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Orlicz Extension of New Sequence Spaces Engendered by the Composition of Binomial Matrix and Double Band Matrix

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Abstract. The present paper is emphasis on introducing Orlicz extension of new sequence spaces (i.e $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$) by way of the composition of binomial matrix and double band matrix, which are $BK-$spaces, moreover we prove that these spaces are linearly isomorphic to the spaces l_∞, c_0 and c. We also derive some inclusion relations. Additionally, we find the Schauder basis for these spaces and finally we also determine the $\alpha-$, $\beta-$ and $\gamma-$ duals of these spaces.

1. Fundamentals and Representations

By w we embodied the clan of all real(or complex) valued sequences. The signs, ℓ_∞, c_0, c and ℓ_p signifies the classical sequence spaces of all bounded, convergent, null and absolutely p-summable sequences, respectively, where $1 \leq p < \infty$.

A $BK-$ space is a Banach sequence space only if each of the maps $p_n : X \to \mathbb{C}$ delineate by $p_n = z_n$ is continuous $\forall\ n \in N$, where X is a sequence space (see [9]).

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By reflecting the above explanation and notion, It is obvious that l_∞, c_0, c and l_p are $BK-$ spaces conferring norm $||z||_\infty = \sup_{k\in\mathbb{N}} |z_k|$ and l_p is a $BK-$ space with $p-$norm given by

$$||z||_{l_p} = \Big(\sum_{k=0}^{\infty} |z_k|^p\Big)^{\frac{1}{p}},$$

where $1 \leq p < \infty$.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex entries, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $z = (z_k) \in X$, the sequence $Az = \{A_n(z)\} \in Y$, where

$$A_n(z) = \sum_k a_{nk} z_k \quad (n \in \mathbb{N}), \tag{1.1}$$

converges for each $n \in \mathbb{N}$. By (X, Y) we denote the class of all matrices A such that $A : X \to Y$. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{z = (z_k) \in w : Az \in X\}. \tag{1.2}$$

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0\ \forall\ n \in \mathbb{N}$ [22].

We mark bs and cs for the sets of all bounded and convergent series, which are defined via the matrix domain of the summation matrix $S = (s_{nk})$ such that

$$bs = (l_\infty)_S \ \text{ and } \ cs = c_S$$

respectively, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n; \\ 0, & k > n \end{cases}$$

$\forall\ n, k \in \mathbb{N}$.

The principle of matrix transformation has an abundant reputation in summability theory given by Cesàro, Nörlund, Borel, etc. For more details on sequence spaces defined by matrix domain of infinite matrices see ([21], [17], [14], [10], [1], [2], [3] and [18], [11], [12], [6] and [7]).

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(z) > 0$ for $z > 0$ and $M(z) \longrightarrow \infty$ as $z \longrightarrow \infty$.

Lindenstrauss and Tzafriri [15] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \Big\{z \in w : \sum_{k=1}^{\infty} M\Big(\frac{|z_k|}{\rho}\Big) < \infty, \ \text{ for some } \ \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||z|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|z_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [15] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lz) \leq kLM(z)$ for all values of $z \geq 0$, $k > 0$ and for $L > 1$.
In second section of this article, we scrutinise Orlicz extension of three new sequence spaces $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ which is generalized by the composition of binomial matrix and double band matrix, where $\mathcal{G}$ is a generalized difference matrix and $\mathcal{M}$ is an Orlicz function. Several interesting inclusion relations between the newly formed sequence spaces are discussed in third section of this article. Afterward, we remark Schauder basis of the spaces $b_0^{r,s}(\mathcal{M},\mathcal{G})$ and $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and determine $\alpha-$, $\beta-$ and $\gamma-$ duals of these spaces in fourth section. We have also shown that the spaces $b_0^{r,s}(\mathcal{M},\mathcal{G})$ and $b_c^{r,s}(\mathcal{M},\mathcal{G})$ are seperable spaces. Moreover, in fifth section we depict some matrix classes related to the spaces $b_c^{r,s}(\mathcal{M},\mathcal{G})$. In final section of this article, the significance of the space is mentioned.

