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Wardowski Contraction on Controlled S-Metric Type Spaces with Fixed Point Results

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Abstract. This article presents the concept of a triple controlled S-metric type space, characterized by three control functions: β, μ, and γ. This extends the idea of controlled S-metric type spaces. We explore several properties and provide illustrative examples. Furthermore, we introduce α_s-admissible mappings and enhance Wardowski's contraction principle by introducing (α_s-$\mathcal{F}$)-contractive mappings specifically designed for triple controlled S-metric type spaces. The article establishes the existence and uniqueness of fixed points within a complete triple controlled S-metric type space. Finally, we apply our main theorem to demonstrate the determination of a unique solution for an mth degree polynomial.

1. Introduction

Banach began exploring metric fixed point theory in 1922 [1]. The Banach contraction principle, a significant topic in this area, attracts considerable interest from authors and researchers due to its wide-ranging applications in mathematics and various scientific disciplines, including computer science, engineering, physics, and economics. The theory of fixed points integrates concepts from topology, analysis, and geometry to explore both the existence and uniqueness of fixed points of function. The Banach Fixed Point Theorem has been expanded in various directions, especially in the advancement of different classes of metric spaces. For instance, Bakhtin [2] introduced b-metric spaces, which subsequently developed into extended forms known as extended b-metric spaces [3]. Additionally, researchers have explored controlled metric type spaces [4], as well as double and triple controlled metric type spaces [5], [6], [7], [8], [9]. Sedghi et al. [10] introduced the concept of S-metric space as a generalization of metric spaces. This concept was further expanded into S_b-metric spaces [11], extended S_b-metric spaces [12], extended S-metric spaces of type (α, β) [13], and controlled S-metric-like spaces [14].

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Researchers have delved into fixed-point theory within diverse metric type spaces, investigating various contraction conditions. For example, Darius Wardowski extended the Banach contraction principle and pioneered $\mathcal{F}$-contraction in 2012 [15], sparking numerous subsequent studies on $\mathcal{F}$-contraction [16], [17], [18], [19].

Furthermore, Samet et al. [20] introduced α-admissible mappings in metric spaces. Priyobarta et al. [21] extended this concept to α_s-admissible mappings within S-metric spaces. In 2016, Gopal et al. [22] introduced (α-$\mathcal{F}$)-contractive mappings, which researchers have utilized to explore fixed point theorems across various complete metric spaces [22], [23], [24], [25].

This article introduces the concept of triple controlled S-metric type spaces, extending the framework of controlled S-metric type spaces. We investigate various properties of this novel space and provide concrete examples to illustrate its characteristics. Additionally, we define α_s-admissible mappings and enhance Wardowski's contraction principle by introducing (α_s-$\mathcal{F}$)-contractive mappings tailored specifically for triple controlled S-metric type spaces. We establish the existence and uniqueness of fixed points within a complete triple controlled S-metric type space. Finally, we demonstrate the practical application of our results by finding a unique real solution to an mth degree polynomial.

2. Preliminaries

We recall some primary results and definitions of S-metric spaces which was proposed by Sedghi et al. [10].

Definition 2.1. *[10] Let $\mathbb{X} \neq \emptyset$, and let $\mathcal{S} : \mathbb{X}^3 \to [0, \infty)$ be a mapping such that for all $\hbar, \wp, \xi \in \mathbb{X}$ and $a \in \mathbb{X}$, it satisfies the following:*

(1) $\mathcal{S}(\hbar, \wp, \xi) = 0$ *iff* $\hbar = \wp = \xi$;
(2) $\mathcal{S}(\hbar, \wp, \xi) \leq \mathcal{S}(\hbar, \hbar, a) + \mathcal{S}(\wp, \wp, a) + \mathcal{S}(\xi, \xi, a)$.

The pair $(\mathbb{X}, \mathcal{S})$ is called S- metric space.

Definition 2.2. *[11] Let $\mathbb{X} \neq \emptyset$ and $b \geq 1$. Let $\mathcal{S} : \mathbb{X}^3 \to [0, \infty)$ be a mapping such that for all $\hbar, \wp, \xi \in \mathbb{X}$, it satisfies the following:*

(1) $\mathcal{S}(\hbar, \wp, \xi) = 0$ *iff* $\hbar = \wp = \xi$;
(2) $\mathcal{S}(\hbar, \wp, \wp) = \mathcal{S}(\wp, \hbar, \hbar)$
(3) $\mathcal{S}(\hbar, \wp, \xi) \leq b[\mathcal{S}(\hbar, \hbar, a) + \mathcal{S}(\wp, \wp, a) + \mathcal{S}(\xi, \xi, a)]$.

Then, the pair $(\mathbb{X}, \mathcal{S})$ is called S_b- metric space.

Definition 2.3. *[12] Let $\mathcal{S} : \mathbb{X}^3 \to [0, \infty)$ be a mapping, where $\mathbb{X} \neq \emptyset$ and consider a function $\theta : \mathbb{X}^3 \to [1, \infty)$, such that for all $\hbar, \wp, \xi \in \mathbb{X}$, it satisfies the following:*

(1) $\mathcal{S}(\hbar, \wp, \xi) = 0$ *iff* $\hbar = \wp = \xi$;
(2) $\mathcal{S}(\hbar, \wp, \xi) \leq \theta(\hbar, \wp, \xi)[\mathcal{S}(\hbar, \hbar, a) + \mathcal{S}(\wp, \wp, a) + \mathcal{S}(\xi, \xi, a)]$.

