n \varsigma + \varsigma\mathcal{S}\mathcal{S} | n \in \mathbb{Z}^+\}$,
(ii) $\mathscr{P}LATI$ generated by "ς" is $<\varsigma>_{lat} = \{x \in \mathcal{S} | x = \sum_n \varsigma + \mathcal{S}\varsigma\mathcal{S} + \mathcal{S}\mathcal{S}\varsigma\mathcal{S}\mathcal{S} | n \in \mathbb{Z}^+\}$,
(iii) $\mathscr{P}LI$ generated by "ς" is $<\varsigma>_l = \{x \in \mathcal{S} | x = \sum_n \varsigma + \mathcal{S}\mathcal{S}\varsigma | n \in \mathbb{Z}^+\}$,
(iv) two sided $\mathscr{P}ID$ generated by "ς" is $<\varsigma>_t = \{x \in \mathcal{S} | x = \sum_n \varsigma + \mathcal{S}\mathcal{S}\varsigma + \varsigma\mathcal{S}\mathcal{S} + \mathcal{S}\mathcal{S}\varsigma\mathcal{S}\mathcal{S} | n \in \mathbb{Z}^+\}$,
(v) $\mathscr{P}ID$ generated by "ς" is $<\varsigma> = \{x \in \mathcal{S} | x = \sum_n \varsigma + \varsigma\mathcal{S}\mathcal{S} + \mathcal{S}\varsigma\mathcal{S} + \mathcal{S}\mathcal{S}\varsigma\mathcal{S}\mathcal{S} + \mathcal{S}\mathcal{S}\varsigma | n \in \mathbb{Z}^+\}$,

(vi) $\mathscr{P}BI$ generated by "ς" is $<\varsigma>_b = \{x \in \mathcal{S} | x = \sum_n \varsigma + \sum_m \varsigma^3 + \varsigma\mathcal{S}\varsigma\mathcal{S}\varsigma | n, m \in \mathbb{Z}^+\}$, where $\sum_n \varsigma$ means that sum of "n" copies of "ς" and $\sum_m \varsigma^3$ means that sum of "m" copies of "ς^3".

Remark 3.1. *Partial ternary semiring $(\mathcal{S}, \sum, \cdot)$ is defined in Definition 3.2 and ternary semirings $(\mathcal{S}, +, \cdot)$ are defined in Definition 2.5.*

Example 3.1. *Consider $\mathcal{S} = \{\backepsilon_1, \backepsilon_2, \backepsilon_3, \backepsilon_4, \backepsilon_5, \backepsilon_6, \backepsilon_7, \backepsilon_8, \backepsilon_9\}$ with "$\sum$" and the ternary product is defined as Definition 3.2.*

$+$	$\backepsilon_1$	$\backepsilon_2$	$\backepsilon_3$	$\backepsilon_4$	$\backepsilon_5$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_9$
$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_2$	$\backepsilon_3$	$\backepsilon_4$	$\backepsilon_5$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_9$
$\backepsilon_2$	$\backepsilon_2$	$\backepsilon_2$	$\backepsilon_5$	$\backepsilon_4$	$\backepsilon_5$	$\backepsilon_9$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_9$
$\backepsilon_3$	$\backepsilon_3$	$\backepsilon_5$	$\backepsilon_3$	$\backepsilon_8$	$\backepsilon_5$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_9$
$\backepsilon_4$	$\backepsilon_4$	$\backepsilon_4$	$\backepsilon_8$	$\backepsilon_4$	$\backepsilon_8$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_7$
$\backepsilon_5$	$\backepsilon_5$	$\backepsilon_5$	$\backepsilon_5$	$\backepsilon_8$	$\backepsilon_5$	$\backepsilon_9$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_9$
$\backepsilon_6$	$\backepsilon_6$	$\backepsilon_9$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_9$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_9$
$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$
$\backepsilon_8$	$\backepsilon_8$	$\backepsilon_8$	$\backepsilon_8$	$\backepsilon_8$	$\backepsilon_8$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_7$
$\backepsilon_9$	$\backepsilon_9$	$\backepsilon_9$	$\backepsilon_9$	$\backepsilon_7$	$\backepsilon_9$	$\backepsilon_9$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_9$

$\cdot$	$\backepsilon_1$	$\backepsilon_2$	$\backepsilon_3$	$\backepsilon_4$	$\backepsilon_5$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_9$
$\backepsilon_1$	a	a	a	a	a	a	a	a	a
$\backepsilon_2$	a	b	a	d	b	a	d	d	b
$\backepsilon_3$	a	c	a	f	c	a	f	f	c
$\backepsilon_4$	a	b	b	d	b	d	d	d	d
$\backepsilon_5$	a	e	a	g	e	a	g	g	e
$\backepsilon_6$	a	c	c	f	c	f	f	f	f
$\backepsilon_7$	a	e	e	g	e	g	g	g	g
$\backepsilon_8$	a	e	b	g	e	d	g	g	h
$\backepsilon_9$	a	e	c	g	e	f	g	g	i

$\cdot$	$\backepsilon_1$	$\backepsilon_2$	$\backepsilon_3$	$\backepsilon_4$	$\backepsilon_5$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_8$	$\backepsilon_9$
a	$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_1$	$\backepsilon_1$
b	$\backepsilon_1$	$\backepsilon_2$	$\backepsilon_1$	$\backepsilon_4$	$\backepsilon_2$	$\backepsilon_1$	$\backepsilon_4$	$\backepsilon_4$	$\backepsilon_2$
c	$\backepsilon_1$	$\backepsilon_3$	$\backepsilon_1$	$\backepsilon_6$	$\backepsilon_3$	$\backepsilon_1$	$\backepsilon_6$	$\backepsilon_6$	$\backepsilon_3$
d	$\backepsilon_1$	$\backepsilon_2$	$\backepsilon_2$	$\backepsilon_4$	$\backepsilon_2$	$\backepsilon_4$	$\backepsilon_4$	$\backepsilon_4$	$\backepsilon_4$
e	$\backepsilon_1$	$\backepsilon_5$	$\backepsilon_1$	$\backepsilon_7$	$\backepsilon_5$	$\backepsilon_1$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_5$
f	$\backepsilon_1$	$\backepsilon_3$	$\backepsilon_3$	$\backepsilon_6$	$\backepsilon_3$	$\backepsilon_6$	$\backepsilon_6$	$\backepsilon_6$	$\backepsilon_6$
g	$\backepsilon_1$	$\backepsilon_5$	$\backepsilon_5$	$\backepsilon_7$	$\backepsilon_5$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_7$
h	$\backepsilon_1$	$\backepsilon_5$	$\backepsilon_2$	$\backepsilon_7$	$\backepsilon_5$	$\backepsilon_4$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_8$
i	$\backepsilon_1$	$\backepsilon_5$	$\backepsilon_3$	$\backepsilon_7$	$\backepsilon_5$	$\backepsilon_6$	$\backepsilon_7$	$\backepsilon_7$	$\backepsilon_9$

Clearly, $(\mathcal{S}, \sum, \cdot)$ and $(\mathcal{S}, +, \cdot)$ are partial ternary semiring and ternary semiring, respectively.

Every RI (LATI, LI, ID, BI) is a $\mathscr{P}RI$ ($\mathscr{P}LATI$, $\mathscr{P}LI$, $\mathscr{P}ID$, $\mathscr{P}BI$). However, the reverse does not hold for this example.