2. Three Sequence Spaces $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$

In this segment of the paper, we contribute some significant material regarding previous studies of composition of Binomial matrix, double band matrix and Euler matrix, so we made an effort to build Orlicz extension of three new sequence spaces. Besides, we show that these Orlicz extensions are linearly isomorphic to c_0, c and l_∞, respectively and also determine some inclusion relations.
In 2005 and 2006 Altay, Başar and Mursaleen were the first who introduce Euler matrix (See [1], [2]) and defined Euler sequence spaces e_0^r, e_c^r and e_∞^r as follows:

$$e_0^r = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k z_k = 0 \right\}$$

$$e_c^r = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k z_k \ \text{ exists } \right\}$$

and

$$e_\infty^r = \left\{ z = (z_k) \in w : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k z_k \right| < \infty \right\}.$$

Subsequently, in 2006 Altay and Polat [3] constructed sequence spaces $e_0^r(\triangle), e_c^r(\triangle)$ and $e_\infty^r(\triangle)$ where $\triangle$ is difference matrix and enhansed the work done by Altay, Başar and Mursaleen as:

$$e_0^r(\triangle) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k (z_k - z_{k-1}) = 0 \right\}$$

$$e_c^r(\triangle) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k (z_k - z_{k-1}) \ \text{ exists } \right\}$$

and

$$e_\infty^r(\triangle) = \left\{ z = (z_k) \in w : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k (z_k - z_{k-1}) \right| < \infty \right\}.$$

Recently, In [4] and [5] Bişgin has defined the Binomial sequence spaces $b_0^{r,s}$, $b_c^{r,s}$ and $b_\infty^{r,s}$ which is a generalization of Altay, Başar and Mursaleen work, as follows:

$$b_0^{r,s} = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k z_k = 0 \right\}$$

$$b_c^{r,s} = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k z_k \text{ exists} \right\}$$

and

$$b_\infty^{r,s} = \left\{ z = (z_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k z_k \right| < \infty \right\},$$

where the Binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined by

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k, & 0 \le k \le n\,; \\ 0, & k > n. \end{cases}$$

$\forall\ r,s \in \mathbb{R}^+$ and $n,k \in \mathbb{N}$. Here we would like to clear that if we take $s+r=1$ we obtain Euler sequence spaces e_0^r, e_c^r and e_∞^r. Afterward, Meng and Song [16] in 2017, gave a new way to above defined Bişgin work and stated the Binomial difference sequence spaces $b_0^{r,s}(\triangle)$, $b_c^{r,s}(\triangle)$ and $b_\infty^{r,s}(\triangle)$ (in case of $m=1$) as follows:

$$b_0^{r,s}(\triangle) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k (z_k - z_{k-1}) = 0 \right\}$$

$$b_c^{r,s}(\triangle) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k (z_k - z_{k-1}) \text{ exists} \right\}$$

and

$$b_\infty^{r,s}(\triangle) = \left\{ z = (z_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k (z_k - z_{k-1}) \right| < \infty \right\},$$

Later on, in 2019 Sönmez [20] introduced new sequence spaces $b_s^{r,s}(\mathcal{G}), b_c^{r,s}(\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{G})$ and defined as:

$$b_0^{r,s}(\mathcal{G}) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k (uz_k + vz_{k-1}) = 0 \right\}$$

$$b_c^{r,s}(\mathcal{G}) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k (uz_k + vz_{k-1}) \text{ exists} \right\}$$

and

$$b_\infty^{r,s}(\mathcal{G}) = \left\{ z = (z_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k (uz_k + vz_{k-1}) \right| < \infty \right\},$$

where $\mathcal{G} = (g_{nk})$ is a generalized difference matrix and is given as

$$g_{nk} = \begin{cases} u, & k = n; \\ v, & k = n-1; \\ 0, & \text{otherwise.} \end{cases}$$

$\forall\, n, k \in \mathbb{N}, u, v \in \mathbb{R}/0$ and $r, s > 0$. Now, by choosing $v = -1$ and $u = 1$, we get the difference matrix.

Now we define three new sequence spaces via Orlicz function $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ as follows:

$$b_0^{r,s}(\mathcal{M},\mathcal{G}) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k M_k(uz_k + vz_{k-1}) = 0 \right\}$$

$$b_c^{r,s}(\mathcal{M},\mathcal{G}) = \left\{ z = (z_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k M_k(uz_k + vz_{k-1}) \text{ exists } \right\}$$

and

$$b_\infty^{r,s}(\mathcal{M},\mathcal{G}) = \left\{ z = (z_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} (s)^{n-k} r^k M_k(uz_k + vz_{k-1}) \right| < \infty \right\},$$