The pair $(\mathbb{X}, \mathcal{S})$ is known as extended S_b- metric space.

Qaralleh et al. [13] extended the concept of extended S_b-metric spaces to include extended S-metric spaces of type (α, β).

Definition 2.4. *[13] Consider a mapping $\mathcal{S} : \mathbb{X}^3 \to [0,\infty)$, where $\mathbb{X}$ be a nonempty set and suppose $\alpha, \beta : \mathbb{X}^3 \to [1,\infty)$ are functions, such that for all $\hbar, \wp, \xi \in \mathbb{X}$, it satisfies the following:*

(1) $\mathcal{S}(\hbar, \wp, \xi) = 0$ *iff* $\hbar = \wp = \xi$;

(2) $\mathcal{S}(\hbar, \wp, \xi) \leq \alpha(\hbar, \wp, \xi)\mathcal{S}(\hbar, \hbar, a) + \beta(\hbar, \wp, \xi)\mathcal{S}(\wp, \wp, a) + \mathcal{S}(\xi, \xi, a)$.

Then, the pair ($\mathbb{X}$, $\mathcal{S}$) is referred to as extended S-metric space of type (α, β).

On the other hand, in 2023, Ekiz et al. [14] introduced the notion of controlled S-metric type spaces.

Definition 2.5. *[14] Let $\mathcal{S} : \mathbb{X}^3 \to [0,\infty)$ be a mapping, where $\mathbb{X}$ be a nonempty set and suppose $\alpha : \mathbb{X}^2 \to [1,\infty)$ is a function so that for all $\hbar, \wp, \xi, a \in \mathbb{X}$, it satisfies the following:*

(1) $\mathcal{S}(\hbar, \wp, \xi) = 0$ *iff* $\hbar = \wp = \xi$;

(2) $\mathcal{S}(\hbar, \wp, \xi) \leq \alpha(\hbar, a)\mathcal{S}(\hbar, \hbar, a) + \alpha(\wp, a)\mathcal{S}(\wp, \wp, a) + \alpha(\xi, a)\mathcal{S}(\xi, \xi, a)$.

The pair ($\mathbb{X}$, $\mathcal{S}$) is referred to as a controlled S-metric type space.

Inspired by the idea of controlled S-metric type spaces, we introduce a novel concept: triple controlled S-metric type spaces, defined as follows:

Definition 2.6. *Consider a mapping $\mathcal{S} : \mathbb{X}^3 \to [0,\infty)$ with $\mathbb{X}$ being a nonempty set, and suppose $\beta, \mu, \gamma : \mathbb{X}^2 \to [1,\infty)$ are mappings such that for all $\hbar, \wp, \xi, a \in \mathbb{X}$, it satisfies these conditions:*

$(T1)$ $\mathcal{S}(\hbar, \wp, \xi) = 0$ *iff* $\hbar = \wp = \xi$;

$(T2)$ $\mathcal{S}(\hbar, \hbar, \xi) = \mathcal{S}(\xi, \xi, \hbar)$;, *for all* $\hbar, \xi \in \mathbb{X}$

$(T3)$ $\mathcal{S}(\hbar, \wp, \xi) \leq \beta(\hbar, a)\mathcal{S}(\hbar, \hbar, a) + \mu(\wp, a)\mathcal{S}(\wp, \wp, a) + \gamma(\xi, a)\mathcal{S}(\xi, \xi, a)$.

The pair ($\mathbb{X}$, $\mathcal{S}$) is referred to as a triple controlled S-metric type space, abbreviated as $\mathcal{TC}$-S-$\mathcal{MTS}$.

Remark 2.1. *By taking $\beta = \mu = \gamma$ in Definition 2.6, it becomes controlled S-metric type space as in Definition 2.5. Hence, our definition of $\mathcal{TC}$-S-$\mathcal{MTS}$ is a generalization of controlled S-metric type space. Moreover, by taking $\beta = \mu = \gamma = 1$, our definition of $\mathcal{TC}$-S-$\mathcal{MTS}$ becomes S-metric space as in Definition 2.1.*

Our next example is inspired by example 1 in [13].

Example 2.1. *Let $\mathbb{X} = \{0, 1, 2\}$, and define $\mathcal{S} : \mathbb{X}^3 \to [0,\infty)$ by*

$$\mathcal{S}(\hbar, \wp, \xi) = \begin{cases} 0 & \text{if } \hbar = \wp = \xi, \\ 1 & \text{if } \hbar \neq \wp \neq \xi, \\ \frac{3}{2} & \text{if } \hbar = \wp, \wp \neq \xi. \end{cases}$$

Let $\beta, \mu, \gamma : \mathbb{X}^2 \to [1,\infty)$ be defined as,

$\beta(\hbar, \wp) = 1 + \hbar + \wp$,

$\mu(\hbar, \wp) = 1 + \hbar\wp$, *and*
$\gamma(\hbar, \wp) = 2 + \hbar + \wp$.

One can easly see that $(\mathbb{X}, \mathcal{S})$ *is a* $\mathcal{TC}$-S-$\mathcal{MTS}$*, since* $\beta \neq \mu \neq \gamma$*, this indicates that* $(\mathbb{X}, \mathcal{S})$ *is not a controlled S-metric type space.*

Next, we discuss the concepts of Cauchy and convergent sequences, completeness, and the concept of the open ball in $\mathcal{TC}$-S-$\mathcal{MTS}$.