Example 3.2. *By Example 3.1, Let $\boldsymbol{Q} = \{\backepsilon_1, \backepsilon_2, \backepsilon_3\}$ and index set $\mathscr{X} = \{1, 2, 3, ...\}$.*
Since, $\sum(\varsigma_i | i \in \mathscr{X}) = \sum(\sum(\varsigma_i | i \in \mathscr{X}_j) | j \in \mathscr{Y})$ and $(\mathscr{X}_j | j \in \mathscr{Y})$ is a partition of $\mathscr{X}$.
If $j \neq k$ then $\mathscr{X}_j \cap \mathscr{X}_k = \emptyset$ and $\mathscr{X} = \cup(\mathscr{X}_j | j \in \mathscr{Y})$.
Suppose that $\varsigma_i = (\backepsilon_1, 0, 0, 0, 0, 0, ...)$, we have $\sum(\varsigma_i | i \in \mathscr{X}) = \backepsilon_1 \in \boldsymbol{Q}$.
Suppose that $\varsigma_i = (0, 0, 0, \backepsilon_2, 0, 0, ...)$, we have $\sum(\varsigma_i | i \in \mathscr{X}) = \backepsilon_2 \in \boldsymbol{Q}$.
Suppose that $\varsigma_i = (0, 0, 0, 0, 0, 0, 0, 0, \backepsilon_3, 0, 0, 0, ...)$, we have $\sum(\varsigma_i | i \in \mathscr{X}) = \backepsilon_3 \in \boldsymbol{Q}$. Hence, $\boldsymbol{Q}$ is a partial addition of $\mathcal{S}$. Also ternary multiplication "$\cdot$" using in the Example 3.1, $\boldsymbol{Q} \cdot \boldsymbol{Q} \cdot \boldsymbol{Q} \subseteq \boldsymbol{Q}$ and $\boldsymbol{Q}\mathcal{S}\boldsymbol{Q}\mathcal{S}\boldsymbol{Q} \subseteq \boldsymbol{Q}$ implies $\boldsymbol{Q}$ is a $\mathscr{P}BI$ of $\mathcal{S}$, but $\boldsymbol{Q} + \boldsymbol{Q} = \{\backepsilon_1, \backepsilon_2, \backepsilon_3, \backepsilon_5\} \nsubseteq \boldsymbol{Q}$. Thus, $\boldsymbol{Q}$ is not a BI of $(\mathcal{S}, +, \cdot)$.

Similarly, $\{\eth_1, \eth_3, \eth_5, \eth_6, \eth_7\}$ *is a* $\mathscr{P}$*RI, but not a RI of* $\mathcal{S}$*.*
Similarly, $\{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5, \eth_6, \eth_7\}$ *is a* $\mathscr{P}$*ID, but not an ID of* $\mathcal{S}$*.*

Definition 3.5. *A* $\mathscr{P}$*BI* P *of* $\mathcal{S}$ *is called a* $\mathscr{P}1P$ *if*
(i) $Q_1Q_2Q_3 \subseteq P$ *implies any one of* $Q_1 \subseteq P, Q_2 \subseteq P$ *and* $Q_3 \subseteq P$*, for any* $\mathscr{P}$*BIs* Q_1, Q_2 *and* Q_3 *of* $\mathcal{S}$*.*
(ii) $\mathscr{P}2P$ *if* $a'\mathcal{S}a''\mathcal{S}a''' \subseteq P$ *implies any one of* $a' \in P$ *or* $a'' \in P$ *or* $a''' \in P$*.*
(iii) $\mathscr{P}3P$ *if* $\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3 \subseteq P$ *implies any one of* $\mathscr{I}_1 \subseteq P$*,* $\mathscr{I}_2 \subseteq P$ *and* $\mathscr{I}_3 \subseteq P$*, for any* $\mathscr{P}$*IDs* $\mathscr{I}_1, \mathscr{I}_2$ *and* $\mathscr{I}_3$ *of* $\mathcal{S}$*.*

Theorem 3.2. *The* $\mathscr{P}$*BI* P *of* $\mathcal{S}$ *is a* $\mathscr{P}2P$ *if and only if* $Z_1Z_2Z_3 \subseteq P$*, where* Z_1 *is a* $\mathscr{P}$*RI,* Z_2 *is a* $\mathscr{P}$*LATI and* Z_3 *is a* $\mathscr{P}$*LI of* $\mathcal{S}$ *implying any one of* $Z_1 \subseteq P$*,* $Z_2 \subseteq P$ *and* $Z_3 \subseteq P$*.*

Proof. Suppose that $Z_1Z_2Z_3 \subseteq P$. To prove that $Z_1 \subseteq P$ or $Z_2 \subseteq P$ or $Z_3 \subseteq P$. Suppose that $Z_1 \nsubseteq P$ and $Z_2 \nsubseteq P$ implies that $\varsigma' \in Z_1$ but $\varsigma' \notin P$ and $\varsigma'' \in Z_2$ but $\varsigma'' \notin P$. To prove that $Z_3 \subseteq P$. For $\varsigma''' \in Z_3$, $\varsigma'\mathcal{S}\varsigma''\mathcal{S}\varsigma''' \subseteq Z_1Z_2Z_3 \subseteq P$. Since P is a $\mathscr{P}2P$ of $\mathcal{S}$ and $\varsigma' \notin P$ and $\varsigma'' \notin P$ implies that $\varsigma''' \in P$. Thus, $Z_3 \subseteq P$.

Conversely, suppose that $\varsigma'\mathcal{S}\varsigma''\mathcal{S}\varsigma''' \subseteq P$. Now $(\varsigma'\mathcal{S}\mathscr{T})(\mathcal{S}\varsigma''\mathcal{S})(\mathcal{S}\mathscr{T}\varsigma''') \subseteq \varsigma'\mathcal{S}\varsigma''\mathcal{S}\varsigma''' \subseteq P$ implies $\varsigma'\mathcal{S}\mathscr{T} \subseteq P$ or $\mathcal{S}\varsigma''\mathcal{S} \subseteq P$ or $\mathcal{S}\mathscr{T}\varsigma''' \subseteq P$. If $\varsigma'\mathcal{S}\mathscr{T} \subseteq P$, then

$$\begin{aligned} <\varsigma'>_r \cdot <\varsigma''>_{lat} \cdot <\varsigma'''>_l &= \left[\left\{\sum_n \varsigma' | n \in \mathbb{Z}^+\right\} + \varsigma'\mathcal{S}\mathscr{T}\right] \cdot \left[\left\{\sum_m \varsigma'' | m \in \mathbb{Z}^+\right\} + \right. \\ &\quad \left. \left[\mathcal{S}\varsigma''\mathcal{S} + \mathcal{S}\mathscr{T}\varsigma''\mathcal{S}\mathscr{T}\right]\right] \cdot \left[\left\{\sum_{m'} \varsigma''' | m' \in \mathbb{Z}^+\right\} + \mathcal{S}\mathscr{T}\varsigma'''\right] \\ &\subseteq \left[\sum_{nmm'} \varsigma'\varsigma''\varsigma'''\right] + \varsigma'\mathcal{S}\varsigma''\mathcal{S}\varsigma''' \\ &\subseteq \varsigma'\mathcal{S}\mathscr{T} \subseteq P. \end{aligned}$$

Thus, $\varsigma' \in P$ or $\varsigma'' \in P$ or $\varsigma''' \in P$.
Similarly, $\mathcal{S}\varsigma''\mathcal{S} \subseteq P$. Let us demonstrate that, $<\varsigma'>_r \cdot <\varsigma''>_{lat} \cdot <\varsigma'''>_l \subseteq [\mathcal{S}\varsigma''\mathcal{S} \cup \mathcal{S}\mathscr{T}\varsigma''\mathcal{S}\mathscr{T}] \subseteq P$.
Suppose $\mathcal{S}\mathscr{T}\varsigma''' \subseteq P$ then $<\varsigma'>_r \cdot <\varsigma''>_{lat} \cdot <\varsigma'''>_l \subseteq \mathcal{S}\mathscr{T}\varsigma''' \subseteq P$. This implies that $\varsigma' \in P$ or $\varsigma'' \in P$ or $\varsigma''' \in P$. □

The following implications hold for $\mathscr{P}1P$ implies $\mathscr{P}2P$ implies $\mathscr{P}3P$. However, these examples do not support the reverse implications.