where $\mathcal{G} = (g_{nk})$ is a generalized difference matrix, which is defined above and $\mathcal{M} = (M_k)$ is an Orlicz function. Here if we take $M_k = (1, 1, 1....1)$ we get generalized difference matrix, $\mathcal{G}$ and also if we take $M_k = (1, 1, 1....1)$, $v = -1$ and $u = 1$, we get the difference matrix, $\triangle$.
In view of notation (1.2) $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ can be redefined via matrix domain of Orlicz function $\mathcal{M}$ and generalized difference matrix $\mathcal{G}$ as follows:

$$b_0^{r,s}(\mathcal{M},\mathcal{G}) = (b_0^{r,s})_{\mathcal{M},\mathcal{G}}, \;\; b_c^{r,s}(\mathcal{M},\mathcal{G}) = (b_c^{r,s})_{\mathcal{M},\mathcal{G}} \;\text{ and }\; b_\infty^{r,s}(\mathcal{M},\mathcal{G}) = (b_\infty^{r,s})_{\mathcal{M},\mathcal{G}} \tag{2.1}$$

Also, by considering the triangular matrix $D^{r,s,u,v,M} = (d_{nk}^{r,s,u,v,M}) = B^{r,s}\mathcal{M}\mathcal{G}$ such that

$$d_{nk}^{r,s,u,v,M} = \begin{cases} \frac{s^{n-k-1}r^k}{(s+r)^n} M_k \left[us \binom{n}{k} + vr \binom{n}{k+1} \right] & 0 \le k \le n\,; \\ 0, & k > n. \end{cases}$$

$\forall\, n, k \in \mathbb{N}$, $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ can be reorganized as follows:

$$b_0^{r,s}(\mathcal{M},\mathcal{G}) = (c_0)_{D^{r,s,u,v,M}}, \;\; b_c^{r,s}(\mathcal{M},\mathcal{G}) = (c)_{D^{r,s,u,v,M}} \;\text{ and }\; b_\infty^{r,s}(\mathcal{M},\mathcal{G}) = (l_\infty)_{D^{r,s,u,v,M}}. \tag{2.2}$$

In this system $D^{r,s,u,v,M}-$ transform of z is given as

$$y_k = (D^{r,s,u,v,M}z)_k = \frac{1}{(s+r)^n} \sum_{i=0}^{k} \binom{k}{i} (s)^{k-i} r^i M_i(uz_i + vz_{i-1}) \tag{2.3}$$

$\forall\, k \in \mathbb{N}$, or by considering another depiction, the sequence $y = (y_k)$ can be redrafted as follows:

$$y_k = (D^{r,s,u,v,M}z)_k = \frac{1}{(s+r)^k} \sum_{i=0}^{k} \left[us \binom{k}{i} + vr \binom{k}{i+1} \right] (s)^{k-i-1} r^i M_i(z_i) \tag{2.4}$$

for all $k \in \mathbb{N}$.

Theorem 2.1. *The spaces $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ are BK-spaces with the norm*

$$||z||_{b_0^{r,s}(\mathcal{M},\mathcal{G})} = ||z||_{b_c^{r,s}(\mathcal{M},\mathcal{G})} = ||z||_{b_\infty^{r,s}(\mathcal{M},\mathcal{G})} = \sup_{k\in\mathbb{N}} |(D^{r,s,u,v,M}z)_k|.$$

Proof. Since (c_0), (c) and (l_∞) sequences are BK- spaces, in addition to this condition (2.2) holds and also $D^{r,s,u,v,M} = (d_{nk}^{r,s,u,v,M})$ is a triangular matrix on relating these three outcomes with the Theorem 4.3.12 of Wilansky [22], we achieve that $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ are BK-spaces. □

Theorem 2.2. *The spaces $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ are linearly isomorphic to l_∞, c_0 and c, respectively, that is $b_0^{r,s}(\mathcal{M},\mathcal{G}) \cong l_\infty$, $b_c^{r,s}(\mathcal{M},\mathcal{G}) \cong c_0$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G}) \cong c$.*

Proof. We only consider the case $b_0^{r,s}(\mathcal{M},\mathcal{G}) \cong c_0$ and the other cases will follow similarly. Thus, to prove the theorem, we must show the existence of linear bijection between $b_0^{r,s}(\mathcal{M},\mathcal{G})$ and c_0.

