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Operation on Generalized Semi Open Sets in Bitopological Spaces

F. H. Khedr[1], M. Azab. Abd-Allah[1], E. A. Abdelgaber[2,*]

[1]*Department of Mathematics, Faculty of Science, Assuit University, Assuit, Egypt*
[2]*Department of Mathematics, Faculty of Science, Minia University, Minia, Egypt*

[*]*Corresponding author:* eman1357977@yahoo.com

Abstract. The aim of this paper is to more contribute the study of operations on bitopological spaces. The concept of operation on generalized semi open sets in bitopological spaces is introduced and studied. Two closure operators related to this concept are introduced and some of their properties and the relation between them are discussed.

1. Introduction

The concept of an operation on a topological space was initiated by Kasahara [11]. More investigation of this concept is given by Asaad, Jankovic and Ogata [4–6, 9, 17]. Several researches developed many concepts of operation on different classes of sets in topological spaces. Operations on the classes of all preopen sets, generalized open sets, semi generalized open sets, $b-$open sets, generalized semi open sets, semi open sets and $\beta-$open sets were introduced and studied in [2], [3], [4], [5], [8], [10], [16] and [18].

Operations on bitopological spaces were discussed in some manner in [1, 12]. A different technique to study operations on bitopological spaces is found in [13]. In [15], the concept of operation on Pc open sets in bitopological spaces is given.

The aim of this paper is to study the concept of operations on generalized semi open sets in bitopological spaces. Firstly, we introduce the concept of a family $ij-GS\kappa O(X)$ of all $ij-gs\kappa-$open sets by using the operation κ on generalized semi open sets in a bitopological space X. Secondly, we introduce the concepts of $ij-\kappa gsCl(A)$ and $ij-gsCl_\kappa(A)$ of any subset A of a bitopological space X and discuss the relation between them. Thirdly, we introduce the concepts of $ij-\kappa gsInt(A)$ and $ij-gsInt_\kappa(A)$ of any subset A of a bitopological space X. Finally, we study the notion of $ij-gs\kappa-$limit point of any subset of a bitopological space and give some properties.

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Throughout this paper, by (X, τ_1, τ_2) or simply X we mean a bitopological space, $i, j = 1, 2, i \neq j$ and *id* denotes the identity map. Let A be a subset of a bitopological space (X, τ_1, τ_2), the closure and the interior of A in the topological space (X, τ_i) are denoted by $i - Cl(A)$ and $i - Int(A)$ respectively.

2. Preliminaries

In this section, we recall some concepts occurring in the papers [7, 13, 14] which will be needed in the sequel.

Definition 2.1 [7]
A subset A of a bitopological space X is called $ij-$semi open if there exists a τ_i-open set U such that $U \subseteq A \subseteq j - Cl(U)$ or equivalently $A \subseteq j - Cl(i - Int(A))$.

The complement of an $ij-$semi open set is called $ij-$semi closed. The family of all $ij-$semi open sets of X is denoted by $ij - SO(X)$. For $x \in X$, the family of all $ij-$semi open sets containing x is denoted by $ij - SO(X, x)$. The $ij-$semi closure of A, denoted by $ij - sCl(A)$, is the intersection of all $ij-$semi closed sets of X containing A .

Definition 2.2 [14]
A subset A of a bitopological space X is called $ij-$generalized semi closed (briefly $ij - gs-$closed) if $ji - sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-open in X.

The set of all $ij-$generalized semi open sets of X is denoted by $ij - GSO(X)$. The complement of an $ij-$generalized semi closed set is called $ij-$generalized semi open set(briefly $ij - gs-$open). If A is $12 - gs-$closed and $21 - gs-$closed, then it is said to be pairwise $gs-$closed. Every τ_j-closed set is $ij-$generalized semi closed .

Definition 2.3 [14]
Let X be a bitopological space and $A \subset X$. The $ij-$generalized semi-closure of A, denoted by $ij - gsCl(A)$, is the intersection of all $ij - gs-$closed sets of X containing A .

Lemma 2.1 [14]
Let X be a bitopological space, $x \in X$ and $A \subset X$. Then $x \in ij - gsCl(A)$ if and only if $U \cap A \neq \phi$, for every $ij - gs-$open set U containing x .

Definition 2.4 [14]
Let X be a bitopological space and $A \subset X$. The $ij-$generalized semi interior of A, denoted by $ij - gsInt(A)$, is the union of all $ij - gs-$open sets of X contained in A.

Definition 2.5 [14]
Let X be a bitopological space, $x \in X$ and $A \subset X$. Then x is called an $ij-$generalized semi-limit point (briefly $ij - gs-$ limit point) of A, if for every $ij - gs-$open set U containing x, $A \cap U \setminus \{x\} \neq \phi$.

The set of all $ij - gs-$limit points of A, denoted by $ij - gsd(A)$, is called the $ij-$generalized semi-derived set of A .

Definition 2.6 [13]
Let X be a bitopological space. An operation γ on $\tau_1 \cup \tau_2$ is a mapping $\gamma : \tau_1 \cup \tau_2 \longrightarrow P(X)$ such

that $V \subseteq V^{\gamma}$ for every $V \in \tau_1 \cup \tau_2$, where V^{γ} denotes the value of γ at V.

Definition 2.7 [13]

A subset A of a bitopological space X is said to be a γ_i-open set if for each $x \in A$, there exists a τ_i-open set U containing x such that $U^{\gamma} \subseteq A$.

The set of all γ_i-open sets of X is denoted by $\tau_{i\gamma}$. Clearly we have $\tau_{i\gamma} \subseteq \tau_i$. The complement of a γ_i-open set is called γ_i-closed.

3. Operation on generalized semi open sets in bitopological spaces

In this section, we introduce the concepts of generalized semi open sets and generalized semi operation closed sets in bitopological spaces and study some of its properties. We show that if κ is $ij - gs\kappa-$regular, then $ij - GS\kappa O(X)$ is a topology on X.

In the following, $P - GSO(X) = 12 - GSO(X) \cup 21 - GSO(X)$.

Definition 3.1

Let X be a bitopological space. An operation κ on $P - GSO(X)$ is a mapping $\kappa : P - GSO(X) \longrightarrow P(X)$ such that $V \subseteq V^{\kappa}$ for every $V \in P - GSO(X)$, where V^{κ} denotes the value of κ at V.

The operators $U^{\kappa} = U$ and $U^{\kappa} = j - Cl(U)$ for $U \in ij - GSO(X)$ are operations on $P - GSO(X)$.

It is known that $\tau_1 \cup \tau_2 \subseteq P - GSO(X)$. Then if we restrict the κ operation to $\tau_1 \cup \tau_2$, we obtain the γ operation in [13].

In the following, we give an example for an operation on generalized semi open sets:

Example 3.1

Let $X = \{a, b, c, d\}$, $\tau_1 = P(X)$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, d\}, \{a, b, d\}\}$, then $\kappa : P - GSO(X) \longrightarrow P(X)$ defined by $U^{\kappa} = 2 - Cl(U)$ for every $U \in P - GSO(X)$ is an operation on $P - GSO(X)$.