Definition 2.7. *Let* $(\mathbb{X}, \mathcal{S})$ *be a* $\mathcal{TC}$-S-$\mathcal{MTS}$ *and let* $\{\hbar_n\}$ *be any sequence in* $\mathbb{X}$.
(1) For $\hbar \in \mathbb{X}$ *with* $\varepsilon > 0$. *Then*

$$B(\hbar, \varepsilon) = \{w \in \mathbb{X}, \mathcal{S}(w, w, \hbar) < \varepsilon\}, \text{ denotes the open ball.}$$

(2) $\{\hbar_n\}$ *converges to a point* w *in* $\mathbb{X}$*, if for every* $\varepsilon > 0$*, you can find an* $N \in \mathbb{N}$*, so* $\mathcal{S}(\hbar_n, \hbar_n, w) < \varepsilon$ *for all* $n \geq N$.
(3) $\{\hbar_n\}$ *is referred to as a Cauchy sequence if for every* $\varepsilon > 0$*, you can find* $N \in \mathbb{N}$ *so that* $\mathcal{S}(\hbar_n, \hbar_n, \hbar_m) < \varepsilon$ *for all* $m, n \geq N$.
(4) The space $(\mathbb{X}, \mathcal{S})$ *is called complete if every Cauchy sequence in* $\mathbb{X}$ *is convergent.*

Lemma 2.1. *Let* $(\mathbb{X}, \mathcal{S})$ *be a* $\mathcal{TC}$-S-$\mathcal{MTS}$ *and let* $\beta, \mu, \gamma : \mathbb{X}^2 \to [1, \infty)$ *be mappings. If the sequence* $\{\hbar_n\}$ *in* $\mathbb{X}$ *is convergent, then the limit is unique.*

Proof. Let $\{\hbar_n\}$ in $\mathbb{X}$ be a convergent sequence and suppose the sequence converges to both x and y with $x \neq y$, i.e., $\lim_{n\to\infty} \mathcal{S}(\hbar_n, \hbar_n, x) = \lim_{n\to\infty} \mathcal{S}(\hbar_n, \hbar_n, y) = 0$. Further, assume the limits; $\lim_{n\to\infty} \beta(x, \hbar_n), \lim_{n\to\infty} \mu(x, \hbar_n)$, and $\lim_{n\to\infty} \gamma(y, \hbar_n)$ exists and finite, then:

$$\mathcal{S}(x, x, y) \leq \beta(x, \hbar_n)\mathcal{S}(x, x, \hbar_n) + \mu(x, \hbar_n)\mathcal{S}(x, x, \hbar_n) + \gamma(y, \hbar_n)\mathcal{S}(y, y, \hbar_n). \tag{2.1}$$

Taking the limit as n tends to ∞ in equation 2.1, we conclude that $\mathcal{S}(x, x, y) = 0$, i.e., $x = y$, a contradiction; thus implying the uniqueness of the limit. □

Samet et al. [20] initially presented the class of α-admissible mappings. For more details, refer to [20], [26].

Definition 2.8. *Let* $T : \mathbb{X} \longrightarrow \mathbb{X}$ *be a mapping with* $\mathbb{X}$ *a nonempty set, and let* $\alpha : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ *be a function. T is called* α*-admissible, if whenever* $\alpha(\hat{x}, \hat{y}) \geq 1$ *implies* $\alpha(T\hat{x}, T\hat{y}) \geq 1$*, for all* $\hat{x}, \hat{y} \in \mathbb{X}$.

Example 2.2. *[20] Consider* $\mathbb{X} = [0, \infty)$*, and let* $T : \mathbb{X} \longrightarrow \mathbb{X}$*, and* $\alpha : \mathbb{X}^2 \to [0, \infty)$ *be given by* $T(x) = \sqrt{x}$*, for all* $x \in \mathbb{X}$. *Define* $\alpha(x, y) = e^{x-y}$ *for* $x \geq y$*, otherwise* $\alpha(x, y) = 0$. *Then, clearly the mapping T is* α*-admissible.*

Priyobarta et al. [21] extended the class of α-admissible mappings in the context of S-metric space as shown below:

Definition 2.9. *[21] Consider the mapping $T : \mathbb{X} \longrightarrow \mathbb{X}$ and let $\alpha_s : \mathbb{X}^3 \to [0, \infty)$ be a function, with $\mathbb{X}$ a nonempty set. T is referred to as α_s-admissible mapping, if for all $\hat{x}, \hat{y}, \hat{z} \in \mathbb{X}$, we have*

$$\alpha_s(\hat{x}, \hat{y}, \hat{z}) \geq 1 \Longrightarrow \alpha_s(T\hat{x}, T\hat{y}, T\hat{z}) \geq 1. \tag{2.2}$$

Example 2.3. *[21] Assume $\mathbb{X} = [0, \infty)$, the mappings $T : \mathbb{X} \longrightarrow \mathbb{X}$, and $\alpha_s : \mathbb{X}^3 \to [0, \infty)$ are defined by $T(u) = 4u$, for all $u \in \mathbb{X}$, and $\alpha_s(x, y, z) = e^{\frac{z}{xy}}$ if $x \geq y \geq z$, $x \neq 0, y \neq 0$, and $\alpha_s(x, y, z) = 0$, if $x < y < z$. Then T is α_s-admissible.*

Darius Wardowski [15] introduced the concept of $\mathcal{F}$–contraction and developed new fixed point theorems for complete metric spaces. Below, we provide the definition.