For, consider the transformation T defined, with the notation (2.3), from $b_0^{r,s}(\mathcal{M},\mathcal{G})$ to c_0 by $T(z) = D^{r,s,u,v,M}z$. Then it is clear that $T(z) = D^{r,s,u,v,M}z \in c_0$ for every $z \in b_0^{r,s}(\mathcal{M},\mathcal{G})$. Also, the linearity of T is obvious. Further, it is trivial that $z = 0$ whenever $Tz = 0$ and hence T is injective. Furthermore, let $y = (y_k) \in c_0$ and define the sequence $z = (z_n)$ by

$$z_n = \frac{1}{u}\sum_{k=0}^{n} M_k\left[\sum_{i=k}^{n}\binom{k}{i}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i}\right]y_k$$

$\forall\ n \in \mathbb{N}$. Then we have

$$\begin{aligned}(D^{r,s,u,v,M}z)_n &= \frac{1}{(s+r)^k}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k M_k(uz_k + vz_{k-1})\\ &= \frac{1}{(s+r)^k}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k M_k\sum_{j=k}^{n}\binom{k}{j}(-s)^{k-j}(r+s)^j r^{-k}y_i\\ &= y_n.\end{aligned}$$

This shows that

$$\lim_{n\to\infty}(D^{r,s,u,v,M}z)_n = \lim_{n\to\infty} y_n = 0.$$

Thus, we deduce that $z = (z_k) \in b_0^{r,s}(\mathcal{M},\mathcal{G})$ and $L(z) = y$. Hence, T is surjective. Further, $\forall\ z \in b_0^{r,s}(\mathcal{M},\mathcal{G})$ we have

$$||L(z)||_\infty = ||D^{r,s,u,v,M}z||_\infty = ||z||_{b_0^{r,s}(\mathcal{M},\mathcal{G})},$$

which means that L is norm preserving, and so is linear bijection. As a result of these facts we have, $b_0^{r,s}(\mathcal{M},\mathcal{G}) \cong c_0$. □

3. The Inclusion Relations

In this section of the paper we make an effort to prove some inclusion relation between the newly defined sequence spaces.

Theorem 3.1. *The inclusions* $\hat{c}_0 \subset b_0^{r,s}(\mathcal{M},\mathcal{G})$, $\hat{c} \subset b_c^{r,s}(\mathcal{M},\mathcal{G})$ *and* $\hat{l_\infty} \subset b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ *hold strictly, where spaces* $\hat{c}_0, \hat{c}$ *and* $\hat{l_\infty}$ *defined in [13].*

Proof. We provide the proof for only $\hat{l_\infty} \subset b_\infty^{r,s}(\mathcal{M},\mathcal{G})$. Let us consider an arbitrary sequence $z = (z_k) \in \hat{l_\infty}$, then we have

$$\begin{aligned} ||z||_{b_\infty^{r,s}(\mathcal{M},\mathcal{G})} &= (D^{r,s,u,v,M}z)_\infty \\ &= \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^k} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} s^{n-k} r^k M_k(uz_k + vz_{k-1}) \right| \\ &\leq \sup_{n\in\mathbb{N}} \left| M_n(uz_n + vz_{n-1}) \right| . \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^k} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} s^{n-k} r^k \right| \\ &= ||z||_{l_\infty}. \end{aligned}$$

This display that $z = (z_k) \in b_\infty^{r,s}(\mathcal{M},\mathcal{G})$. Thus the inclusion $\hat{l_\infty} \subset b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ holds. Now we also need to show that the inclusion $\hat{l_\infty} \subset b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ is strict, for this we define $z = (z_k)$ such that $z_k = \frac{1}{u} \sum_{k=0}^{i} (-\frac{v}{u})^{k-i} (-\frac{s+r}{r})^i$ for all $k \in \mathbb{N}$ here we take $\mathcal{M} = (M_k) = (1, 1, 1, 1........)$. as a result of which $Gz = ((-\frac{s+r}{r})^k) \notin \hat{l_\infty}$ on the other hand $(D^{r,s,u,v,M}z) = ((-\frac{r}{s+r})^k) \in \hat{l_\infty}$. As a significance of which $z = (z_k) \in b_\infty^{r,s}(\mathcal{M},\mathcal{G}) \setminus \hat{l_\infty}$. Thus inclusion $\hat{c}_0 \subset b_0^{r,s}(\mathcal{M},\mathcal{G})$, is strict. □

Theorem 3.2. *The inclusions* $b_0^{r,s}(\mathcal{M},\mathcal{G}) \subset b_c^{r,s}(\mathcal{M},\mathcal{G}) \subset b_\infty^{r,s}(\mathcal{M},\mathcal{G})$.