Definition 3.2

A subset A of a bitopological space X is said to be an $ij - gs\kappa-$open set if for every $x \in A$, there exists an $ij - gs-$open set U containing x such that $U^{\kappa} \subseteq A$.

If A is $12 - gs\kappa-$open and $21 - gs\kappa-$open, then it is called pairwise $gs\kappa-$open. The set of all $ij - gs\kappa-$open sets of X is denoted by $ij - GS\kappa O(X)$. The complement of an $ij - gs\kappa-$open set is $ij - gs\kappa-$closed. If B is $12 - gs\kappa-$closed and $21 - gs\kappa-$closed, then B is called pairwise $gs\kappa-$closed. In any bitopological space, X and ϕ are $ij - gs\kappa-$open sets.

If $U^{\kappa} = U$ for $U \in ij - GSO(X)$, then the concept of $ij - gs\kappa-$open set coincide with the concept of $ij - gs-$open set [14]. In case $U^{\kappa} = j - Cl(U)$ for $U \in ij - GSO(X)$, $ij - gs\kappa-$open set is called $ij - gs\theta-$open.

Remark 3.1

Intersection of any two $ij - gs\kappa-$open subsets of a bitopological space X need not be $ij - gs\kappa-$open as can be shown by the following example:-

Example 3.2

Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{b\}, \{c, d\}, \{b, c, d\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$ and let κ be an operation on $P-GSO(X)$ defined by $U^{\kappa}=2-Cl(A)$ for every $U \in P-GSO(X)$. Here $\{a,b\}$ and $\{a,c,d\}$ are $12-gs\kappa-$open sets but $\{a,b\} \cap \{a,c,d\}=\{a\}$ is not $12-gs\kappa-$open set.

If $U^{\kappa}=U$ for $U \in ij-GSO(X)$, then the intersection of two $ij-gs-$open subsets of a bitopological space X is $ij-gs-$open [14].

Remark 3.2

The families $ij-GSO(X)$ and $ij-GS\kappa O(X)$ are independent to each other as can be shown by the following example:-

Example 3.3

Let $X=\{a,b,c,d\}$, $\tau_1=\{X,\phi,\{b\},\{c,d\},\{b,c,d\}\}$, $\tau_2=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c,d\}\}$ and κ be an operation on $P-GSO(X)$ defined by $U^{\kappa}=2-Cl(A)$ for every $U \in P-GSO(X)$. Here $\{a\}$ is $21-gs-$open but not $21-gs\kappa-$open and $\{a,c,d\}$ is $21-gs\kappa-$open but not $21-gs-$open.

Theorem 3.1

Let X be a bitopological space and κ be an operation on $P-GSO(X)$. Then the following results are true:-

(1) Arbitrary union of $ij-gs\kappa-$open subsets of X is also $ij-gs\kappa-$open.

(2) Arbitrary intersection of $ij-gs\kappa-$closed subsets of X is also $ij-gs\kappa-$closed.

(3) Every γ_j-open subset of X is $ij-gs\kappa-$open.

Proof.

(1) Let $\{B_{\alpha} : \alpha \in \Delta\}$ be a collection of $ij-gs\kappa-$open subsets of X and $x \in B=\cup B_{\alpha}$, then $x \in B_{\alpha}$ for some α. Since B_{α} is an $ij-gs\kappa-$open set, then there exists an $ij-gs-$open set U containing x such that $U^{\kappa} \subseteq B_{\alpha}$. Since $B_{\alpha} \subseteq \cup B_{\alpha}=B$, then $U^{\kappa} \subseteq B$. Therefore, for every $x \in B$, there exists an $ij-gs-$open set U containing x such that $U^{\kappa} \subseteq B$ which implies that B is $ij-gs\kappa-$open. Thus $\cup B_{\alpha}$ is $ij-gs\kappa-$open.

(2) Follows from (1).

(3) Let V be a γ_j-open subset of X containing x, then there exists a τ_j-open set U containing x such that $U^{\gamma} \subseteq V$. Since every τ_j-open is $ij-gs-$open, then U is $ij-gs-$open. Since V is γ_j-open subset of X, then $\gamma=\kappa$ which implies that $U^{\gamma}=U^{\kappa}$. Therefore, $U^{\kappa} \subseteq V$ and U is $ij-gs-$open which proves that V is $ij-gs\kappa-$open.

Example 3.4

Let X and κ be defined as in Example 3.2. Let an operation $\gamma : \tau_1 \cup \tau_2 \longrightarrow P(X)$ defined by $U^{\gamma}=2-Cl(U)$, for every $U \in \tau_1 \cup \tau_2$. Here every $\tau_{2\gamma}$ is $12-gs\kappa-$open and every $\tau_{1\gamma}$ is $21-gs\kappa-$open.

In Example 3.4, if $U^{\gamma}=1-Cl(U)$, then γ is not the restriction of κ on $\tau_1 \cup \tau_2$. So, not every $\tau_{2\gamma}$ is $12-gs\kappa-$open because $\{a\} \in \tau_{2\gamma}$ but not $12-gs\kappa-$open.

Definition 3.3

Let X be a bitopological space and κ, μ be operations on $P-GSO(X)$. Then X is said to be an $ij-gs(\kappa,\mu)-$regular space if for every $x \in X$ and every $ij-gs-$open set U containing x, there exists an $ij-gs-$open set V containing x such that $V^{\kappa} \subseteq U^{\mu}$.

If X is $12-gs(\kappa,\mu)-$regular space and $21-gs(\kappa,\mu)-$regular space, then X is called pairwise $gs(\kappa,\mu)-$regular space.

If $U^{\mu}=U$ for $U\in ij-GSO(X)$, then $ij-gs(\kappa,\mu)-$regular space is called $ij-gs\kappa-$regular. If $U^{\kappa}=U^{\mu}=U$ for $U\in ij-GSO(X)$, then $ij-gs(\kappa,\mu)-$regular space is called $ij-gs-$regular. If $U^{\kappa}=U^{\mu}=j-Cl(U)$ for $U\in ij-GSO(X)$, then $ij-gs(\kappa,\mu)-$regular space is called $ij-gs\theta-$regular.

Definition 3.4

Let X be a bitopological space. An operation κ on $P-GSO(X)$ is said to be an $ij-gs\kappa-$regular operation if for every $x\in X$ and for every pair of $ij-gs-$open subsets A,B of X containing x, there exists an $ij-gs-$open set W containing x such that $W^{\kappa}\subseteq A^{\kappa}\cap B^{\kappa}$.

If κ is $12-gs\kappa-$regular operation and $21-gs\kappa-$regular operation, then κ is called pairwise $gs\kappa-$regular operation.

If $U^{\kappa}=U$ for $U\in ij-GSO(X)$, then $ij-gs\kappa-$regular operation is called $ij-gs-$regular operation. In case $U^{\kappa}=j-Cl(U)$ for $U\in ij-GSO(X)$, then $ij-gs\kappa-$regular operation is called $ij-gs\theta-$regular.