Example 3.3. *In Example 3.1, Clearly* $P = \{\eth_1, \eth_2, \eth_4\}$ *is a* $\mathscr{P}2P$*. Now,* $\{\eth_1, \eth_3, \eth_5\} \cdot \{\eth_1, \eth_3, \eth_6\} \cdot \{\eth_1, \eth_4, \eth_6\} = \{\eth_1\} \subseteq P$*, but* $\{\eth_1, \eth_3, \eth_5\} \nsubseteq P$ *and* $\{\eth_1, \eth_3, \eth_6\} \nsubseteq P$ *and* $\{\eth_1, \eth_4, \eth_6\} \nsubseteq P$*. This implies that* P *is not a* $\mathscr{P}1P$*.*

Example 3.4. *Consider* $\mathcal{S} = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5, \mho_6\}$ *with the following compositions:*

$+$	$\mho_1$	$\mho_2$	$\mho_3$	$\mho_4$	$\mho_5$	$\mho_6$
$\mho_1$	$\mho_1$	$\mho_2$	$\mho_3$	$\mho_4$	$\mho_5$	$\mho_6$
$\mho_2$	$\mho_2$	$\mho_2$	$\mho_3$	$\mho_4$	$\mho_5$	$\mho_6$
$\mho_3$	$\mho_3$	$\mho_3$	$\mho_3$	$\mho_6$	$\mho_5$	$\mho_6$
$\mho_4$	$\mho_4$	$\mho_4$	$\mho_6$	$\mho_4$	$\mho_5$	$\mho_6$
$\mho_5$	$\mho_5$	$\mho_5$	$\mho_5$	$\mho_5$	$\mho_5$	$\mho_5$
$\mho_6$	$\mho_6$	$\mho_6$	$\mho_6$	$\mho_6$	$\mho_5$	$\mho_6$

$\cdot$	$\mho_1$	$\mho_2$	$\mho_3$	$\mho_4$	$\mho_5$	$\mho_6$
$\mho_1$	a	a	a	a	a	a
$\mho_2$	a	b	c	b	c	c
$\mho_3$	a	b	c	b	c	c
$\mho_4$	a	d	e	d	e	e
$\mho_5$	a	d	e	d	e	e
$\mho_6$	a	d	e	d	e	e

$\cdot$	$\mho_1$	$\mho_2$	$\mho_3$	$\mho_4$	$\mho_5$	$\mho_6$
a	$\mho_1$	$\mho_1$	$\mho_1$	$\mho_1$	$\mho_1$	$\mho_1$
b	$\mho_1$	$\mho_2$	$\mho_3$	$\mho_2$	$\mho_3$	$\mho_3$
c	$\mho_1$	$\mho_2$	$\mho_3$	$\mho_2$	$\mho_3$	$\mho_3$
d	$\mho_1$	$\mho_4$	$\mho_5$	$\mho_4$	$\mho_5$	$\mho_5$
e	$\mho_1$	$\mho_4$	$\mho_5$	$\mho_4$	$\mho_5$	$\mho_5$
f	$\mho_1$	$\mho_4$	$\mho_5$	$\mho_4$	$\mho_5$	$\mho_5$

Clearly $P = \{\mho_1, \mho_3\}$ is a $\mathscr{P}3P$. Now, $\mho_2\mathcal{S}\mho_5\mathcal{S}\mho_6 = \{\mho_1, \mho_3\} \subseteq P$ but $\mho_2 \notin P$,$\mho_5 \notin P$ and $\mho_6 \notin P$, implies P is not a $\mathscr{P}2P$ of $\mathcal{S}$.

Definition 3.6. *(i) A subset $\mathscr{M}$ of $\mathcal{S}$ is represent a $\mathscr{P}$-m_1-system if for any $\varsigma^{'}, \varsigma^{''}, \varsigma^{'''} \in \mathscr{M}$, $\exists\, \varsigma_1^{'} \in <\varsigma^{'}>_b$, $\varsigma_1^{''} \in <\varsigma^{''}>_b$ and $\varsigma_1^{'''} \in <\varsigma^{'''}>_b$ such that $\varsigma_1^{'} \cdot \varsigma_1^{''} \cdot \varsigma_1^{'''} \in \mathscr{M}$.*
(ii) A subset $\mathscr{M}$ of $\mathcal{S}$ is represent a $\mathscr{P}$-m_2-system if for any $\varsigma^{'}, \varsigma^{''}, \varsigma^{'''} \in \mathscr{M}$, $\exists\, \varsigma_1^{'} \in <\varsigma^{'}>_r$, $\varsigma_1^{''} \in <\varsigma^{''}>_{lat}$ and $\varsigma_1^{'''} \in <\varsigma^{'''}>_l$ such that $\varsigma_1^{'} \cdot \varsigma_1^{''} \cdot \varsigma_1^{'''} \in \mathscr{M}$.
(iii) A subset $\mathscr{M}$ of $\mathcal{S}$ is represent a $\mathscr{P}$-m_3-system if for any $\varsigma^{'}, \varsigma^{''}, \varsigma^{'''} \in \mathscr{M}$, $\exists\, \varsigma_1^{'} \in <\varsigma^{'}>$, $\varsigma_1^{''} \in <\varsigma^{''}>$ and $\varsigma_1^{'''} \in <\varsigma^{'''}>$ such that $\varsigma_1^{'} \cdot \varsigma_1^{''} \cdot \varsigma_1^{'''} \in \mathscr{M}$.

Lemma 3.1. *Let P be a $\mathscr{P}BI$ of $\mathcal{S}$. Then, P is a $\mathscr{P}1P$ ($\mathscr{P}2P$, $\mathscr{P}3P$) if and only if $\mathcal{S} \setminus P$ is a $\mathscr{P}$-m_1-system ($\mathscr{P}$-m_2-system, $\mathscr{P}$-m_3-system) of $\mathcal{S}$.*

Proof. Let P be the $\mathscr{P}1P$ of $\mathcal{S}$ and let $\varsigma^{'}, \varsigma^{''}, \varsigma^{'''} \in \mathcal{S} \setminus P$. Hence, $<\varsigma^{'}>_b \cdot <\varsigma^{''}>_b \cdot <\varsigma^{'''}>_b \nsubseteq P$. Then there exist $\varsigma_1^{'} \in <\varsigma^{'}>_b$, $\varsigma_1^{''} \in <\varsigma^{''}>_b$ and $\varsigma_1^{'''} \in <\varsigma^{'''}>_b$ such that $\varsigma_1^{'} \cdot \varsigma_1^{''} \cdot \varsigma_1^{'''} = \left\{ \sum_{n_1} \varsigma^{'} + \sum_{n_2} (\varsigma^{'})^3 + \mho_1 \right\} \cdot \left\{ \sum_{n_1^{'}} \varsigma^{''} + \sum_{n_2^{'}} (\varsigma^{''})^3 + \mho_2 \right\} \cdot \left\{ \sum_{n_1^{''}} \varsigma^{'''} + \sum_{n_2^{''}} (\varsigma^{'''})^3 + \mho_3 \right\}$, where $\mho_1 = \sum_{(i,j)} \varsigma^{'} r_i \varsigma^{'} r_j \varsigma^{'}$ and $\mho_2 = \sum_{(i,j)} \varsigma^{''} r_i^{'} \varsigma^{''} r_j^{'} \varsigma^{''}$ and $\mho_3 = \sum_{(i,j)} \varsigma^{'''} r_i^{''} \varsigma^{'''} r_j^{''} \varsigma^{'''}$ for $n_1, n_2, n_1^{'}, n_2^{'}, n_1^{''}, n_2^{''} \in \mathbb{N}$ and $r_i, r_j, r_i^{'}, r_j^{'}, r_i^{''}, r_j^{''} \in \mathcal{S}$. Now, $\mho_1 \cdot \mho_2 \cdot \mho_3 \in <\varsigma^{'}>_b \cdot <\varsigma^{''}>_b \cdot <\varsigma^{'''}>_b \nsubseteq P$. Thus, $\varsigma_1^{'} \cdot \varsigma_1^{''} \cdot \varsigma_1^{'''} \notin P$. Hence, $\mathcal{S} \setminus P$ is a $\mathscr{P}$-m_1-system.