Definition 2.10. *[15] Assume $\mathcal{F}$ denotes the family of all functions $F : (0, \infty) \to (-\infty, \infty)$ satisfying the following:*

(W1) F is a strictly increasing function.

(W2) Let $\{t_n\}$ be any sequence of positive real numbers, then this holds;

$$\lim_{n\to\infty} t_n = 0 \text{ iff } \lim_{n\to\infty} F(t_n) = -\infty.$$

(W3) $\lim_{t\to 0^+} t^k F(t) = 0$, for some $k \in (0, 1)$.

Example 2.4. *Consider the functions $G(s) = ln(s)$, and $M(s) = \frac{-1}{\sqrt{s}}$, for $s > 0$. Then both functions $G(s)$, and $M(s)$, satisfies the conditions $(W1), (W2)$, and $(W3)$. Therefore, $G(s)$, and $M(s)$ belong to $\mathcal{F}$, for further details consult [15].*

Many authors have modified the $\mathcal{F}$ -contraction mapping, which Wardowski introduced [15], according to the metric space setting, consult [27], [28]. We present a modified $\mathcal{F}$-contraction mapping that suits our $\mathcal{TC}$-S-$\mathcal{MTS}$.

Definition 2.11. *Assume $(\mathbb{X}, \mathcal{S})$ is a $\mathcal{TC}$-S-$\mathcal{MTS}$, where $\mathbb{X} \neq \emptyset$. The mapping $T : \mathbb{X} \to \mathbb{X}$ is referred to as a modified $\mathcal{F}$-contraction mapping, if there exists a function $F \in \mathcal{F}$ and some constant $\tau > 0$ so this holds;*

$$\mathcal{S}(Tx, Ty, Tz) > 0 \implies \tau + F(\mathcal{S}(Tx, Ty, Tz)) \leq F(\mathcal{S}(x, y, z)), \text{for all } x, y, z \in \mathbb{X}. \tag{2.3}$$

Next, we present the novel concept of (α_s-$\mathcal{F}$)-contractive mappings within the $\mathcal{TC}$-S-$\mathcal{MTS}$ framework as follows.

Definition 2.12. *Assume $(\mathbb{X}, \mathcal{S})$ is a $\mathcal{TC}$-S-$\mathcal{MTS}$, where $\mathbb{X} \neq \emptyset$. A mapping $T : \mathbb{X} \to \mathbb{X}$ is said to be an (α_s-$\mathcal{F}$)-contraction mapping, if there exists a function $\alpha_s : \mathbb{X}^3 \to [0, \infty)$, $F \in \mathcal{F}$, and some constant $\tau > 0$ so this holds;*

$$\tau + \alpha_s(x, y, z) F(\mathcal{S}(Tx, Ty, Tz)) \leq F(\mathcal{S}(x, y, z)), \tag{2.4}$$

for $x, y, z \in \mathbb{X}$, with $\mathcal{S}(Tx, Ty, Tz) > 0$.

3. Main Results

In this section, we establish the existence and uniqueness of fixed points in a complete $\mathcal{TC}$-S-$\mathcal{MTS}$ $(\mathbb{X}, \mathcal{S})$ using the $(\alpha_s$-$\mathcal{F})$-contraction mappings.

Theorem 3.1. *Let $(\mathbb{X}, \mathcal{S})$ be a complete $\mathcal{TC}$-S-$\mathcal{MTS}$, where $\mathbb{X} \neq \emptyset$. Let $T : \mathbb{X} \to \mathbb{X}$ be $(\alpha_s$-$\mathcal{F})$-contraction mapping. Assume the following holds:*

1. *There is $\hbar_0 \in \mathbb{X}$, such that $\alpha_s(\hbar_0, \hbar_0, T\hbar_0) \geq 1$.*
2. *T is α_s-admissible.*
3. *For $\hbar_0 \in \mathbb{X}$, the sequence $\{\hbar_n\}$, is defined by $\hbar_n = T^n \hbar_0$, furthermore these hold:*

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{\gamma(\hbar_{n+1}, \hbar_m)[\beta(\hbar_{n+1}, \hbar_{n+2}) + \mu(\hbar_{n+1}, \hbar_{n+2})]}{[\beta(\hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})]} < 1. \tag{3.1}$$

In addition, for each $\hbar \in \mathbb{X}$, the limits;

$$\lim_{n\to\infty} \beta(\hbar, \hbar_n)\ ,\ \lim_{n\to\infty} \mu(\hbar_n, \hbar) \text{ and } \lim_{n\to\infty} \gamma(\hbar_n, \hbar) \text{ exists and finite.} \tag{3.2}$$

Then, T possesses a fixed point. For the uniqueness of the fixed point, assume both u, and v are fixed points such that $\alpha_s(u, u, v) \geq 1$, then T posses a unique fixed point in $\mathbb{X}$.

Proof. Select $\hbar_0 \in \mathbb{X}$ so $\alpha(\hbar_0, T\hbar_0) \geq 1$. A sequence $\{\hbar_n\}$ is formed by $T\hbar_0 = \hbar_1, T^2\hbar_0 = T\hbar_1 = \hbar_2$. Therefore, for any $n \in \mathbb{N}$, we have

$$T^n\hbar_0 = T^{n-1}\hbar_1 = \cdots = T\hbar_{n-1} = \hbar_n.$$

In addition, $T^n\hbar_0 \neq T^{n+1}\hbar_0$ holds for all $n \geq 0$.