Definition 3.5

Let X be a bitopological space. An operation κ on $P-GSO(X)$ is said to be an $ij-gs\kappa-$open operation if for every $x\in X$ and for every $ij-gs-$open subset U of X containing x, there exists an $ij-gs\kappa-$open subset S of X containing x such that $S\subseteq U^{\kappa}$.

If κ is $12-gs\kappa-$open operation and $21-gs\kappa-$open operation, then κ is called pairwise $gs\kappa-$open.

If $U^{\kappa}=U$ for $U\in ij-GSO(X)$, then $ij-gs\kappa-$open operation is called $ij-gs-$open operation. In case $U^{\kappa}=j-Cl(U)$ for $U\in ij-GSO(X)$, then $ij-gs\kappa-$open operation is called $ij-gs\theta-$open.

Remark 3.3

There exists an operation κ on $P-GSO(X)$ which is pairwise $gs\kappa-$regular but not pairwise $gs\kappa-$open and there exists an operation κ which is pairwise $gs\kappa-$open but not pairwise $gs\kappa-$ regular. In Example 3.1, the operation κ is pairwise $gs\kappa-$regular but not pairwise $gs\kappa-$open because there exists $a\in X$ and $\{a\}$ is $12-gs-$open set containing a but there is no $12-gs\kappa-$open set S contain a such that $S\subseteq\{a\}^{\kappa}$. In Example 3.2, the operation κ is pairwise $gs\kappa-$open but not pairwise $gs\kappa-$regular because there exist $\{a\}$ and $\{a,b\}$ pair of $12-gs-$open sets containing a but there is no $12-gs-$open set U containing a and such that $U^{\kappa}\subseteq\{a\}^{\kappa}\cap\{a,b\}^{\kappa}$.

In the following proposition, we show that the set $ij-GS\kappa O(X)$ forms a topology on X when κ is $ij-gs\kappa-$regular.

Proposition 3.1

Let X be a bitopological space and κ be an $ij-gs\kappa-$regular operation on $P-GSO(X)$. Then:-

(1) If A and B are $ij-gs\kappa-$open subsets of X, then $A\cap B$ is $ij-gs\kappa-$open.

(2) $ij-GS\kappa O(X)$ is a topology on X.

Proof.

(1)Let A and B be $ij-gs\kappa-$open subsets of X and $x\in A\cap B$, then $x\in A$ and $x\in B$. Since A and B are $ij-gs\kappa-$open, then there exist two $ij-gs-$open sets U and V containing x such that $U^{\kappa}\subseteq A$ and

$V^\kappa \subseteq B$. Therefore, $U^\kappa \cap V^\kappa \subseteq A \cap B$. Since κ is $ij-gs\kappa-$regular, then there exists an $ij-gs-$open set W containing x such that $W^\kappa \subseteq U^\kappa \cap V^\kappa$. Therefore, $W^\kappa \subseteq A \cap B$ which proves that $A \cap B$ is $ij-gs\kappa-$open.

(2) Follows from (1), Theorem 3.1(1) and Definition 3.2.

Remark 3.4

The condition in Proposition 3.1 is necessary. In Example 3.2, the operation κ is not $12-gs\kappa-$regular and $12-GS\kappa O(X)$ is not a topology on X because $\{a,b\}$ and $\{a,c,d\}$ are $12-gs\kappa-$open sets but $\{a,b\} \cap \{a,c,d\} = \{a\}$ is not $12-gs\kappa-$open.

4. $ij-\kappa gsCl(A)$ and $ij-gsCl_\kappa(A)$

In this section, we introduce new two types of closure operators in bitopological spaces, namely, $ij-\kappa gsCl$ and $ij-gsCl_\kappa$. We study some of their properties and the relation between them.

Definition 4.1

Let X be a bitopological space, $x \in X$, $A \subset X$ and κ be an operation on $P-GSO(X)$. Then x is said to be an $ij-\kappa gs$ closure point of A if $U^\kappa \cap A \neq \phi$ for every $ij-gs-$open set U containing x. The set of all $ij-\kappa gs$ closure points of A, denoted by $ij-\kappa gsCl(A)$, is called the $ij-\kappa gs$ closure of A.

If $U^\kappa = U$ for $U \in ij-GSO(X)$, then the concept of $ij-\kappa gsCl(A)$ coincide with the concept of $ij-gsCl(A)$ [14]. In case $U^\kappa = j-Cl(U)$ for $U \in ij-GSO(X)$, the $ij-\kappa gsCl(A)$ is called $ij-\theta gs$ closure of A (briefly $ij-\theta gsCl(A)$).

Definition 4.2

Let X be a bitopological space, $A \subset X$ and κ be an operation on $P-GSO(X)$. Then we define $ij-gsCl_\kappa(A)$ as the intersection of all $ij-gs\kappa-$closed subsets of X containing A.

$ij-gsCl_\kappa(A) = \cap\{F \subseteq X : A \subseteq F, F^c \in ij-GS\kappa O(X)\}$

If $U^\kappa = U$ for $U \in ij-GSO(X)$, then the concept of $ij-gsCl_\kappa(A)$ coincide with the concept of $ij-gsCl(A)$ [14]. In case $U^\kappa = j-Cl(U)$ for $U \in ij-GSO(X)$, $ij-gsCl_\kappa(A)$ is called $ij-gsCl_\theta(A)$.

Theorem 4.1

Let A be a subset of a bitopological space X, $y \in X$ and κ be an operation on $P-GSO(X)$. Then $y \in ij-gsCl_\kappa(A)$ if and only if $V \cap A \neq \phi$ for every $V \in ij-GS\kappa O(X)$ such that $y \in V$.

Proof.

Let $F = \{y \in X : V \cap A \neq \phi$ for every $V \in ij-GS\kappa O(X)$ and $y \in V\}$. We need to prove that $F = ij-gsCl_\kappa(A)$. Let $x \notin F$, then there exists an $ij-gs\kappa-$open set V containing x such that $V \cap A = \phi$. Therefore, V^c is $ij-gs\kappa-$closed and $A \subseteq V^c$ which implies that $ij-gsCl_\kappa(A) \subseteq V^c$. Since $x \in V$, then $x \notin V^c$ which implies that $x \notin ij-gsCl_\kappa(A)$. Thus $ij-gsCl_\kappa(A) \subseteq F$. Conversely, let $x \notin ij-gsCl_\kappa(A) = \cap\{E : A \subseteq E, E^c \in ij-GS\kappa O(X)\}$, then there exists an $ij-gs\kappa-$closed set E such that $A \subseteq E$ but $x \notin E$, then $x \in E^c$ and E^c is $ij-gs\kappa-$open. Since $A \subseteq E$, then $E^c \cap A = \phi$. Hence we have E^c is an $ij-gs\kappa-$open set containing x such that $E^c \cap A = \phi$ which implies that $x \notin F$. Therefore, $F \subseteq ij-gsCl_\kappa(A)$. Thus $F = ij-gsCl_\kappa(A)$.

In the following, we give the relation between $ij-\kappa gsCl(A)$ and $ij-gsCl_\kappa(A)$ for any subset A of a bitopological space X. Also, we study some of its properties.