Conversely, Let $\mathcal{S} \setminus P$ be a $\mathscr{P}$-m_1-system. Suppose that $Q_1 \cdot Q_2 \cdot Q_3 \subseteq P$ for the $\mathscr{P}BIs$ Q_1, Q_2 and Q_3 of $\mathcal{S}$. Let us arrive at a contradiction. Let $\varsigma_1^{''} \in Q_1 \setminus P$, $\varsigma_2^{''} \in Q_2 \setminus P$ and $\varsigma_3^{''} \in Q_3 \setminus P$. Hence, $\varsigma_1^{''}, \varsigma_2^{''}, \varsigma_3^{''} \in \mathcal{S} \setminus P$ implies $<\varsigma_1^{''}>_b \cdot <\varsigma_2^{''}>_b \cdot <\varsigma_3^{''}>_b \nsubseteq P$, which is a contradiction. Thus, $Q_1 \subseteq P$ or $Q_2 \subseteq P$ or $Q_3 \subseteq P$. Therefore, P is a $\mathscr{P}1P$ of $\mathcal{S}$. Similarly, we can prove the other cases. □

The following implications hold for $\mathscr{P}$-m_1-system implying that the $\mathscr{P}$-m_2-system implies $\mathscr{P}$-m_3-system. It is clear that this example does not support the reverse implications.

Example 3.5. *By Example 3.1,* $\mathscr{M} = \{\eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9\}$ *is a* $\mathscr{P}$-m_2*-system, but not a* $\mathscr{P}$-m_1*-system. For* $\eth_5, \eth_6, \eth_7 \in \mathscr{M}$*, but there is no* $x_1 \in <\eth_5>_b$*,* $y_1 \in <\eth_6>_b$ *and* $\varsigma_1 \in <\eth_7>_b$ *such that* $x_1 \cdot y_1 \cdot \varsigma_1 \in \mathscr{M}$*. Since* $<\eth_5>_b \cdot <\eth_6>_b \cdot <\eth_7>_b = \{\eth_1, \eth_3, \eth_5\} \cdot \{\eth_1, \eth_3, \eth_6\} \cdot \{\eth_1, \eth_6, \eth_7\} = \{\eth_1\} \notin \mathscr{M}$.

Example 3.6. *By Example 3.4,* $\mathscr{M} = \{\mho_2, \mho_4, \mho_5, \mho_6\}$ *is a* $\mathscr{P}$-m_3*-system, but not a* $\mathscr{P}$-m_2*-system by* $\mho_2 \mathcal{S} \mho_4 \mathcal{S} \mho_6 = \{\mho_1, \mho_3\} \notin \mathscr{M}$.

Lemma 3.2. *Every* $\mathscr{P}$-m_2*-system is a* $\mathscr{P}$-m*-system and vice versa.*

Proof. Let $a, b, c \in \mathscr{M}$, $\exists\, x' \in <a>_r, y' \in <b>_{lat}$ and $z' \in <c>_l$ such that $x' \cdot y' \cdot z' \in \mathscr{M}$. Now,

$$\begin{aligned}
x' \cdot y' \cdot z' &= \Big[\sum_{n_1} a + ar_1r_2\Big] \cdot \Big[\sum_{n_2} b + r_3br_4 + r_5r_6br_7r_8\Big] \cdot \Big[\sum_{n_3} c + r_9r_{10}c\Big] \\
&= \Big[\sum_{n_1n_2} ab + (\sum_{n_1} a)r_3br_4 + (\sum_{n_1} a)r_5r_6br_7r_8 + ar_1r_2a(\sum_{n_2} b) + \\
&\quad ar_1r_2r_3br_4 + ar_1r_2r_5r_6br_7r_8\Big] \cdot \Big[\sum_{n_3} c + r_9r_{10}c\Big] \\
&= \Big[\sum_{n_1n_2n_3} abc + (\sum_{n_1} a)r_3br_4(\sum_{n_3} c) + (\sum_{n_1} a)r_5r_6br_7r_8(\sum_{n_3} c) + ar_1r_2a(\sum_{n_2n_3} bc) \\
&\quad + ar_1r_2r_3br_4(\sum_{n_3} c) + ar_1r_2r_5r_6br_7r_8(\sum_{n_3} c) + (\sum_{n_1n_2} ab)r_9r_{10}c + \\
&\quad (\sum_{n_1} a)r_3br_4r_9r_{10}c + (\sum_{n_1} a)r_5r_6br_7r_8r_9r_{10}c + ar_1r_2a(\sum_{n_2} b)r_9r_{10}c + \\
&\quad ar_1r_2r_3br_4r_9r_{10}c + ar_1r_2r_5r_6br_7r_8r_9r_{10}c\Big] \\
&= \sum_{n_1n_2n_3} abc + ar'br''c \in \mathscr{M}.
\end{aligned}$$

Again $a, b, \sum_{n_1n_2n_3} abc + ar'br''c \in \mathscr{M}$, $\exists\, x'' \in <a>_r, y'' \in <b>_{lat}$ and $z'' \in <\sum_{n_1n_2n_3} abc + ar'br''c>_l$ such that $x'' \cdot y'' \cdot z'' \in \mathscr{M}$. Since, $x'' \cdot y'' \cdot z'' = ar_{11}br_{12}c \in a\mathcal{S}b\mathcal{S}c$. Therefore, $ar_{11}br_{12}c = x'' \cdot y'' \cdot z'' \in \mathscr{M}$. Hence, $\mathscr{M}$ is a $\mathscr{P}$-m-system.

Conversely, let $a, b, c \in \mathscr{M}$, $\exists\, r_1, r_2 \in \mathcal{S}$ such that $ar_1br_2c \in \mathscr{M}$. Let $ar_1 = a_1$ and $r_2c = c_1$, $\exists\, a_1 \in <a>_r$, $b \in <b>_{lat}$ and $c_1 \in <c>_l$ such that $a_1 \cdot b \cdot c_1 \in \mathscr{M}$. Hence, $\mathscr{M}$ is a $\mathscr{P}$-m_2-system. □

Definition 3.7. *(i) Let* Q *be a* $\mathscr{P}$*BI of* $\mathcal{S}$ *and let* $L^Q = \{x \in Q | \mathcal{S}\mathscr{T}x \subseteq Q\}$ *and which is related to* $H^Q = \{y \in L^Q | y\mathcal{S}\mathscr{T} \subseteq L^Q\}$.
(ii) $R^Q = \{x \in Q | x\mathcal{S}\mathscr{T} \subseteq Q\}$ *and which is related to* $H^Q = \{y \in R^Q | \mathcal{S}\mathscr{T}y \subseteq R^Q\}$.

Lemma 3.3. *Let* Q *be a* $\mathscr{P}$*BI of* $\mathcal{S}$*. Prove that* L^Q *is a* $\mathscr{P}$*LI of* $\mathcal{S}$ *such that* $L^Q \subseteq Q$.

Proof. Let $\varsigma_i \in L^{Q}$. Then $\varsigma_i \in Q$ and $\mathcal{S}\mathscr{T}\varsigma_i \subseteq Q$, $\forall i$. Since Q is a $\mathscr{P}BI$ of $\mathcal{S}$, then $\sum_i \varsigma_i \in Q$ and $(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n) \in Q$. Now, $\mathcal{S}\mathscr{T}(\sum_i \varsigma_i) \subseteq Q$. Thus, $\sum_i \varsigma_i \in L^{Q}$. Now, $\mathcal{S}\mathscr{T}(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n) = (\mathcal{S}\mathscr{T}\varsigma_1) \cdot (\varsigma_2 \cdot ... \cdot \varsigma_n) \subseteq (\mathcal{S}\mathscr{T}\varsigma_1) \cdot (\mathcal{S}\mathscr{T}\varsigma_2) \cdot (\varsigma_3 ... \cdot \varsigma_n) \subseteq (\mathcal{S}\mathscr{T}\varsigma_1) \cdot (\mathcal{S}\mathscr{T}\varsigma_2) \cdot ... \cdot (\mathcal{S}\mathscr{T}\varsigma_n) \subseteq Q$. Thus, $(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n) \in L^{Q}$. Let $x \in L^{Q}$ and $r_1, r_2 \in \mathcal{S}$. Since $r_1 r_2 x \in \mathcal{S}\mathscr{T}x \subseteq Q$, we have $r_1 r_2 x \in Q$ and $\mathcal{S}\mathscr{T} r_1 r_2 x \subseteq \mathcal{S}\mathscr{T}\mathcal{S}\mathscr{T}x \subseteq \mathcal{S}\mathscr{T}x \subseteq Q$. Thus, $r_1 r_2 x \in L^{Q}$. Hence, L^{Q} is a $\mathscr{P}LI$ of $\mathcal{S}$ and $L^{Q} \subseteq Q$. □