Proof. It is a well recognized fact that every null sequence is also convergent and every convergent sequence is also bounded. So, the inclusion $b_0^{r,s}(\mathcal{M},\mathcal{G}) \subset b_c^{r,s}(\mathcal{M},\mathcal{G}) \subset b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ holds. Now it remains to show that the this inclusion is strict, for this we define two sequence spaces $z = (z_k)$ and $y = (y_k)$ such that $z_k = \frac{1-(-\frac{v}{u})^{k+1}}{u+v}$ and $y_k = \frac{1}{u} \sum_{k=0}^{i} (-\frac{v}{u})^{k-i} (-\frac{r+2s}{r})^i$ $\forall\, k \in \mathbb{N}$ here we take $\mathcal{M} = (M_k) = (1, 1, 1, 1........)$. Then we clearly perceive that $(D^{r,s,u,v,M}z) = e \in c \setminus c_0$ and $(D^{r,s,u,v,M}y) = ((-1)^k) \in l_\infty \backslash c$. thus $z = (z_k) \in b_c^{r,s}(\mathcal{M},\mathcal{G}) \backslash b_0^{r,s}(\mathcal{M},\mathcal{G})$ and $y = (y_k) \in b_\infty^{r,s}(\mathcal{M},\mathcal{G}) \backslash b_c^{r,s}(\mathcal{M},\mathcal{G})$. On combining these facts we conclude that the inclusion $b_0^{r,s}(\mathcal{M},\mathcal{G}) \subset b_c^{r,s}(\mathcal{M},\mathcal{G}) \subset b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ is strict. □

Theorem 3.3. $c \subset b_0^{r,s}(\mathcal{M},\mathcal{G})$ *embraces strictly, whenever* $u + v = 0$.

Proof. It is understandable that $MGz \in c_0$ whenever $z \in c$ and $\mathcal{M} = (M_k) = (1, 1, 1, 1........)$. It is also known that when $r, s > 0$ then the binomial matrix is regular. On carteling these facts, we attain that $B^{r,s}MGz \in c_0$ whenever $z \in c$ and $\mathcal{M} = (M_k) = (1, 1, 1, 1........)$, namely $z \in b_0^{r,s}(\mathcal{M},\mathcal{G})$, whenever $z \in c$ and $\mathcal{M} = (M_k) = (1, 1, 1, 1........)$. So the inclusion $c \subset b_0^{r,s}(\mathcal{M},\mathcal{G})$

holds. For proving the strictness we define $z = (z_k)$ such that $z_k = (-1)^k\Big[\frac{1-(\frac{v}{u})^{k+1}}{u-v}\Big]$ for all $k \in \mathbb{N}$, and $\mathcal{M} = (M_k) = (1, 1, 1, 1........)$, then we see that $z = (z_k) \notin c$ but $(D^{r,s,u,v,M}z) = ((\frac{s-r}{s+r})^k) \in c_0$, that means $z \in b_0^{r,s}(\mathcal{M}, \mathcal{G})$. This shows that $c \subset b_0^{r,s}(\mathcal{M}, \mathcal{G})$ is strict. □

4. The Schauder Basis and $\alpha-, \beta-$ and $\gamma-$ Duals

In this segment of the paper, we find the Schauder Basis of Orlicz extension of binomial difference sequence spaces $b_0^{r,s}(\mathcal{M}, \mathcal{G})$ and $b_c^{r,s}(\mathcal{M}, \mathcal{G})$. Addition to this we also determine $\alpha-, \beta-$ and $\gamma-$ Duals of $b_0^{r,s}(\mathcal{M}, \mathcal{G})$, $b_c^{r,s}(\mathcal{M}, \mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M}, \mathcal{G})$.
A sequence $u = (u_k) \in X$ is called a Schauder basis for a normed space $(X, \|.\|_X)$ if, $\forall\ z = (z_k) \in X$ $\exists$ a unique sequence (λ_k) of scalars such that $z = \sum_k \lambda_k u_k$ i.e

$$\lim_{n\to\infty} \Big\| z - \sum_{k=0}^{n} \lambda_k u_k \Big\| \to 0.$$

Theorem 4.1. *For all $k \in \mathbb{N}$, let $\xi_k = (D^{r,s,u,v,M}z)_k$ and consider the sequence $a = (a_k)$ for fixed $k \in \mathbb{N}$ and defined as $a_k = M_k \frac{1-(-\frac{v}{u})}{u+v}$ and $a^{(k)}(r, s, u, v, M) = \{a_n^{(k)}(r, s, u, v, M)\}_{n\in\mathbb{N}}$ is given by*

$$a_n^{(k)}(r, s, u, v, M) = \begin{cases} 0, 0 \le n < k; \\ \sum_{k=0}^{n} M_k \Big[\frac{1}{u} \sum_{i=k}^{n} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} -\ \ v \\ u \end{pmatrix}^{n-i} (-s)^{i-k}(r+s)^k r^{-i}\Big], k \le n \end{cases}$$