Proposition 4.1

Let A be a subset of a bitopological space X and κ be an operation on $P-GSO(X)$. Then $A \subseteq ij-gsCl(A) \subseteq ij-\kappa gsCl(A) \subseteq ij-gsCl_\kappa(A)$.

Proof.

By definition, we have $A \subseteq ij-gsCl(A)$. Let $x \in ij-gsCl(A)$, then $A \cap V \neq \phi$ for every $ij-gs-$open set V containing x. Since $V \subseteq V^\kappa$, then $A \cap V^\kappa \neq \phi$, for every $ij-gs-$open set V containing x. Therefore, $x \in ij-\kappa gsCl(A)$. Thus $ij-gsCl(A) \subseteq ij-\kappa gsCl(A)$. Let $x \notin ij-gsCl_\kappa(A)$, then there exists an $ij-gs\kappa-$open set U containing x such that $U \cap A = \phi$. Hence there exists an $ij-gs-$open set W containing x such that $W^\kappa \subseteq U$. Therefore, $W^\kappa \cap A = \phi$ which implies that $x \notin ij-\kappa gsCl(A)$. Then $ij-\kappa gsCl(A) \subseteq ij-gsCl_\kappa(A)$.

Thus $A \subseteq ij-gsCl(A) \subseteq ij-\kappa gsCl(A) \subseteq ij-gsCl_\kappa(A)$.

Remark 4.1

Generally $ij-gsCl_\kappa(A) \neq ij-\kappa gsCl(A)$. In Example 3.1, let $A = \{a\}$, then $12-\kappa gsCl(A) = \{a, c\}$ and $12-gsCl_\kappa(A) = \{a, b, c, d\}$.

Theorem 4.2

Let A, B be subsets of a bitopological space X and κ be an operation on $P-GSO(X)$. Then the following statements are true:-

(a) $ij-gsCl_\kappa(A)$ is an $ij-gs\kappa-$closed set and $A \subseteq ij-gsCl_\kappa(A)$.

(b) A is $ij-gs\kappa-$closed if and only if $A = ij-gsCl_\kappa(A)$.

(c) If $A \subseteq B$, then $ij-gsCl_\kappa(A) \subseteq ij-gsCl_\kappa(B)$.

(d) $ij-gsCl_\kappa(A) \cup ij-gsCl_\kappa(B) \subseteq ij-gsCl_\kappa(A \cup B)$.

(e) If κ is an $ij-gs\kappa-$regular operation, then $ij-gsCl_\kappa(A) \cup ij-gsCl_\kappa(B) = ij-gsCl_\kappa(A \cup B)$.

(f) $ij-gsCl_\kappa(A \cap B) \subseteq ij-gsCl_\kappa(A) \cap ij-gsCl_\kappa(B)$.

(g) $ij-gsCl_\kappa(ij-gsCl_\kappa(A)) = ij-gsCl_\kappa(A)$.

Proof.

(a) From Theorem 3.1 (2) and Definition 4.2.

(b) From Definition 4.2 and (a).

(c) Let $A \subseteq B$ and $x \notin ij-gsCl_\kappa(B)$, then there exists an $ij-gs\kappa-$open set V containing x such that $V \cap B = \phi$. Since $A \subseteq B$, then $V \cap A = \phi$. Therefore, $x \notin ij-gsCl_\kappa(A)$ which proves that $ij-gsCl_\kappa(A) \subseteq ij-gsCl_\kappa(B)$.

(d) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then from (c) we have $ij-gsCl_\kappa(A) \subseteq ij-gsCl_\kappa(A \cup B)$ and $ij-gsCl_\kappa(B) \subseteq ij-gsCl_\kappa(A \cup B)$. Therefore, $ij-gsCl_\kappa(A) \cup ij-gsCl_\kappa(B) \subseteq ij-gsCl_\kappa(A \cup B)$.

(e) Let κ be an $ij-gs\kappa-$regular operation and $x \notin ij-gsCl_\kappa(A) \cup ij-gsCl_\kappa(B)$, then $x \notin ij-gsCl_\kappa(A)$ and $x \notin ij-gsCl_\kappa(B)$. Therefore, there exist two $ij-gs\kappa-$open sets U and V containing x such that $U \cap A = \phi$ and $V \cap B = \phi$. Since κ is an $ij-gs\kappa-$regular operation, then $U \cap V$ is $ij-gs\kappa-$open. But $(U \cap V) \cap (A \cup B) = (U \cap V \cap A) \cup (U \cap V \cap B) = \phi$. Hence we have $(U \cap V) \cap (A \cup B) = \phi$

and $U \cap V$ is an $ij-gs\kappa-$open set containing x which implies $x \notin ij-gsCl_{\kappa}(A \cup B)$. Therefore, $ij-gsCl_{\kappa}(A \cup B) \subseteq ij-gsCl_{\kappa}(A) \cup ij-gsCl_{\kappa}(B)$. Thus, by (d) $ij-gsCl_{\kappa}(A \cup B) = ij-gsCl_{\kappa}(A) \cup ij-gsCl_{\kappa}(B)$.

(f) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by (c), $ij-gsCl_{\kappa}(A \cap B) \subseteq ij-gsCl_{\kappa}(A)$ and $ij-gsCl_{\kappa}(A \cap B) \subseteq ij-gsCl_{\kappa}(B)$. Thus $ij-gsCl_{\kappa}(A \cap B) \subseteq ij-gsCl_{\kappa}(A) \cap ij-gsCl_{\kappa}(B)$.

(g) From (a), $A \subseteq ij-gsCl_{\kappa}(A)$, then we have $ij-gsCl_{\kappa}(A) \subseteq ij-gsCl_{\kappa}(ij-gsCl_{\kappa}(A))$. Now, we need to prove that $ij-gsCl_{\kappa}(ij-gsCl_{\kappa}(A)) \subseteq ij-gsCl_{\kappa}(A)$. Let $x \in ij-gsCl_{\kappa}(ij-gsCl_{\kappa}(A))$, then $V \cap ij-gsCl_{\kappa}(A) \neq \phi$ for every $ij-gs\kappa-$open set V containing x. Therefore, there exists a point $y \in X$ such that $y \in V \cap ij-gsCl_{\kappa}(A)$ which implies that $y \in V$ and $y \in ij-gsCl_{\kappa}(A)$. Hence V is an $ij-gs\kappa-$open set containing y and $y \in ij-gsCl_{\kappa}(A)$, then $V \cap A \neq \phi$. Thus $V \cap A \neq \phi$ for every $ij-gs\kappa-$open set V containing x which proves that $x \in ij-gsCl_{\kappa}(A)$. Then $ij-gsCl_{\kappa}(ij-gsCl_{\kappa}(A)) \subseteq ij-gsCl_{\kappa}(A)$. Thus $ij-gsCl_{\kappa}(ij-gsCl_{\kappa}(A)) = ij-gsCl_{\kappa}(A)$.

Theorem 4.3

Let A, B be subsets of a bitopological space X and κ be an operation on $P-GSO(X)$. Then the following results are true:-

(a) $A \subseteq ij-\kappa gsCl(A)$.