As T is an α_s-admissible mapping, this implies $\alpha_s(\hbar_n, \hbar_n, \hbar_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Since T is $(\alpha_s$-$\mathcal{F})$-contraction mapping, we obtain

$$\begin{aligned} \tau + F(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})) &= \tau + F(\mathcal{S}(T\hbar_{n-1}, T\hbar_{n-1}, T\hbar_n)). \\ &\leq \tau + \alpha_s(\hbar_{n-1}, \hbar_{n-1}, \hbar_n) F(\mathcal{S}(T\hbar_{n-1}, T\hbar_{n-1}, T\hbar_n)). \\ &\leq F(\mathcal{S}(\hbar_{n-1}, \hbar_{n-1}, \hbar_n)), \end{aligned}$$

which implies

$$\begin{aligned} F(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})) &\leq F(\mathcal{S}(\hbar_{n-1}, \hbar_{n-1}, \hbar_n)) - \tau. \\ &\leq F(\mathcal{S}(\hbar_{n-2}, \hbar_{n-2}, \hbar_{n-1})) - 2\tau. \\ &\leq \cdots \leq F(\mathcal{S}(\hbar_0, \hbar_0, \hbar_1)) - n\tau. \end{aligned} \tag{3.3}$$

Letting $n \to \infty$ in equation 3.3, and with $\tau > 0$, we have

$$\lim_{n\to\infty} F(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})) = -\infty. \tag{3.4}$$

As $F \in \mathcal{F}$, utilizing (W2) of Definition 2.10, we deduce that $\lim_{n\to\infty} \mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}) = 0$, and by (W3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}))^k F((\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})) = 0. \tag{3.5}$$

From equation 3.3, we obtain

$$F(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})) - F(\mathcal{S}(\hbar_0, \hbar_0, \hbar_1)) \leq -n\tau.$$

Thus for any n, we have

$$\begin{aligned}(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}))^k F(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}) - (\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})^k F(\mathcal{S}(\hbar_0, \hbar_0, \hbar_1))\\ \leq -n\tau(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}))^k \leq 0.\end{aligned} \tag{3.6}$$

Making n tends to infinity in equation 3.6, we have

$$\lim_{n\to\infty} n(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}))^k = 0. \tag{3.7}$$

This gives, $\lim_{n\to\infty} n^{1/k}(\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})) = 0$, hence there exists some $n_0 \in \mathbb{N}$, such that

$$\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}) \leq \frac{1}{n^{1/k}}, \text{ for all } n \geq n_0. \tag{3.8}$$

To demonstrate $\{\hbar_n\}$ is a Cauchy sequence, hence for all natural numbers $m, n \in \mathbb{N}$ with condition $n < m$, we obtain

$$\begin{aligned}\mathcal{S}(\hbar_n, \hbar_n, \hbar_m) &\leq \beta(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})\\ &+ \gamma(\hbar_{n+1}, \hbar_m)\mathcal{S}(\hbar_{n+1}, \hbar_{n+1}, \hbar_m).\\ &\leq \beta(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})\\ &+ \gamma(\hbar_{n+1}, \hbar_m)[\beta(\hbar_{n+1}, \hbar_{n+2})\mathcal{S}(\hbar_{n+1}, \hbar_{n+1}, \hbar_{n+2}) + \mu(\hbar_{n+1}, \hbar_{n+2})\mathcal{S}(\hbar_{n+1}, \hbar_{n+1}, \hbar_{n+2})\\ &+ \gamma(\hbar_{n+2}, \hbar_m)\mathcal{S}(\hbar_{n+2}, \hbar_{n+2}, \hbar_m)].\\ &\leq \beta(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})\\ &+ \gamma(\hbar_{n+1}, \hbar_m)\beta(\hbar_{n+1}, \hbar_{n+2})\mathcal{S}(\hbar_{n+1}, \hbar_{n+1}, \hbar_{n+2})\\ &+ \gamma(\hbar_{n+1}, \hbar_m)\mu(\hbar_{n+1}, \hbar_{n+2})\mathcal{S}(\hbar_{n+1}, \hbar_{n+1}, \hbar_{n+2})\\ &+ \gamma(\hbar_{n+1}, \hbar_m)\gamma(\hbar_{n+2}, \hbar_m)[\beta(\hbar_{n+2}, \hbar_{n+3})\mathcal{S}(\hbar_{n+2}, \hbar_{n+2}, \hbar_{n+3})\\ &+ \mu(\hbar_{n+2}, \hbar_{n+3})\mathcal{S}(\hbar_{n+2}, \hbar_{n+2}, \hbar_{n+3}) + \gamma(\hbar_{n+3}, \hbar_m)\mathcal{S}(\hbar_{n+3}, \hbar_{n+3}, \hbar_m)].\\ &\vdots\end{aligned}$$

$$\begin{aligned}\mathcal{S}(\hbar_n, \hbar_n, \hbar_m) \leq{}& \beta(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})\\ &+ \sum_{i=n+1}^{m-2}[\beta(\hbar_i, \hbar_{i+1}) + \mu(\hbar_i, \hbar_{i+1})]\mathcal{S}(\hbar_i, \hbar_i, \hbar_{i+1})\left(\prod_{j=n+1}^{i}\gamma(\hbar_j, \hbar_m)\right)\\ &+ \prod_{i=n+1}^{m-1}\gamma(\hbar_i, \hbar_m)\mathcal{S}(\hbar_{m-1}, \hbar_{m-1}, \hbar_m).\end{aligned}$$

$$\begin{aligned}\mathcal{S}(\hbar_n, \hbar_n, \hbar_m) \leq{}& [\beta(\hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})]\mathcal{S}(\hbar_n, \hbar_n, \hbar_{n+1})\\ &+ \sum_{i=n+1}^{m-1}[\beta(\hbar_i, \hbar_{i+1}) + \mu(\hbar_i, \hbar_{i+1})]\mathcal{S}(\hbar_i, \hbar_i, \hbar_{i+1})\left(\prod_{j=n+1}^{i}\gamma(\hbar_j, \hbar_m)\right)\end{aligned} \tag{3.9}$$