Then the following conditions holds:
(a) The Schauder basis of $b_0^{r,s}(\mathcal{M}, \mathcal{G})$ is $a^{(k)}(r, s, u, v, M)_{k\in\mathbb{N}}$ and all $z = (z_k) \in b_0^{r,s}(\mathcal{M}, \mathcal{G})$ can be uniquely written as

$$z = \sum_k \xi_k a^{(k)}(r, s, u, v, M).$$

(b)The Schauder basis of $b_c^{r,s}(\mathcal{M}, \mathcal{G})$ is the sequence $a, a^{(0)}(r, s, u, v, M), a^{(1)}(r, s, u, v, M), ...$ and all $z = (z_k) \in b_c^{r,s}(\mathcal{M}, \mathcal{G})$ can be uniquely written as

$$z = la + \sum_k |\xi_k - l| a^{(k)}(r, s, u, v, M).$$

where $l = \lim_{k\to\infty} (D^{r,s,u,v,M}z)_k.$

Proof. (a) It is definitely understand that $D^{r,s,u,v,M}a^{(k)}(r, s, u, v, M) = e^{(k)} \in a_0\ \forall\ k \in \mathbb{N}$, where $e^{(k)} = (0, 0, 0, \cdots, 1, 0, \cdots)$ with 1 in the k^{th} place. Then we can easily accomplish $a^{(k)}(r, s, u, v, M) \subset b_0^{r,s}(\mathcal{M}, \mathcal{G})$ holds.
Let $z = (z_k) \in b_0^{r,s}(\mathcal{M}, \mathcal{G})$. We can mark for all $m \in \mathbb{N}$ as

$$z^{[m]} = \sum_{k=0}^{m} \xi_k a^{(k)}(r, s, u, v, M).$$

Then by putting the matrix $D^{r,s,u,v,M} = (d_{nk}^{r,s,u,v,M})$ to $z^{[m]}$, we have

$$D^{r,s,u,v,M}z^{[m]} = \sum_{k=0}^{m} \xi_k D^{r,s,u,v,M} a^{(k)}(r,s,u,v,M) = \sum_{k=0}^{m} (D^{r,s,u,v,M}z)_k e^{(k)}$$

and $\forall\ n, m \in \mathbb{N}$ we have

$$D^{r,s,u,v,M}(z - z^{[m]})_n = \begin{cases} 0, & 0 \leq n \leq m; \\ (D^{r,s,u,v,M}z)_n, & n > m. \end{cases}$$

For every $\varepsilon > 0\ \exists\ m_0 = m_0^{\varepsilon} \in \mathbb{N}$ s.t.,

$$|(D^{r,s,u,v,M}z)_m| < \frac{\varepsilon}{2}$$

$\forall\ m_0 \leq m$. On description of this

$$\|z - z^{[m]}\|_{b_0^{r,s}(\mathcal{M},\mathcal{G})} = \sup_{m \leq n} |(D^{r,s,u,v,M}z)_n| \leq \sup_{m_0 \leq n} |(D^{r,s,u,v,M}z)_n| \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $m_0 \leq m$. Which implies

$$z = \sum_k \xi_k a^{(k)}(r,s,u,v,M).$$

Now, let us assume that there exist an another representation of $z = (z_k)$ such that

$$z = \sum_k \vartheta_k a^{(k)}(r,s,u,v,M).$$

Then by the continuity of the transformation T which is defined in the proof of the theorem 2.2, we get

$$(D^{r,s,u,v,M}z)_n = \sum_k \vartheta_k [D^{r,s,u,v,M} a^{(k)}(r,s,u,v,M)]_n = \sum_k \vartheta_k e_n^{(k)} = \vartheta_n$$

$\forall\ n \in \mathbb{N}$. This implies that $(D^{r,s,u,v,M}z)_n = \xi_n$ for all $n \in \mathbb{N}$. Therefore all $z = (z_k) \in b_0^{r,s}(\mathcal{M},\mathcal{G})$ has a unique representation.