(b) $ij-\kappa gsCl(\phi) = ij-gsCl_{\kappa}(\phi) = \phi$ and $ij-\kappa gsCl(X) = ij-gsCl_{\kappa}(X) = X$.

(c) If X is an $ij-gs\kappa-$regular space, then $ij-\kappa gsCl(A) = ij-gsCl(A)$.

(d) If $A \subseteq B$, then $ij-\kappa gsCl(A) \subseteq ij-\kappa gsCl(B)$.

(e) $ij-\kappa gsCl(A) \cup ij-\kappa gsCl(B) \subseteq ij-\kappa gsCl(A \cup B)$.

(f) If κ is an $ij-gs\kappa-$regular operation, then $ij-\kappa gsCl(A) \cup ij-\kappa gsCl(B) = ij-\kappa gsCl(A \cup B)$.

(g) $ij-\kappa gsCl(A \cap B) \subseteq ij-\kappa gsCl(A) \cap ij-\kappa gsCl(B)$.

(h) If κ is an $ij-gs\kappa-$open operation, then $ij-\kappa gsCl(A) = ij-gsCl_{\kappa}(A)$ and $ij-\kappa gsCl(ij-\kappa gsCl(A)) = ij-\kappa gsCl(A)$ hold.

Proof.

(a) Let $x \notin ij-\kappa gsCl(A)$, then there exists an $ij-gs-$open set U containing x such that $U^{\kappa} \cap A = \phi$. Since $U \subseteq U^{\kappa}$, then $U \cap A = \phi$. But $x \in U$, then $x \notin A$. Therefore, $A \subseteq ij-\kappa gsCl(A)$.

(b) From definitions and Proposition 4.1.

(c) Let X be an $ij-gs\kappa-$regular space. We have $ij-gsCl(A) \subseteq ij-\kappa gsCl(A)$ from Proposition 4.1. Let $x \notin ij-gsCl(A)$, then there exists an $ij-gs-$open set U containing x such that $U \cap A = \phi$. Since X is an $ij-gs\kappa-$regular space, then there exists an $ij-gs-$open set V containing x such that $V^{\kappa} \subseteq U$ which implies that $V^{\kappa} \cap A = \phi$. Therefore, $x \notin ij-\kappa gsCl(A)$ which proves that $ij-\kappa gsCl(A) \subseteq ij-gsCl(A)$. Thus $ij-\kappa gsCl(A) = ij-gsCl(A)$.

(d) Let $A \subseteq B$ and $x \notin ij-\kappa gsCl(B)$, then there exists an $ij-gs-$open set U containing x such that $U^{\kappa} \cap B = \phi$. Since $A \subseteq B$, then $U^{\kappa} \cap A = \phi$. Therefore, $x \notin ij-\kappa gsCl(A)$ which proves that $ij-\kappa gsCl(A) \subseteq ij-\kappa gsCl(B)$.

(e) We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then from (d), we have $ij-\kappa gsCl(A) \subseteq ij-\kappa gsCl(A \cup B)$ and $ij-\kappa gsCl(B) \subseteq ij-\kappa gsCl(A \cup B)$. Therefore, $ij-\kappa gsCl(A) \cup ij-\kappa gsCl(B) \subseteq ij-\kappa gsCl(A \cup B)$.

(f) Let κ be an $ij-gs\kappa-$regular operation and $x \notin ij-\kappa gsCl(A) \cup ij-\kappa gsCl(B)$, then $x \notin ij-\kappa gsCl(A)$ and $x \notin ij-\kappa gsCl(B)$. Therefore, there exist two $ij-gs-$open sets U and V containing x such that $U^\kappa \cap A = \phi$ and $V^\kappa \cap B = \phi$. Since κ is an $ij-gs\kappa-$regular operation, then there exists an $ij-gs-$open set W containing x such that $W^\kappa \subseteq U^\kappa \cap V^\kappa$. Now, $(U^\kappa \cap V^\kappa) \cap (A \cup B) = (U^\kappa \cap V^\kappa \cap A) \cup (U^\kappa \cap V^\kappa \cap B) = \phi$. Since $W^\kappa \subseteq U^\kappa \cap V^\kappa$, then $W^\kappa \cap (A \cup B) = \phi$. Therefore, $x \notin ij-\kappa gsCl(A \cup B)$ which implies that $ij-\kappa gsCl(A \cup B) \subseteq ij-\kappa gsCl(A) \cup ij-\kappa gsCl(B)$. Thus, by (e) $ij-\kappa gsCl(A \cup B) = ij-\kappa gsCl(A) \cup ij-\kappa gsCl(B)$.
(g) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then from (d) $ij-\kappa gsCl(A \cap B) \subseteq ij-\kappa gsCl(A)$ and $ij-\kappa gsCl(A \cap B) \subseteq ij-\kappa gsCl(B)$. Thus $ij-\kappa gsCl(A \cap B) \subseteq ij-\kappa gsCl(A) \cap ij-\kappa gsCl(B)$.
(h) Let κ be an $ij-gs\kappa-$open operation.

(i) We have $ij-\kappa gsCl(A) \subseteq ij-gsCl_\kappa(A)$ from Proposition 4.1. Now, Let $y \notin ij-\kappa gsCl(A)$, then there exists an $ij-gs-$open set U containing y such that $U^\kappa \cap A = \phi$. Since κ is an $ij-gs\kappa-$open operation, then there exists an $ij-gs\kappa-$ open set W containing y such that $W \subseteq U^\kappa$ which implies that $W \cap A = \phi$. Therefore, $y \notin ij-gsCl_\kappa(A)$ which proves that $ij-gsCl_\kappa(A) \subseteq ij-\kappa gsCl(A)$. Thus $ij-gsCl_\kappa(A) = ij-\kappa gsCl(A)$.

(ii) From Theorem 4.2 (g), we have $ij-gsCl_\kappa(ij-gsCl_\kappa(A)) = ij-gsCl_\kappa(A)$ and from $h(i)$, we have $ij-gsCl_\kappa(A) = ij-\kappa gsCl(A)$. Then $ij-\kappa gsCl(ij-\kappa gsCl(A)) = ij-\kappa gsCl(A)$.

In the following proposition, we show that the set $ij-\kappa gsCl(A)$ satisfies the Kuratowski closure axioms.

Proposition 4.2

Let X be a bitopological space. If κ is an $ij-gs\kappa-$regular and an $ij-gs\kappa-$open operation on $P-GSO(X)$, then $ij-\kappa gsCl(A)$ satisfies the Kuratowski closure axioms.

Proof. Follows from Theorem 4.3 (a), (b), (f) and (h).

Remark 4.2

Let X be a bitopological space. If κ is not an $ij-gs\kappa-$open operation, then $ij-\kappa gsCl(ij-\kappa gsCl(A)) \neq ij-\kappa gsCl(A)$. In Example 3.1, κ is not $12-gs\kappa-$open operation. If $A = \{a\}$, then $12-\kappa gsCl(A) = \{a, c\}$ and $12-\kappa gsCl(12-\kappa gsCl(A)) = \{a, b, c, d\}$ which shows that $12-\kappa gsCl(12-\kappa gsCl(A)) \neq 12-\kappa gsCl(A)$ in general.