Applying equation 3.8 into inequality 3.9, it becomes

$$\begin{aligned}\mathcal{S}(\hbar_n, \hbar_n, \hbar_m) \leq{}& [\beta(\hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})](\frac{1}{n^{1/k}})\\ &+ \sum_{i=n+1}^{m-1}[\beta(\hbar_i, \hbar_{i+1}) + \mu(\hbar_i, \hbar_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=n+1}^{i}\gamma(\hbar_j, \hbar_m)\right)\\ \leq{}& [\beta(\hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})](\frac{1}{n^{1/k}})\\ &+ \sum_{i=1}^{m-1}[\beta(\hbar_i, \hbar_{i+1}) + \mu(\hbar_i, \hbar_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\gamma(\hbar_j, \hbar_m)\right)\end{aligned} \tag{3.10}$$

Let $L_p = \sum_{i=1}^{p-1}[\beta(\hbar_i, \hbar_{i+1}) + \mu(\hbar_i, \hbar_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=1}^{p}\gamma(\hbar_j, \hbar_m)\right)$.

Hence, inequality 3.10 can be written as

$$\mathcal{S}(\hbar_n, \hbar_n, \hbar_m) \leq [\beta(\hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})](\frac{1}{n^{1/k}}) + (L_{m-1} - L_n) \tag{3.11}$$

Applying the ratio test on inequality 3.11 and then taking the limit as n and m tends to infinity while employing equations 3.1, we obtain $\lim_{n,m\to\infty}[L_{m-1} - L_n] = 0$. Moreover, employing 3.17, implies $\lim_{n\to\infty}\beta(\hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})](\frac{1}{n^{1/k}}) = 0$.

Hence, we have shown that

$$\lim_{n,m\to\infty}\mathcal{S}(\hbar_n, \hbar_n, \hbar_m) = 0.$$

We conclude $\{\hbar_n\}$ is a Cauchy sequence. From the completeness of $(\mathbb{X}, \mathcal{S})$, therefore the sequence will converge to some $u \in \mathbb{X}$, i.e.

$$\lim_{n\to\infty}\mathcal{S}(\hbar_n, \hbar_n, u) = 0. \tag{3.12}$$

In the following paragraph, we will illustrate that u is a fixed point of the mapping T, i.e. $Tu = u$. Initially, we will show that $\lim_{n\to\infty}\mathcal{S}(T\hbar_n, T\hbar_n, Tu) = 0$.

Assume that $\mathcal{S}(T\hbar_n, T\hbar_n, Tu) > 0$ for all n. By Definition 2.11, we have

$$\tau + F(\mathcal{S}(T\hbar_n, T\hbar_n, Tu)) \leq F(\mathcal{S}(\hbar_n, \hbar_n, u)) \tag{3.13}$$

Taking the limit as n tends to infinity in equation 3.13, and using equation 3.12, and (W2) from Definition 2.10, we obtain $\lim_{n\to\infty} F(\mathcal{S}(T\hbar_n, T\hbar_n, Tu)) = -\infty$. Again, by Definition 2.10 this implies $\lim_{n\to\infty} \mathcal{S}(T\hbar_n, T\hbar_n, Tu) = 0$.

To show u is a fixed point, observe

$$\begin{aligned}\mathcal{S}(Tu, Tu, u) = \mathcal{S}(u, u, Tu) &\leq \beta(u, \hbar_{n+1})\mathcal{S}(u, u, \hbar_{n+1}) + \mu(u, \hbar_{n+1})\mathcal{S}(u, u, \hbar_{n+1}) \\ &+ \gamma(Tu, \hbar_{n+1})\mathcal{S}(Tu, Tu, T\hbar_n). \\ &\leq \beta(u, \hbar_{n+1})\mathcal{S}(u, u, \hbar_{n+1}) + \mu(u, \hbar_{n+1})\mathcal{S}(u, u, \hbar_{n+1}) \\ &+ \gamma(Tu, \hbar_{n+1})\mathcal{S}(T\hbar_n, T\hbar_n, Tu).\end{aligned}$$

As n tends to ∞ in the preceding inequality, we conclude that $\mathcal{S}(Tu, Tu, u) = 0$, implying $Tu = u$. Now, we proceed to establish the uniqueness of the fixed point. Suppose there exists two fixed points, u and v, with $u \neq v$ and $\alpha_s(u, u, v) \geq 1$. Since $Tu = u \neq v = Tv$, it implies that $\mathcal{S}(Tu, Tu, Tv) > 0$. Given that T is a (α_s-$\mathcal{F}$)-contraction mapping, utilizing equation 2.4, we obtain

$$\begin{aligned}\tau + F(\mathcal{S}(Tu, Tu, Tv)) &\leq \tau + \alpha_s(u, u, v)F(\mathcal{S}(Tu, Tu, Tv)). \\ &\leq F(\mathcal{S}(u, u, v)) = F(\mathcal{S}(Tu, Tu, Tv))\end{aligned}$$