(b) The inclusion $a, a^{(k)}(r,s,u,v,M) \subset b_c^{r,s}(\mathcal{M},\mathcal{G})$ clearly embrace as from the above defined fragment (a) we identified that $a^{(k)}(r,s,u,v,M) \subset b_c^{r,s}(\mathcal{M},\mathcal{G})$ and also $D^{r,s,u,v,M}a = e \in c$. For an arbitrary $z = (z_k) \in b_c^{r,s}(\mathcal{M},\mathcal{G})$ we built a sequence $y = (y_k)$ such that $y = z - la$, where $l = \lim_{\xi_k}$, then it is clear that $y = (y_k) \in b_c^{r,s}(\mathcal{M},\mathcal{G})$ and by the above defined part (a) $y = (y_k)$ has a unique representation. This drag us to $z = (z_k)$ has a unique representation of the form

$$z = la + \sum_k |\xi_k - l| a^{(k)}(r,s,u,v,M).$$

□

Furthermore if we associate Theorem 2.1 and Theorem 4.1 we get corollary 4.2.

Corollary 4.1. *The sequence spaces $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_0^{r,s}(\mathcal{M},\mathcal{G})$ are separable.*

A set defined by

$$N(X,Y) = \{b = b_k \in w : bz = (b_k z_k) \in Y \text{ for all } z = (z_k) \in X\}$$

is called the multiplier space of the sequence spaces X and Y. Then the $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence space X are defined by the aid of the notation of multiplier spaces such that

$$X^{\alpha} = N(X, l_1), \quad X^{\beta} = N(X, cs), \text{ and } X^{\gamma} = N(X, bs),$$

respectively

Now, we carry on with some declaration which are practice in the next lemma (see [[19]]).

$$\sup_{K\in\mathcal{F}} \sum_n |\sum_{k\in K} b_{nk}| < \infty, \tag{4.1}$$

$$\sup_{K\in\mathcal{F}} \sum_k |b_{nk}| < \infty, \tag{4.2}$$

$$\lim_{n\to\infty} \sum_k |b_{nk}| = \sum_k |\lim_{n\to\infty} b_{nk}|, \tag{4.3}$$

$$\lim_{n\to\infty} b_{nk} = \mu_k, \text{ for all } k \in \mathbb{N}, \tag{4.4}$$

$$\lim_{n\to\infty} \sum_k b_{nk} = \mu. \tag{4.5}$$

Here $\mathcal{F}$ represents the set of all finite subsets of $\mathbb{N}$.

Lemma 4.1. *[19] Let $B = (b_{nk})$ be an infinite matrix. Then the following declaration embraces:*
(i) $B = (b_{nk}) \in (c_0 : l_1) = (c : l_1) = (l_\infty : l_1) \Leftrightarrow$ (4.1) hold.
(ii)$B = (b_{nk}) \in (c_0 : l_\infty) = (c : l_\infty) = (l_\infty : l_\infty) \Leftrightarrow$ (4.2) hold.
(iii)$B = (b_{nk}) \in (c_0 : c) \Leftrightarrow$ (4.2) and (4.4) holds.
(iv)$B = (b_{nk}) \in (c : c) \Leftrightarrow$ (4.2),(4.4) and (4.5) holds.
(v)$B = (b_{nk}) \in (l_\infty : c) \Leftrightarrow$ (4.3) and (4.4) holds.
(vi)$B = (b_{nk}) \in (c : c_0) \Leftrightarrow$ (4.2),(4.4) and (4.5) holds with $\mu_k = 0$, for all $k \in \mathbb{N}$ and $\mu = 0$.

Theorem 4.2. *The $\alpha-$ dual of the Binomial sequence spaces $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ is*

$$a_1^{r,s,u,v,M} = \left\{ b = (b_k) \in w : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} M_k \left[\frac{1}{u} \sum_{i=k}^{n} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} -\frac{v}{u} \end{pmatrix}^{n-i} (-s)^{i-k}(r+s)^k r^{-i} b_n \right] \right| < \infty \right\}$$

Proof. To prove this theorem we need to keep in mind the sequence which is defined in the proof of Theorem 2.2 so

$$b_n z_(n) = \sum_{k=0}^{n} M_k \left[\frac{1}{u} \sum_{i=k}^{n} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} -\frac{v}{u} \end{pmatrix}^{n-i} (-s)^{i-k}(r+s)^k r^{-i} b_n \right] y_k = \sum_{k=0}^{n} u_{nk}^{r,s,u,v,M} y_k = (U^{r,s,u,v,M} y)_n$$