Theorem 4.4

Let X be a bitopological space and κ be an $ij-gs\kappa-$regular operation on $P-GSO(X)$. Then $ij-gsCl_\kappa(A) \cap U \subseteq ij-gsCl_\kappa(A \cap U)$ for every $ij-gs\kappa-$open set U and every subset A of X.

Proof.

Let $x \in ij-gsCl_\kappa(A) \cap U$ and U be an $ij-gs\kappa-$open set, then $x \in ij-gsCl_\kappa(A)$ and $x \in U$. Let V be an $ij-gs\kappa-$open set containing x, then $U \cap V$ is $ij-gs\kappa-$open because κ is an $ij-gs\kappa-$regular operation. Hence we have $U \cap V$ is an $ij-gs\kappa-$open set containing x and $x \in ij-gsCl_\kappa(A)$ which implies that $A \cap (U \cap V) \neq \phi$. Therefore, $(A \cap U) \cap V \neq \phi$ and V is an $ij-gs\kappa-$open set containing x which proves that $x \in ij-gsCl_\kappa(A \cap U)$. Thus $ij-gsCl_\kappa(A) \cap U \subseteq ij-gsCl_\kappa(A \cap U)$ for every $ij-gs\kappa-$open set U and every subset A of X.

Theorem 4.5
Let A be any subset of a bitopological space X and κ be an operation on $P-GSO(X)$. Then the following statements are equivalent:-
(a) A is $ij-gs\kappa-$closed set.
(b) $ij-\kappa gsCl(A)=A$.
(c) $ij-gsCl_{\kappa}(A)=A$.
Proof.
$(a)\Rightarrow(b)$ Let A be an $ij-gs\kappa-$closed set. We have $A\subseteq ij-\kappa gsCl(A)$. Now, let $x\notin A$, then $x\in A^c$. Therefore, A^c is an $ij-gs\kappa-$open set containing x which implies that there exists an $ij-gs-$open set U containing x such that $U^{\kappa}\subseteq A^c$. Hence $U^{\kappa}\cap A=\phi$ which implies that $x\notin ij-\kappa gsCl(A)$. Hence $ij-\kappa gsCl(A)\subseteq A$. Thus $ij-\kappa gsCl(A)=A$.
$(b)\Rightarrow(c)$ Let $ij-\kappa gsCl(A)=A$. We have $A\subseteq ij-gsCl_{\kappa}(A)$. Now, let $x\notin A$, then $x\notin ij-\kappa gsCl(A)$. Hence there exists an $ij-gs-$open set U containing x such that $U^{\kappa}\cap A=\phi$. Therefore, $U^{\kappa}\subseteq A^c$. Since $x\notin A$, then $x\in A^c$ which proves that A^c is an $ij-gs\kappa-$ open set. Then A^c is an $ij-gs\kappa-$open set containing x such that $A^c\cap A=\phi$. Therefore, $x\notin ij-gsCl_{\kappa}(A)$ which implies that $ij-gsCl_{\kappa}(A)\subseteq A$. Thus $ij-gsCl_{\kappa}(A)=A$.
$(c)\Rightarrow(a)$ Let $ij-gsCl_{\kappa}(A)=A$ and $x\notin A$, then $x\notin ij-gsCl_{\kappa}(A)$. Therefore, there exists an $ij-gs\kappa-$open set U containing x such that $U\cap A=\phi$. Hence there exists an $ij-gs-$open set V containing x such that $V^{\kappa}\subseteq U$ which implies that $V^{\kappa}\cap A=\phi$. Therefore, $V^{\kappa}\subseteq A^c$. Since $x\notin A$, then $x\in A^c$. Therefore, A^c is $ij-gs\kappa-$open which proves that A is $ij-gs\kappa-$closed.

5. $ij-\kappa gsInt(A)$ AND $ij-gsInt_{\kappa}(A)$

In this section, we introduce the notions of $ij-\kappa gsInt(A)$ and $ij-gsInt_{\kappa}(A)$ for any subset A of a bitopological space X. We study some of their properties and discuss the relation between them.
Definition 5.1
Let X be a bitopological space, $x\in X$, $A\subset X$ and κ be an operation on $P-GSO(X)$. Then $x\in A$ is said to be an $ij-\kappa gs$ interior point of A if there exists an $ij-gs-$open set V containing x such that $V^{\kappa}\subseteq A$.

The set of all $ij-\kappa gs$ interior points of A, denoted by $ij-\kappa gsInt(A)$, is called the $ij-\kappa gs$ interior of A.

If $U^{\kappa}=U$ for $U\in ij-GSO(X)$, then the concept of $ij-\kappa gsInt(A)$ coincide with the concept of $ij-gsInt(A)$ [14]. In case $U^{\kappa}=j-Cl(U)$ for $U\in ij-GSO(X)$, $ij-\kappa gsInt(A)$ is called $ij-\theta gsInt(A)$.
Definition 5.2
Let X be a bitopological space and κ be an operation on $P-GSO(X)$. Then we define $ij-gsInt_{\kappa}(A)$ as the union of all $ij-gs\kappa-$open sets contained in A.

$ij-gsInt_{\kappa}(A)=\cup\{U:U\subseteq A,U\in ij-GS\kappa O(X)\}$

If $U^\kappa = U$ for $U \in ij-GSO(X)$, then the concept of $ij-gsInt_\kappa(A)$ coincide with the concept of $ij-gsInt(A)$ [14]. In case $U^\kappa = j-Cl(U)$ for $U \in ij-GSO(X)$, then the $ij-gsInt_\kappa(A)$ is called $ij-gsInt_\theta(A)$.

In the following lemma, we show the relation between $ij-gsCl_\kappa(A)$ and $ij-gsInt_\kappa(A)$.

Lemma 5.1

For any subset A of a bitopological space X, we have:

(i) $ij-gsCl_\kappa(A^c) = (ij-gsInt_\kappa(A))^c$,

(ii) $ij-gsInt_\kappa(A^c) = (ij-gsCl_\kappa(A))^c$.

Proof.

(i) Let $x \notin ij-gsCl_\kappa(A^c)$, then there exists an $ij-gs\kappa-$open set U containing x such that $U \cap A^c = \phi$. Hence $x \in U \subseteq A$ which implies that $x \in ij-gsInt_\kappa(A)$. Therefore, $x \notin (ij-gsInt_\kappa(A))^c$. Thus $(ij-gsInt_\kappa(A))^c \subseteq ij-gsCl_\kappa(A^c)$. Conversely, let $x \notin (ij-gsInt_\kappa(A))^c$, then $x \in ij-gsInt_\kappa(A)$. Hence, there exists an $ij-gs\kappa-$open set U containing x such that $U \subseteq A$ which implies that $U \cap A^c = \phi$. Therefore, $x \notin ij-gsCl_\kappa(A^c)$ which proves that $ij-gsCl_\kappa(A^c) \subseteq (ij-gsInt_\kappa(A))^c$. Thus $ij-gsCl_\kappa(A^c) = (ij-gsInt_\kappa(A))^c$.