This implies $\tau \leq 0$, leading to a contradiction. Therefore, $u = v$, implying the uniqueness of the fixed point. □

Let $(\mathbb{X}, \mathcal{S})$ be a $\mathcal{TC}$-S-$\mathcal{MTS}$, in case $\beta = \mu = \gamma$, then $(\mathbb{X}, \mathcal{S})$ becomes controlled S-metric type space as in definition 2.5. Hence, our next theorem presents an alternate proof for the existence of a fixed point in a complete controlled S-metric type space under the (α_s-$\mathcal{F}$)-contractive mapping; compare this with Theorem 1 in [14].

Theorem 3.2. *Assume $(\mathbb{X}, \mathcal{S})$ is a complete controlled S-metric type space, where $\mathbb{X}$ is a nonempty set. Let $T : \mathbb{X} \to \mathbb{X}$ be (α_s-$\mathcal{F}$)-contraction mapping, such that the following holds:*

(1) *T is α_s-admissible.*
(2) *There is $\hbar_0 \in \mathbb{X}$, so that $\alpha_s(\hbar_0, \hbar_0, T\hbar_0) \geq 1$.*
(3) *For $\hbar_0 \in \mathbb{X}$, the sequence $\{\hbar_n\}$, is defined by $\hbar_n = T^n\hbar_0$, moreover, assume these hold*

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{\beta(\hbar_{n+1}, \hbar_m)\beta(\hbar_{n+1}, \hbar_{n+2})}{\beta(\hbar_n, \hbar_{n+1})} < 1. \tag{3.14}$$

And

$$\lim_{n\to\infty} \beta(\hbar, \hbar_n) \text{, exists and finite.} \tag{3.15}$$

Then, T possesses a fixed point. Furthermore, if there exist two fixed points of T in $\mathbb{X}$, denoted as u and v with $\alpha_s(u,u,v) \geq 1$, then T has a unique fixed point in $\mathbb{X}$.

Proof. Repeat the proof of Theorem 3.1 by taking $\beta = \mu = \gamma$. □

A corollary to our main theorem is mentioned next.

Corollary 3.1. *Assume $(\mathbb{X}, \mathcal{S})$ is a complete $\mathcal{TC}$-S-$\mathcal{MTS}$, where $\mathbb{X} \neq \emptyset$. The mapping $T : \mathbb{X} \to \mathbb{X}$ satisfies the following conditions:*

$$\tau + F(\mathcal{S}(Tx, Ty, Tz)) \leq F(\mathcal{S}(x, y, z)), \text{for all } x, y, z \in \mathbb{X}.$$

Let $\hbar_0 \in \mathbb{X}$, the sequence $\{\hbar_n\}$, is defined by $\hbar_n = T^n \hbar_0$, and assume these hold

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{\gamma(\hbar_{n+1}, \hbar_m)[\beta(\hbar_{n+1}, \hbar_{n+2}) + \mu(\hbar_{n+1}, \hbar_{n+2})]}{[\beta(\hbar_n, \hbar_{n+1}) + \mu(\hbar_n, \hbar_{n+1})]} < 1. \tag{3.16}$$

In addition, for each $\hbar \in \mathbb{X}$, the limits;

$$\lim_{n\to\infty} \beta(\hbar, \hbar_n), \lim_{n\to\infty} \mu(\hbar_n, \hbar) \text{ and } \lim_{n\to\infty} \gamma(\hbar_n, \hbar) \text{ exists and finite.} \tag{3.17}$$

Then, T possesses a unique fixed point.

Proof. Define $\alpha_s : \mathbb{X}^3 \to [0, \infty)$ by $\alpha_s(x, y, z) = 1$, for all $x, y, z \in \mathbb{X}$. Then, the result is obtained by repeating the proof of Theorem 3.1 with the defined α_s.

□

4. Application

In conclusion, we emphasize the significance of Theorem 3.1 in determining a unique real solution for an mth degree polynomial. While there are various approaches for solving root-finding problems, including numerical techniques, the application of fixed point results, as demonstrated below, offers a straightforward solution.

Theorem 4.1. *For $m \geq 3$ any natural number, the below equation*

$$x^m - (m^4 - 1)x^{m+1} - m^4 x + 1 = 0, \tag{4.1}$$

possess a unique solution in the interval $[-1, 1]$.

Proof. If $|x| > 1$, then equation 4.1 have no solution; thus, $|x| \leq 1$. Let $\mathbb{X} = [-1, 1]$, then for any $x, y \in \mathbb{X}$, let $\mathcal{S} : \mathbb{X}^3 \to [0, +\infty)$, be given by $\mathcal{S}(x, y, z) = |x - y| + |y - z|$. The three functions $\beta, \mu, \gamma : \mathbb{X}^2 \to [1, \infty)$ are defined by

$$\beta(x, y) = max\{x, y\} + 2, \ \ \mu(x, y) = 1, \text{ and}$$

$$\gamma(x, y) = \begin{cases} max\{x, y\} + \frac{9}{10} & \text{if } x, y \in [0, 1], \\ 1 & \text{otherwise} \end{cases}$$

It can be shown that $(\mathbb{X}, \mathcal{S})$ is a complete $\mathcal{TC}$-S-$\mathcal{MTS}$.