Lemma 3.4. *Let Q be a $\mathscr{P}BI$ of $\mathcal{S}$. Then, H^{Q} is a partial subring of $\mathcal{S}$.*

Proof. Let $\varsigma_i \in H^{Q}$. Then $\varsigma_i \in L^{Q}$ and $\varsigma_i \mathcal{S}\mathscr{T} \subseteq L^{Q}$, $\forall i$. Since $\varsigma_i \in L^{Q}$, $\varsigma_i \in Q$ and $\mathcal{S}\mathscr{T}\varsigma_i \subseteq Q$, $\forall i$. Since Q is the partial subring of $\mathcal{S}$ and $\varsigma_i \in Q$. We have $\sum_i \varsigma_i \in Q$ and $(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n) \in Q$. Now, $\mathcal{S}\mathscr{T}(\sum_i \varsigma_i) \subseteq Q$ implies $\sum_i \varsigma_i \in L^{Q}$. Now, $(\sum_i \varsigma_i)\mathcal{S}\mathscr{T} \subseteq L^{Q}$ implies $\sum_i \varsigma_i \in H^{Q}$. Now, $\mathcal{S}\mathscr{T}(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n) = (\mathcal{S}\mathscr{T}\varsigma_1) \cdot (\varsigma_2 \cdot ... \cdot \varsigma_n) \subseteq (\mathcal{S}\mathscr{T}\varsigma_1) \cdot (\mathcal{S}\mathscr{T}\varsigma_2) \cdot (\varsigma_3 ... \cdot \varsigma_n) \subseteq (\mathcal{S}\mathscr{T}\varsigma_1) \cdot (\mathcal{S}\mathscr{T}\varsigma_2) \cdot ... \cdot (\mathcal{S}\mathscr{T}\varsigma_n) \subseteq Q$ implies $(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n) \in L^{Q}$ and $(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n)\mathcal{S}\mathscr{T} = (\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_{n-1}) \cdot (\varsigma_n \mathcal{S}\mathscr{T}) \subseteq (\varsigma_1 \mathcal{S}\mathscr{T}) \cdot (\varsigma_2 \mathcal{S}\mathscr{T}) \cdot ... \cdot (\varsigma_n \mathcal{S}\mathscr{T}) \subseteq L^{Q}$. Thus, $(\varsigma_1 \cdot \varsigma_2 \cdot ... \cdot \varsigma_n) \in H^{Q}$. □

Lemma 3.5. *Let Q be a $\mathscr{P}LI$ of $\mathcal{S}$. Then, $L^{Q} = Q$.*

Proof. Clearly, $L^{Q} \subseteq Q$. Let $x \in Q$, but Q is a $\mathscr{P}LI$ of $\mathcal{S}$. Now, $\mathcal{S}\mathscr{T}x \subseteq Q$ implies $x \in L^{Q}$. Thus, $Q \subseteq L^{Q}$. Hence, $L^{Q} = Q$. □

Theorem 3.3. *Let Q be a $\mathscr{P}BI$ of a $\mathcal{S}$. Then, H^{Q} is the unique largest two-sided $\mathscr{P}ID$ of $\mathcal{S}$ and which is contained in Q.*

Proof. Let Q be the $\mathscr{P}BI$ of $\mathcal{S}$. First, we that H^{Q} is a two sided $\mathscr{P}ID$ of $\mathcal{S}$. Since $H^{Q} \subseteq L^{Q} \subseteq Q$. Let $\varsigma_i \in H^{Q}, \forall i \in I$ and $y_1, y_2 \in \mathcal{S}$. Then $\varsigma_i \in H^{Q} \subseteq Q \implies \varsigma_i \in Q$. Since $\varsigma_i \in L^{Q}$, we have $\mathcal{S}\mathcal{S}\varsigma_i \subseteq Q$ and $\varsigma_i \mathcal{S}\mathcal{S} \subseteq L^{Q}$, $\forall i \in I$. Because Q is the $\mathscr{P}BI$ of $\mathcal{S}$, then $\sum_i \varsigma_i \in Q$. Since $\varsigma_i \in L^{Q}$, $\sum_i \varsigma_i \in L^{Q}$, $\mathcal{S}\mathcal{S}(\sum_i \varsigma_i) \subseteq L^{Q} \subseteq Q$ and $(\sum_i \varsigma_i)\mathcal{S}\mathcal{S} \subseteq L^{Q}$. Hence, $\sum_i \varsigma_i \in H^{Q}$. Since $x \in L^{Q}$, then $y_1 y_2 x \in \mathcal{S}\mathcal{S}x \subseteq Q$ and $\mathcal{S}\mathcal{S}y_1 y_2 x \subseteq \mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S}x \subseteq \mathcal{S}\mathcal{S}x \subseteq Q \implies y_1 y_2 x \in L^{Q}$. Moreover $x y_1 y_2 \in x\mathcal{S}\mathcal{S} \subseteq L^{Q}$. Therefore $x y_1 y_2 \in L^{Q}$ and $y_1 y_2 x \in L^{Q}$. To prove that $x y_1 y_2 \in H^{Q}$ and $y_1 y_2 x \in H^{Q}$. Now, $x y_1 y_2 \mathcal{S}\mathcal{S} \subseteq x\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S} \subseteq x\mathcal{S}\mathcal{S} \subseteq L^{Q} \implies x y_1 y_2 \in H^{Q}$. Moreover $y_1 y_2 x \mathcal{S}\mathcal{S} \subseteq \mathcal{S}\mathcal{S}x\mathcal{S}\mathcal{S} \subseteq \mathcal{S}\mathcal{S}L^{Q} \subseteq L^{Q} \implies y_1 y_2 x \in H^{Q}$, since L^{Q} is a $\mathscr{P}LI$ of $\mathcal{S}$. Hence, H^{Q} is a two-sided $\mathscr{P}ID$ of $\mathcal{S}$. To prove H^{Q} is the largest two sided $\mathscr{P}ID$ of $\mathcal{S}$. Let I be any $\mathscr{P}ID$ of $\mathcal{S}$ and $I \subseteq Q$. Let $i \in I$. Consequently, $i \in Q$ and $\mathcal{S}\mathcal{S}i \subseteq I \subseteq Q$. Hence, $\mathcal{S}\mathcal{S}i \subseteq Q \implies i \in L^{Q}$. Hence, $I \subseteq L^{Q}$. Next, $i \in L^{Q}$ and $i\mathcal{S}\mathcal{S} \subseteq I \subseteq L^{Q} \implies i \in H^{Q}$. Hence, $I \subseteq H^{Q}$. □

Theorem 3.4. *Let Q be a $\mathscr{P}BI$ of $\mathcal{S}$. If Q is a $\mathscr{P}1P$ ($\mathscr{P}2P$) of $\mathcal{S}$, then H^{Q} is a $\mathscr{P}PID$ of $\mathcal{S}$.*

Proof. Let Q be P1P of $\mathcal{S}$. To prove that H^{Q} is a PPID of $\mathcal{S}$. Suppose that $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ be the PBIs of $\mathcal{S}$ such that $\mathscr{B}_1 \cdot \mathscr{B}_2 \cdot \mathscr{B}_3 \subseteq H^{Q}$. By Theorem 3.3, H^{Q} is the largest PID of $\mathcal{S}$ such that $H^{Q} \subseteq Q$. Thus $\mathscr{I}_1 \subseteq \mathscr{B}_1 \subseteq H^{Q}$ or $\mathscr{I}_2 \subseteq \mathscr{B}_2 \subseteq H^{Q}$ or $\mathscr{I}_3 \subseteq \mathscr{B}_3 \subseteq H^{Q}$ for the IDs $\mathscr{I}_1, \mathscr{I}_2$ and $\mathscr{I}_3$. □

In the following examples, we show that the converse of the Theorem 3.4 is not true.