(ii) Similar to the proof of (i).

Theorem 5.1

Let A, B be subsets of a bitopological space X and κ be an operation on $P-GSO(X)$. Then the following results are true:-

(a) $ij-gsInt_\kappa(A)$ is $ij-gs\kappa-$open and $ij-gsInt_\kappa(A) \subseteq A$.

(b) $ij-gsInt_\kappa(\phi) = \phi$ and $ij-gsInt_\kappa(X) = X$.

(c) A is $ij-gs\kappa-$open if and only if $ij-gsInt_\kappa(A) = A$.

(d) If $A \subseteq B$, then $ij-gsInt_\kappa(A) \subseteq ij-gsInt_\kappa(B)$.

(e) $ij-gsInt_\kappa(A) \cup ij-gsInt_\kappa(B) \subseteq ij-gsInt_\kappa(A \cup B)$.

(f) $ij-gsInt_\kappa(A \cap B) \subseteq ij-gsInt_\kappa(A) \cap ij-gsInt_\kappa(B)$.

(g) If κ is an $ij-gs\kappa-$regular operation, then $ij-gsInt_\kappa(A) \cap ij-gsInt_\kappa(B) = ij-gsInt_\kappa(A \cap B)$.

(h) $ij-gsInt_\kappa(ij-gsInt_\kappa(A)) = ij-gsInt_\kappa(A)$.

Proof.

(a) From Theorem 3.1 (1) and Definition 5.2.

(b) From Definition 3.2 and Definition 5.2.

(c) From Definition 5.2 and (a).

(d) Let $A \subseteq B$, then $ij-gsInt_\kappa(A) = \cup\{U : U \subseteq A, U \in ij-GS\kappa O(X)\} \subseteq \cup\{U : U \subseteq B, U \in ij-GS\kappa O(X)\} = ij-gsInt_\kappa(B)$.

(e) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then from (d) $ij-gsInt_\kappa(A) \subseteq ij-gsInt_\kappa(A \cup B)$ and $ij-gsInt_\kappa(B) \subseteq ij-gsInt_\kappa(A \cup B)$. Therefore, $ij-gsInt_\kappa(A) \cup ij-gsInt_\kappa(B) \subseteq ij-gsInt_\kappa(A \cup B)$.

(f) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then from (d) $ij-gsInt_\kappa(A \cap B) \subseteq ij-gsInt_\kappa(A)$ and $ij-gsInt_\kappa(A \cap B) \subseteq ij-gsInt_\kappa(B)$. Thus $ij-gsInt_\kappa(A \cap B) \subseteq ij-gsInt_\kappa(A) \cap ij-gsInt_\kappa(B)$.

(g) Since we have $ij-gsCl_\kappa(A^c) \cup ij-gsCl_\kappa(B^c) = ij-gsCl_\kappa(A^c \cup B^c)$ when κ is an $ij-gs\kappa-$regular operation from Theorem 4.2 (e), then $(ij - gsCl_\kappa(A^c) \cup ij - gsCl_\kappa(B^c))^c = (ij - gsCl_\kappa(A^c \cup B^c))^c$. Therefore, $(ij - gsCl_\kappa(A^c))^c \cap (ij - gsCl_\kappa(B^c))^c = ij - gsInt_\kappa(A^c \cup B^c)^c$. Thus $ij - gsInt_\kappa(A) \cap ij - gsInt_\kappa(B) = ij - gsInt_\kappa(A \cap B)$.

(h) $ij - gsInt_\kappa(ij - gsInt_\kappa(A)) = (ij - gsCl_\kappa(ij - gsInt_\kappa(A))^c)^c = (ij - gsCl_\kappa(ij - gsCl_\kappa(A^c)))^c = (ij - gsCl_\kappa(A^c))^c = ij - gsInt_\kappa(A^c)^c = ij - gsInt_\kappa(A)$.

Theorem 5.2

Let X be a bitopological space and κ be an $ij - gs\kappa-$regular operation on $P - GSO(X)$. Then $ij - gsInt_\kappa(B \cup F) \subseteq ij - gsInt_\kappa(B) \cup F$ for every $ij - gs\kappa-$closed set F and every subset B of X.

Proof.

Let F be $ij - gs\kappa-$ closed and A subset of X, then F^c is $ij - gs\kappa-$open. Hence from Theorem 4.4, we have $ij - gsCl_\kappa(A) \cap F^c \subseteq ij - gsCl_\kappa(A \cap F^c)$. Then $(ij - gsCl_\kappa(A \cap F^c))^c \subseteq (ij - gsCl_\kappa(A) \cap F^c)^c$. Therefore, $ij - gsInt_\kappa(A \cap F^c)^c \subseteq (ij - gsCl_\kappa(A))^c \cup F$. Then $ij - gsInt_\kappa(A^c \cup F) \subseteq ij - gsInt_\kappa(A^c) \cup F$. Put $A^c = B$ which implies that $ij - gsInt_\kappa(B \cup F) \subseteq ij - gsInt_\kappa(B) \cup F$, for every $ij - gs\kappa-$closed set F and every subset B of X.

6. $ij - gs\kappa d(A)$

In this section, we introduce the concept of $ij - gs\kappa d(A)$ for any subset A of a bitopological space X. We study some of its properties and give the relation between $ij - gs\kappa d(A)$ and $ij - gsCl_\kappa(A)$.

Definition 6.1

Let X be a bitopological space, $x \in X$ and $A \subset X$. Then x is called an $ij - gs\kappa-$ limit point of A if $A \cap U \setminus \{x\} \neq \phi$ for every $ij - gs\kappa-$open set U containing x.

The set of all $ij - gs\kappa-$ limit points of A, denoted by $ij - gs\kappa d(A)$, is called the $ij - gs\kappa$ derived set of A.

If $U^\kappa = U$, for $U \in ij - GSO(X)$, then the concept of $ij - gs\kappa d(A)$ coincide with the concept of $ij - gsd(A)$ [14]. In case $U^\kappa = j - Cl(U)$ for $U \in ij - GSO(X)$, then $ij - gs\kappa d(A)$ is called $ij - gs\theta$ derived set of A (briefly $ij - gs\theta d(A)$).

In the following, we show the relation between $ij - gs\kappa d(A)$ and $ij - gsCl_\kappa(A)$ for any subset A of a bitopological space X.

Lemma 6.1

Let A and B be subsets of a bitopological space X. Then the following statements are true:-

(1)If $A \subseteq B$, then $ij - gs\kappa d(A) \subseteq ij - gs\kappa d(B)$.

(2) $ij - gs\kappa d(A) \subseteq ij - gsCl_\kappa(A)$.

Proof.

(1) Let $A \subseteq B$ and $x \in ij - gs\kappa d(A)$, then $A \cap U \setminus \{x\} \neq \phi$ for every $ij - gs\kappa-$open set U containing x. Therefore, $B \cap U \setminus \{x\} \neq \phi$ for every $ij - gs\kappa-$open set U containing x. Thus $x \in ij - gs\kappa d(B)$ which proves that $ij - gs\kappa d(A) \subseteq ij - gs\kappa d(B)$.