Define the mapping $T : \mathbb{X} \rightarrow \mathbb{X}$ by

$$Tx = \frac{x^m + 1}{(m^4 - 1)x^m + m^4}, \tag{4.2}$$

Since $m \geq 3$, we will fix our $m = 3$ to make the computation easier, but using this method, one can show the results hold for any $m \geq 3$. Hence equation 4.2 becomes

$$Tx = \frac{x^3 + 1}{(3^4 - 1)x^3 + 3^4} = \frac{x^3 + 1}{80x^3 + 81}. \tag{4.3}$$

Let $\alpha : \mathbb{X}^3 \rightarrow (-\infty, +\infty)$, and $F : (0, +\infty) \rightarrow (-\infty, +\infty)$ be defined by,

$$\alpha_s(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

and $F(t) = ln(t)$. One can easily show that T is α_s-admissible, since if $\alpha_s(x, y, z) \geq 1$, means $x, y, z \in [0, 1]$, so $Tx, Ty, Tz \in [0, 1]$, consequently, $\alpha_s(Tx, Ty, Tz) \geq 1$. To show T is (α_s-$\mathcal{F}$)-contraction mapping; Observe that

$$\begin{aligned} |Tx - Ty| &= |\frac{x^3 + 1}{80x^3 + 81} - \frac{y^3 + 1}{80y^3 + 81}|. \\ &= |\frac{x^3 - y^3}{(80x^3 + 81)(80y^3 + 81)}|. \\ &\leq \frac{|x - y|}{81}. \end{aligned}$$

Hence

$$\mathcal{S}(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| \leq \frac{1}{81}|x - y| + \frac{1}{81}|y - z| = \frac{1}{81}\mathcal{S}(x, y, z).$$

Hence

$$ln(81) + \alpha_s(x, y, z)ln(\mathcal{S}(Tx, Ty, Tz)) \leq ln(81) + ln(\mathcal{S}(Tx, Ty, Tz)) \leq ln(\mathcal{S}(x, y, z)) \tag{4.4}$$

By taking $\tau = ln(81) > 0$, in 2.4, equation 4.4 becomes;

$$\tau + \alpha_s(x, y, z)F(\mathcal{S}(Tx, Ty, Tz)) \leq F(\mathcal{S}(x, y, z)).$$

Hence, T is (α_s-$\mathcal{F}$)-contraction mapping.

Finally, we show that equation 3.1 of Theorem 3.1 holds. Let $x_0 \in \mathbb{X}$ such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$, so $x_0 \in [0, 1]$. Form the sequence $x_n = T^n x_0$. We study the behavior of $T^n x_0$ as n tends to infinity. The sequence $\{x_n\}$ is decreasing as n tends to infinity. Since, $x_1 = Tx_0 = \frac{x_0^3+1}{80x_0^3+81}$, and

$$x_2 = Tx_1 = T(\frac{x_0^3 + 1}{80x_0^3 + 81}) = \frac{x_1^3 + 1}{80x_1^3 + 81} = \frac{(\frac{x_0^3+1}{80x_0^3+81})^3 + 1}{80(\frac{x_0^3+1}{80x_0^3+81})^3 + 81}. \tag{4.5}$$

Continuing in this manner, shows that $\{x_n\}$ is a decreasing sequence, and each $x_n \in [0, 1]$. Let

$$\begin{aligned} R_{n,m} &= \frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]}. \\ &= \frac{(max\{x_{n+1}, x_m\} + \frac{9}{10})[(max\{x_{n+1}, x_{n+2}\} + 2) + 1]}{[(max\{x_n, x_{n+1}\} + 2) + 1)]}. \\ &= \frac{(max\{x_{n+1}, x_m\} + \frac{9}{10})[max\{x_{n+1}, x_{n+2}\} + 3]}{[max\{x_n, x_{n+1}\} + 3]}. \end{aligned}$$

Hence, one can easily see that

$$\sup_{m\geq 1} \lim_{n\to\infty} R_{n,m} = \sup_{m\geq 1} \lim_{n\to\infty} \frac{(max\{x_{n+1}, x_m\} + \frac{9}{10})[(max\{x_{n+1}, x_{n+2}\} + 3]}{[(max\{x_n, x_{n+1}\} + 3)]} < 1. \tag{4.6}$$

Furthermore, $\lim_{n\to\infty} \beta(x, x_n) = \lim_{n\to\infty} = max\{x, x_n\} + 2$ exists and finite, also both $\lim_{n\to\infty} \mu(x_n, x)$ and $\lim_{n\to\infty} \gamma(x_n, x)$ exist and finite. Thus, all the conditions of Theorem 3.1 are fulfilled. Therefore, T possesses a unique fixed point in $\mathbb{X}$, which is a unique real solution of equation 4.1. □

5. Conclusions

This article introduced the concept of a triple controlled S-metric type space, which extends the idea of a controlled S-metric type space. We also introduced the concept of α_s-admissible mappings and refined Wardowski's contraction principle by defining (α_s-$\mathcal{F}$)-contractive mappings within the framework of triple controlled S-metric type spaces. Furthermore, we established the existence and uniqueness of fixed points in a complete triple controlled S-metric type space. To illustrate our findings, we demonstrated a unique real solution for a polynomial of degree m. Exploring applications of these concepts in various mathematical and applied contexts could be a promising direction for future research.

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