Example 3.7. *By Example 3.1,* $Q = \{\eth_1, \eth_3, \eth_5\}$ *is a* $\mathscr{P}$*BI and* $H^Q = \{\eth_1, \eth_5\}$ *is a* $\mathscr{P}$*PID, but* Q *is not a* $\mathscr{P}$*1P of* $\mathcal{S}$*. For the* $\mathscr{P}$*BIs* $Q_1 = \{\eth_1, \eth_2, \eth_3\}$ *and* $Q_2 = \{\eth_1, \eth_3, \eth_6\}$ *and* $Q_3 = \{\eth_1, \eth_4, \eth_6\}$*. Now,* $Q_1 \cdot Q_2 \cdot Q_3 = \{\eth_1\} \subseteq Q$ *but* $Q_1 \nsubseteq Q$ *and* $Q_2 \nsubseteq Q$ *and* $Q_3 \nsubseteq Q$*.*

Example 3.8. *By Example 3.1,* $Q = \{\eth_1, \eth_4, \eth_7\}$ *is a* $\mathscr{P}$*BI and* $H^Q = \{\eth_1, \eth_7\}$ *is a* $\mathscr{P}$*PID, but* Q *is not a* $\mathscr{P}$*2P of* $\mathcal{S}$*. For* $\eth_2, \eth_6, \eth_8 \in \mathcal{S}$ *and* $\eth_2 \mathcal{S} \eth_6 \mathcal{S} \eth_8 = \{\eth_1, \eth_4\} \subseteq Q$ *but* $\eth_2 \notin Q, \eth_6 \notin Q$ *and* $\eth_8 \notin Q$*.*

Theorem 3.5. *The* $\mathscr{P}$*BI* Q *is a* $\mathscr{P}$*3P of* $\mathcal{S}$ *if and only if* H^Q *is a* $\mathscr{P}$*PID of* $\mathcal{S}$*.*

Proof. Let Q be a $\mathscr{P}$*BI* of $\mathcal{S}$ and Q be $\mathscr{P}$*3P* of $\mathcal{S}$. To prove that H^Q is a $\mathscr{P}$*PID* of $\mathcal{S}$. Suppose that $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ are the $\mathscr{P}$*IDs* of $\mathcal{S}$ such that $\mathscr{A}_1 \cdot \mathscr{A}_2 \cdot \mathscr{A}_3 \subseteq H^Q$. By Theorem 3.3, H^Q is the largest two sided $\mathscr{P}$*ID* of $\mathcal{S}$ such that $H^Q \subseteq Q$. Thus $\mathscr{A}_1 \subseteq H^Q$ or $\mathscr{A}_2 \subseteq H^Q$ or $\mathscr{A}_3 \subseteq H^Q$.

Conversely, suppose that H^Q is a $\mathscr{P}$*PID* of $\mathcal{S}$. To prove that Q is a $\mathscr{P}$*3P* of $\mathcal{S}$. For the $\mathscr{P}$*IDs* $\mathscr{I}_1, \mathscr{I}_2$ and $\mathscr{I}_3$ of $\mathcal{S}$ such that $\mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq Q$. To show that $\mathscr{I}_1 \subseteq Q$ or $\mathscr{I}_2 \subseteq Q$ or $\mathscr{I}_3 \subseteq Q$. Now, $\mathscr{I}_1 \cdot \mathscr{I}_2 \cdot \mathscr{I}_3 \subseteq H^Q$. This implies that $\mathscr{I}_1 \subseteq H^Q \subseteq Q$ or $\mathscr{I}_2 \subseteq H^Q \subseteq Q$ or $\mathscr{I}_3 \subseteq H^Q \subseteq Q$. Hence, Q is a $\mathscr{P}$*3P* of $\mathcal{S}$. □

Theorem 3.6. *Let* $\mathscr{M}$ *be a* $\mathscr{P}$*-*m_3*-system and* Q *be a* $\mathscr{P}$*BI of* $\mathcal{S}$ *with* $Q \cap \mathscr{M} = \emptyset$*. Then there exists a* $\mathscr{P}$*3P* P *of* $\mathcal{S}$ *containing* Q *with* $P \cap \mathscr{M} = \emptyset$*.*

Proof. Let $\mathscr{X} = \left\{\mathscr{J} | \mathscr{J} \text{ is a } \mathscr{P}BI \text{ with } Q \subseteq \mathscr{J} \text{ and } \mathscr{J} \cap \mathscr{M} = \emptyset\right\}$. Clearly, $\mathscr{X}$ is non-empty. According to Zorn's lemma, there exists a maximal element P in $\mathscr{X}$ and $P \cap \mathscr{M} = \emptyset$. To prove that P is a $\mathscr{P}$*3P* of $\mathcal{S}$. Using Theorem 3.5, we prove that H^P is a $\mathscr{P}$*PID* in $\mathcal{S}$. Since $H^P \subseteq P$ and $P \cap \mathscr{M} = \emptyset$ implies that $H^P \cap \mathscr{M} = \emptyset$.

Case-(i): Suppose that H^P is the largest $\mathscr{P}$*ID* in $\mathcal{S}$ such that $H^P \cap \mathscr{M} = \emptyset$. Suppose $< \varsigma' >< \varsigma'' >< \varsigma''' > \subseteq H^P$. Then $< \varsigma' > \subseteq H^P$ or $< \varsigma'' > \subseteq H^P$ or $< \varsigma''' > \subseteq H^P$. By proving at a contradiction approach, If $< \varsigma' > \nsubseteq H^P$, $< \varsigma'' > \nsubseteq H^P$ and $< \varsigma''' > \nsubseteq H^P$, then ${\varsigma_1}' \in < \varsigma' > \backslash H^P$, ${\varsigma_1}'' \in < \varsigma'' > \backslash H^P$ and ${\varsigma_1}''' \in < \varsigma''' > \backslash H^P$. Then $< {\varsigma_1}' > \subseteq < \varsigma' >$, $< {\varsigma_1}'' > \subseteq < \varsigma'' >$ and $< {\varsigma_1}''' > \subseteq < \varsigma''' >$. If $< \varsigma' >< \varsigma'' >< \varsigma''' > \subseteq H^P$ then $< {\varsigma_1}' >< {\varsigma_1}'' >< {\varsigma_1}''' > \subseteq < \varsigma' >< \varsigma'' >< \varsigma''' > \subseteq H^P$. By the maximal property of P, $(H^P + < {\varsigma_1}' >) \cap \mathscr{M} \neq \emptyset$ and $(H^P + < {\varsigma_1}'' >) \cap \mathscr{M} \neq \emptyset$ and $(H^P + < {\varsigma_1}''' >) \cap \mathscr{M} \neq \emptyset$. Thus, $(H^P + < {\varsigma_1}' >)(H^P + < {\varsigma_1}'' >)(H^P + < {\varsigma_1}''' >) \subseteq H^P$. Since $\mathscr{M}$ is a $\mathscr{P}$-m_3-system, for $\varsigma_1, \varsigma_2 \in \mathscr{M}$, then there exist $\varsigma_1 \in (H^P + < {\varsigma_1}' >) \cap \mathscr{M}$ and $\varsigma_2 \in (H^P + < {\varsigma_1}'' >) \cap \mathscr{M}$ and $\varsigma_3 \in (H^P + < {\varsigma_1}''' >) \cap \mathscr{M}$ such that ${\varsigma_1}'{\varsigma_2}'{\varsigma_3}' \in \mathscr{M}$, where $\varsigma_1' \in < \varsigma_1 >, \varsigma_2' \in < \varsigma_2 >$ and $\varsigma_3' \in < \varsigma_3 >$. If $\varsigma_1 \in (H^P + < {\varsigma_1}' >)$, then $\varsigma_1' = l' + \mho_1$ for some $l' \in H^P$ and $\mho_1 \in < {\varsigma_1}' >$ and if $\varsigma_2 \in (H^P + < {\varsigma_1}'' >)$, then $\varsigma_2' = l'' + \mho_2$ for some $l'' \in H^P$ and $\mho_2 \in < {\varsigma_1}'' >$. If $\varsigma_3 \in (H^P + < {\varsigma_1}''' >)$, then $\varsigma_3' = l''' + \mho_3$ for some $l''' \in H^P$ and $\mho_3 \in < {\varsigma_1}''' >$. Now, $\varsigma_1' \cdot \varsigma_2' \cdot \varsigma_3' \in (l' + \mho_1) \cdot (l'' + \mho_2) \cdot (l''' + \mho_3) = l'l''l''' + l'\mho_2 l''' + \mho_1 l'' l''' + \mho_1\mho_2 l''' + l'l''\mho_3 + l'\mho_2\mho_3 + \mho_1 l''\mho_3 + \mho_1\mho_2\mho_3 \in H^P + < \varsigma' > \cdot < \varsigma'' > \cdot < \varsigma''' >$. If $< \varsigma' > \cdot < \varsigma'' > \cdot < \varsigma''' > \subseteq H^P$, then $\varsigma_1' \cdot \varsigma_2' \cdot \varsigma_3' \in H^P$. Thus, $H^P \cap \mathscr{M} \neq \emptyset$, which is a contradiction. Hence, $< \varsigma' > \cdot < \varsigma'' > \cdot < \varsigma''' > \nsubseteq H^P$. Hence, H^P is a $\mathscr{P}$*PID* of $\mathcal{S}$. By Theorem 3.5, P is a $\mathscr{P}$*3P* of $\mathcal{S}$.