(2) Let $x \notin ij - gsCl_\kappa(A)$, then there exists an $ij - gs\kappa-$open set U containing x such that $A \cap U = \phi$. Therefore, $A \cap U \setminus \{x\} = \phi$. Thus $x \notin ij - gs\kappa d(A)$ which proves that $ij - gs\kappa d(A) \subseteq ij - gsCl_\kappa(A)$.

Proposition 6.1
Let A be any subset of bitopological space X. Then $ij-gsCl_{\kappa}(A)=A\cup ij-gs\kappa d(A)$.
Proof.
From Lemma 6.1 (2), we have $ij-gs\kappa d(A)\subseteq ij-gsCl_{\kappa}(A)$ which implies that $A\cup ij-gs\kappa d(A)\subseteq A\cup ij-gsCl_{\kappa}(A)$. Since $A\subseteq ij-gsCl_{\kappa}(A)$, then $A\cup ij-gsCl_{\kappa}(A)=ij-gsCl_{\kappa}(A)$. Hence $A\cup ij-gs\kappa d(A)\subseteq ij-gsCl_{\kappa}(A)$. Conversely, let $x\notin A\cup ij-gs\kappa d(A)$, then $x\notin A$ and $x\notin ij-gs\kappa d(A)$. Therefore, there exists an $ij-gs\kappa-$open set U containing x such that $A\cap U\setminus\{x\}=\phi$. Since $x\notin A$, then $A\cap U=\phi$. Hence $x\notin ij-gsCl_{\kappa}(A)$. Therefore, $ij-gsCl_{\kappa}(A)\subseteq A\cup ij-gs\kappa d(A)$. Thus $ij-gsCl_{\kappa}(A)=A\cup ij-gs\kappa d(A)$.
Proposition 6.2
Let X be a bitopological space and κ be an $ij-gs\kappa-$regular operation on $P-GSO(X)$. Then $ij-gs\kappa d(A\cup B)=ij-gs\kappa d(A)\cup ij-gs\kappa d(B)$ for every subsets A and B of X.
Proof.
Since $A\subseteq A\cup B$ and $B\subseteq A\cup B$, then by Lemma 6.1 (1), we have $ij-gs\kappa d(A)\subseteq ij-gs\kappa d(A\cup B)$ and $ij-gs\kappa d(B)\subseteq ij-gs\kappa d(A\cup B)$. Hence $ij-gs\kappa d(A)\cup ij-gs\kappa d(B)\subseteq ij-gs\kappa d(A\cup B)$. Conversely, let $x\notin ij-gs\kappa d(A)\cup ij-gs\kappa d(B)$, then $x\notin ij-gs\kappa d(A)$ and $x\notin ij-gs\kappa d(B)$. Therefore, there exist two $ij-gs\kappa-$open sets U and V containing x such that $A\cap U\setminus\{x\}=\phi$ and $B\cap V\setminus\{x\}=\phi$. Since κ is an $ij-gs\kappa-$regular operation and U,V are $ij-gs\kappa-$open sets, then $U\cap V$ is $ij-gs\kappa-$open. Therefore, $(U\cap V)\setminus\{x\}\cap(A\cup B)=(U\setminus\{x\}\cap V\setminus\{x\})\cap(A\cup B)=\phi$ and $U\cap V$ is an $ij-gs\kappa-$open set containing x which implies that $x\notin ij-gs\kappa d(A\cup B)$. Then $ij-gs\kappa d(A\cup B)\subseteq ij-gs\kappa d(A)\cup ij-gs\kappa d(B)$. Thus $ij-gs\kappa d(A\cup B)=ij-gs\kappa d(A)\cup ij-gs\kappa d(B)$.

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References

[1] M.E. Abd El-Monsef, A.M. Kozae, B.M. Taher, Compactification in Bitopological Spaces, Arab J. Sci. Eng. 22 (1997), 99–105.

[2] N. Ahmad, B.A. Asaad, More Properties of an Operation on Semi-Generalized Open Sets, Italian J. Pure Appl. Math. 39 (2018), 608–627

[3] T.V. An, D.X. Cuong, H. Maki, On Operation-Preopen Sets in Topological Spaces, Sci. Math. Japon. e-2008 (2008), 241–260.

[4] B.A. Asaad, Some Applications of Generalized Open Sets via Operations, New Trends Math. Sci. 1 (2017), 145–157. https://doi.org/10.20852/ntmsci.2017.132.

[5] B.A. Asaad, N. Ahmad, Operation on Semi Generalized Open Sets With Its Separation Axioms, Int. J. Pure Appl. Math. 118 (2018), 701–711.

[6] B.A. Asaad, Z.A. Ameen, Some Properties of an Operation on $g\alpha$-Open Sets, New Trends Math. Sci. 2 (2019), 150–158. https://doi.org/10.20852/ntmsci.2019.353.

[7] S. Bose, Semi Open Sets, Semi Continuity and Semi Open Mappings in Bitopological Spaces, Bull. Calcutta Math. Soc. 73 (1981), 237–246.

[8] E. Rosas, C. Carpintero, N. Rajesh, Operation Approaches on b-Open Sets and Applications, Bol. Soc. Paran. Mat. 30 (2012), 21–33. https://doi.org/10.5269/bspm.v30i1.13013.

[9] D.S. Jankovic, On Functions with α-Closed Graphs, Glas. Mat. 18 (1983), 141–148.

[10] R. Jayashree, Operation Approach of gs-Open Sets in Topological Spaces, Thesis, Avinashilingam Deemed University for Women, 2019. http://hdl.handle.net/10603/302913.

[11] S. Kasahara, Operation-Compact Spaces, Math. Japon. 24 (1979), 97–105. https://cir.nii.ac.jp/crid/1572261549591870720.

[12] F.H. Khedr, Operation on Bitopologies, Delta J. Sci. 8 (1984), 309–320.

[13] F.H. Khedr, K.M. Abdelhakiem, Operations on Bitopological Spaces, Fasc. Math. 45 (2010), 47–57.

[14] F. Khedr, H. Al-Saadi, On Pairwise Semi-Generalized Closed Sets, J. King Abdulaziz Univ.: Sci. 21 (2009), 269–295. https://doi.org/10.4197/sci.21-2.7.

[15] F.H. Khedr, O.R. Sayed, Operation Pc-Open Sets and Operation Pc-Separation Axioms in Bitopological Spaces, Tatra Mount. Math. Publ. 85 (2023), 101–120. https://doi.org/10.2478/tmmp-2023-0027.

[16] G.S.S. Krishnan, Operation Approaches on Semi-Open Sets and Applications, Kochi J. Math. 2 (2007), 21–33.

[17] H. Ogata, Operations on Topological Spaces and Associated Topology, Math. Japon. 36 (1991), 175–184.

[18] S. Tahiliani, Operation Approach to β-Open Sets and Applications, Math. Comm. 16 (2011), 577–591.