Case-(ii): If H^P is not a largest $\mathscr{P}$*ID* in $\mathcal{S}$, then there is a maximal ID P' in $\mathcal{S}$ such that $H^P \subseteq P'$ and $P' \cap \mathscr{M} = \emptyset$ and apply case-(i). Thus, $H^{P'}$ is a $\mathscr{P}$*PID*. Hence, P' is a $\mathscr{P}$*3P* of $\mathcal{S}$. □

4. Different $\mathscr{P}SPBIs$

In this section, we introduce three different $\mathscr{P}SPBIs$ of $\mathcal{S}$.

Definition 4.1. *(i) A $\mathscr{P}BI$ P of $\mathcal{S}$ is called a* **partial 1-semiprime** *($\mathscr{P}1SP$) if $\mathcal{Q}^3 \subseteq P$, implies $\mathcal{Q} \subseteq P$ for any $\mathscr{P}BI$ $\mathcal{Q}$ of $\mathcal{S}$.*
(ii) **partial 2-semiprime** *($\mathscr{P}2SP$) if $x'\mathcal{S}x'\mathcal{S}x' \subseteq P$ implies $x' \in P$.*
(iii) **partial 3-semiprime** *($\mathscr{P}3SP$) if $\mathscr{I}^3 \subseteq P$ implies $\mathscr{I} \subseteq P$, for any $\mathscr{P}ID$ $\mathscr{I}$ of $\mathcal{S}$.*

Theorem 4.1. *A $\mathscr{P}BI$ P of $\mathcal{S}$ is $\mathscr{P}2SP$ if and only if ${Z_1}^3 \subseteq P$ $({Z_2}^3 \subseteq P, {Z_3}^3 \subseteq P)$, with Z_1 is a $\mathscr{P}RI$ (Z_2 is a $\mathscr{P}LATI$ and Z_3 is a $\mathscr{P}LI$) of $\mathcal{S}$ implies $Z_1 \subseteq P$ $(Z_2 \subseteq P, Z_3 \subseteq P)$.*

Proof. Suppose that $Z_1^3 \subseteq P$. To prove that $Z_1 \subseteq P$. For $\varsigma' \in Z_1$, $\varsigma'\mathcal{S}\varsigma'\mathcal{S}\varsigma' \subseteq Z_1^3 \subseteq P$. Because P is a ($\mathscr{P}2SP$) of $\mathcal{S}$ implies that $\varsigma' \in P$. Thus, $Z_1 \subseteq P$.
Conversely, suppose that $\varsigma'\mathcal{S}\varsigma'\mathcal{S}\varsigma' \subseteq P$.
Now $(\varsigma'\mathcal{S}\mathscr{T})(\varsigma'\mathcal{S}\mathscr{T})(\varsigma'\mathcal{S}\mathscr{T}) \subseteq (\varsigma'\mathcal{S}\mathscr{T})\mathcal{S}(\varsigma'\mathcal{S}\mathscr{T})\mathcal{S}(\varsigma'\mathcal{S}\mathscr{T}) \subseteq \varsigma'\mathcal{S}\varsigma'\mathcal{S}\varsigma' \subseteq P$ implies $\varsigma'\mathcal{S}\mathscr{T} \subseteq P$. If $\varsigma'\mathcal{S}\mathscr{T} \subseteq P$, then

$$\begin{aligned} <\varsigma'>_r \cdot <\varsigma'>_{rt} \cdot <\varsigma'>_{rt} &= \left[\left\{\sum_n \varsigma' | n \in \mathbb{Z}^+\right\} + \varsigma'\mathcal{S}\mathscr{T}\right] \cdot \left[\left\{\sum_m \varsigma' | n \in \mathbb{Z}^+\right\} + \varsigma'\mathcal{S}\mathscr{T}\right] \\ &\quad \left[\left\{\sum_{m'} \varsigma' | n \in \mathbb{Z}^+\right\} + \varsigma'\mathcal{S}\mathscr{T}\right] \\ &\subseteq \left[\sum_{nmm'} \varsigma'\varsigma'\varsigma'\right] + \varsigma'\mathcal{S}\varsigma'\mathcal{S}\varsigma' \\ &\subseteq \varsigma'\mathcal{S}\mathscr{T} \subseteq P. \end{aligned}$$

Thus, $\varsigma' \in P$. □

The following implications hold for $\mathscr{P}1SP$ implies $\mathscr{P}2SP$ implies $\mathscr{P}3SP$. Some examples showing that the reverse implications may not be valid.

Example 4.1. *In Example 3.1, Clearly, $P = \{\eth_1, \eth_4, \eth_6\}$ is a $\mathscr{P}2SP$, but P is not a $\mathscr{P}1SP$. For $\mathscr{P}BI$ $\mathcal{Q} = \{\eth_1, \eth_3\}$ and $\mathcal{Q}^3 \subseteq P$ but $\mathcal{Q} \nsubseteq P$.*

Example 4.2. *By Example 3.4 and routine calculation, $P = \{\mho_1, \mho_5\}$ is a $\mathscr{P}3SP$ of $\mathcal{S}$. Now, $\mho_6\mathcal{S}\mho_6\mathcal{S}\mho_6 = \{\mho_1, \mho_5\} \subseteq P$ but $\mho_6 \notin P$ implies P is not a $\mathscr{P}2SP$ of $\mathcal{S}$.*

Definition 4.2. *(i) A subset $\mathscr{N}$ of $\mathcal{S}$ is represent a $\mathscr{P}$-n_1-system if for any $\varsigma_1 \in \mathscr{N}$, $\exists\, \varsigma', \varsigma'', \varsigma''' \in <\varsigma_1>_b$ such that $\varsigma' \cdot \varsigma'' \cdot \varsigma''' \in \mathscr{N}$.*
(ii) A subset $\mathscr{N}$ of $\mathcal{S}$ is represent a $\mathscr{P}$-n_2-system if for any $\varsigma_2 \in \mathscr{N}$, $\exists\, \varsigma', \varsigma'', \varsigma''' \in <\varsigma_2>_r$ or $\varsigma', \varsigma'', \varsigma''' \in <\varsigma_2>_{lat}$ or $\varsigma', \varsigma'', \varsigma''' \in <\varsigma_2>_l$ such that $\varsigma' \cdot \varsigma'' \cdot \varsigma''' \in \mathscr{N}$.
(iii) A subset $\mathscr{N}$ of $\mathcal{S}$ is represent a $\mathscr{P}$-n_3-system if for any $\varsigma_3 \in \mathscr{N}$, $\exists\, \varsigma', \varsigma'', \varsigma''' \in <\varsigma_3>$ such that $\varsigma' \cdot \varsigma'' \cdot \varsigma''' \in \mathscr{N}$.

Theorem 4.2. *Let P be the $\mathscr{P}BI$ of $\mathcal{S}$. Then, P is a $\mathscr{P}1SP$ ($\mathscr{P}2SP$, $\mathscr{P}3SP$) if and only if $\mathcal{S}\setminus P$ is a $\mathscr{P}$-$n_1-system$($\mathscr{P}$-$n_2-system$, $\mathscr{P}$-$n_3-system$).*

Proof. Let P be the partial 1-semiprime of $\mathcal{S}$ and let $\varsigma \in \mathcal{S}\setminus P$. Hence, $<\varsigma>_b \cdot <\varsigma>_b \cdot <\varsigma>_b \nsubseteq P$. Then there exist $\varsigma', \varsigma'', \varsigma''' \in <\varsigma>_b$such that $\varsigma' \cdot \varsigma'' \cdot \varsigma''' = \left\{\sum_{n_1}\varsigma + \sum_{n_2}\varsigma^3 + \mho_1\right\} \cdot \left\{\sum_{n_1'}\varsigma + \sum_{n_2'}\varsigma^3 + \mho_2\right\} \cdot \left\{\sum_{n_1''}\varsigma + \sum_{n_2''}\varsigma^3 + \mho_3\right\}$, where $\mho_1 = \sum_{(i,j)}\varsigma r_i \varsigma r_j \varsigma$ and $\mho_2 = \sum_{(i,j)}\varsigma r_i' \varsigma r_j' \varsigma$ and $\mho_3 = \sum_{(i,j)}\varsigma r_i'' \varsigma r_j'' \varsigma$ for $n_1, n_2, n_1', n_2', n_1'', n_2'' \in \mathbb{N}$ and $r_i, r_j, r_i', r_j', r_i'', r_j'' \in \mathcal{S}$. Now, $\mho_1 \cdot \mho_2 \cdot \mho_3 \in <\varsigma>_b \cdot <\varsigma>_b \cdot <\varsigma>_b \nsubseteq P$. Thus, $\varsigma' \cdot \varsigma'' \cdot \varsigma''' \notin P$. Hence, $\mathcal{S}\setminus P$ is a partial-n_1-system.

Conversely, Let $\mathcal{S}\setminus P$ be a partial-n_1-system. Suppose that $Q^3 \subseteq P$ for the partial bi-ideal Q of $\mathcal{S}$. Let us arrive at a contradiction. Let $\varsigma \in Q\setminus P$. Hence, $\varsigma \in \mathcal{S}\setminus P$ implies $<\varsigma>_b \cdot <\varsigma>_b \cdot <\varsigma>_b \nsubseteq P$, which is a contradiction. Thus, $Q \subseteq P$. Therefore, P is a partial 1-semiprime of $\mathcal{S}$. Similarly, we can prove the other cases. □

The following implications hold for $\mathscr{P}$-n_1-system implying that the $\mathscr{P}$-n_2-system implies $\mathscr{P}$-n_3-system. It is impossible to prove the reverse of the implications using the following example.

Example 4.3. *By Example 3.1,$\mathscr{N} = \{\eth_2, \eth_3, \eth_5, \eth_7, \eth_8, \eth_9\}$ is a $\mathscr{P}$-n_2- system, but not a $\mathscr{P}$-n_1-system. For $\eth_3 \in \mathscr{N}$, there is no $\varsigma_1, \varsigma_2, \varsigma_3 \in <\eth_3>_b$ such that $\varsigma_1 \cdot \varsigma_2 \cdot \varsigma_3 \in \mathscr{N}$. Since $<\eth_3>_b \cdot <\eth_3>_b \cdot <\eth_3>_b = \{\eth_1\} \notin \mathscr{N}$.*

Example 4.4. *By Example 3.4, $\mathscr{N} = \{\mho_2, \mho_3, \mho_4, \mho_6,\}$ is a $\mathscr{P}$-n_3-system, but not a $\mathscr{P}$-n_2-system of $\mathcal{S}$. For $\mho_6 \in \mathscr{N}$ and $\mho_6\mathcal{S}\mho_6\mathcal{S}\mho_6 = \mho_5 \notin \mathscr{N}$.*

Corollary 4.1. *If Q is a $\mathscr{P}1SP$ ($\mathscr{P}2SP$) of $\mathcal{S}$, then H^Q is a $\mathscr{P}SPID$ of $\mathcal{S}$.*

Proof. Let Q be ($\mathscr{P}1SP$) of $\mathcal{S}$. To prove that H^Q is a $\mathscr{P}SPID$ of $\mathcal{S}$. Suppose that $\mathscr{B}$ is the $\mathscr{P}SPBI$ of $\mathcal{S}$ such that $\mathscr{B}^3 \subseteq H^Q$. By Theorem 3.3, H^Q is the largest $\mathscr{P}PID$ of $\mathcal{S}$ such that $H^Q \subseteq Q$. Thus $\mathscr{I} \subseteq \mathscr{B} \subseteq H^Q$ for the ID $\mathscr{I}$. □

Using this example, we show that the converse of the above corollary is not true.

Example 4.5. *By Example 3.1 and routine computation, $H^Q = \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_7\}$,
$Q = \{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5, \eth_6, \eth_7\}$ and $Q_1 = \{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8\}$. Clearly, H^Q is a $\mathscr{P}SPID$, but Q is not a $\mathscr{P}1SP$ of $\mathcal{S}$ by ${Q_1}^3 \subseteq Q$ but $Q_1 \nsubseteq Q$.
By Example 3.1, taking $H^Q = \{\eth_1, \eth_4, \eth_6\}$ is a $\mathscr{P}SPID$ of $\mathcal{S}$. Let $Q = \{\eth_1, \eth_4, \eth_6, \eth_7\}$ be a $\mathscr{P}BI$ and $\eth_8\mathcal{S}\eth_8\mathcal{S}\eth_8 = \{\eth_1, \eth_4, \eth_6, \eth_7\} \subseteq Q$ but $\eth_8 \notin Q$. This implies that Q is not a $\mathscr{P}2SP$ of $\mathcal{S}$.*

Theorem 4.3. *The $\mathscr{P}BI$ Q is a $\mathscr{P}3SP$ of $\mathcal{S}$ if and only if H^Q is a $\mathscr{P}SPID$ of $\mathcal{S}$.*

Proof. Let Q be a $\mathscr{P}3SP$ of $\mathcal{S}$. To prove that H^Q is a $\mathscr{P}SPID$ of $\mathcal{S}$, suppose that $\mathscr{A}$ is the $\mathscr{P}SPID$ of $\mathcal{S}$ such that $\mathscr{A}^3 \subseteq H^Q$. According to Theorem 3.3, H^Q is the largest $\mathscr{P}SPID$ of $\mathcal{S}$ such that $H^Q \subseteq Q$. Thus, $\mathscr{A} \subseteq H^Q$.

Conversely, suppose that H^Q is a $\mathscr{P}SPID$ of $\mathcal{S}$. To prove that Q is a $\mathscr{P}3SPID$ of $\mathcal{S}$. For the PID

$\mathscr{I}$ of $\mathcal{S}$ such that $\mathscr{I}^3 \subseteq Q$. To show that $\mathscr{I} \subseteq Q$. Now, $\mathscr{I}^3 \subseteq H^Q$. This implies that $\mathscr{I} \subseteq H^Q \subseteq Q$. Hence, Q is a $\mathscr{P}3SPID$ of $\mathcal{S}$. □

5. Conclusion

In this article, we study $\mathscr{P}1P$, $\mathscr{P}2P$, $\mathscr{P}3P$, $\mathscr{P}1SP$, $\mathscr{P}2SP$ and $\mathscr{P}3SP$ as well as some characterization of $\mathscr{P}BI$. Some of their fundamental characteristics have been discussed and some have been described using $\mathscr{P}PBI$ and $\mathscr{P}SPBI$. In addition, we demonstrated how to construct generators of $\mathscr{P}LI$, $\mathscr{P}LATI$, $\mathscr{P}RI$, $\mathscr{P}ID$ and $\mathscr{P}BI$ like elements and subsets. In the future, we will use $\mathscr{P}PBI$ to characterize partial hyper semirings and partially ternary hyper semirings. There are also several other types of $\mathscr{P}PBI$ like the maximum and minimal $\mathscr{P}BI$.

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