$\forall\ n \in \mathbb{N}$. when $z = (z_k) \in b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ or $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ we have $bz = (b_n z_n) \in l_1$ iff $U^{r,s,u,v,M} y \in l_1$ whenever $y = (y_k) \in c_0, c$ or l_∞. This indicate that $b = (b_n) \in \{b_0^{r,s}(\mathcal{M},\mathcal{G})\}^{\alpha} =$

$\{b_c^{r,s}(\mathcal{M},\mathcal{G})\}^\alpha = \{b_\alpha^{r,s}(\mathcal{M},\mathcal{G})\}^\alpha$ if and only if $U^{r,s,u,v,M} \in (c_0 : l_1) = (c : l_1) = (l_\infty : l_1)$. By associating this outcome and Lemma 4.3 (i) we conclude that

$$b = (b_n) \in \{b_0^{r,s}(\mathcal{M},\mathcal{G})\}^\alpha \Leftrightarrow \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} M_k \left[\frac{1}{u} \sum_{i=k}^{n} \binom{i}{k} \left(-\frac{v}{u}\right)^{n-i} (-s)^{i-k}(r+s)^k r^{-i} b_n \right] \right| < \infty.$$

This proves that $b = (b_n) \in \{b_0^{r,s}(\mathcal{M},\mathcal{G})\}^\alpha = \{b_c^{r,s}(\mathcal{M},\mathcal{G})\}^\alpha = \{b_\alpha^{r,s}(\mathcal{M},\mathcal{G})\}^\alpha = a_1^{r,s,u,v,M}$. □

Theorem 4.3. *Let us consider four sets $a_2^{r,s,u,v,M}, a_3^{r,s,u,v,M}, a_4^{r,s,u,v,M}$ and $a_5^{r,s,u,v,M}$ defined as follows:*

$$a_2^{r,s,u,v,M} = \left\{ b = (b_k) \in w : \sup_{n\in\mathbb{N}} \sum_k \left| v_{nk}^{r,s,u,v,M} \right| < \infty \right\}.$$

$$a_3^{r,s,u,v,M} = \left\{ b = (b_k) \in w : \lim_{n\to\infty} v_{nk}^{r,s,u,v,M} \ \textit{exists for all}\ k \in \mathbb{N}. \right\}$$

$$a_4^{r,s,u,v,M} = \left\{ b = (b_k) \in w : \lim_{n\to\infty} \sum_k \left| v_{nk}^{r,s,u,v,M} \right| = \sum_k \left| \lim_{n\to\infty} v_{nk}^{r,s,u,v,M} \right| < \infty \right\}.$$

and

$$a_5^{r,s,u,v,M} = \left\{ b = (b_k) \in w : \lim_{n\to\infty} \sum_k v_{nk}^{r,s,u,v,M} \right| < \infty \right\}.$$

Where the matrix $V = v_{nk}^{r,s,u,v,M}$ is defined by sequence $b = (b_n)$ by

$$v_{nk}^{r,s,u,v,M} = \begin{cases} \sum_{k=0}^{n} M_k \left[\frac{1}{u} \sum_{j=k}^{i} \sum_{i=k}^{n} \binom{j}{k} \left(-\frac{v}{u}\right)^{i-j} (-s)^{j-k}(r+s)^k r^{-j} a_i \right], & 0 \le k \le n \\ 0, & n < k; \end{cases}$$

for all $n, k \in \mathbb{N}$. Then the following holds:
(i) $\{b_0^{r,s}(\mathcal{M},\mathcal{G})\}^\beta = a_2^{r,s,u,v,M} \cap a_3^{r,s,u,v,M}$;
(ii) $\{b_c^{r,s}(\mathcal{M},\mathcal{G})\}^\beta = a_2^{r,s,u,v,M} \cap a_3^{r,s,u,v,M} \cap a_5^{r,s,u,v,M}$;
(iii) $\{b_\infty^{r,s}(\mathcal{M},\mathcal{G})\}^\beta = a_3^{r,s,u,v,M} \cap a_4^{r,s,u,v,M}$;
(iv) $\{b_0^{r,s}(\mathcal{M},\mathcal{G})\}^\gamma = \{b_c^{r,s}(\mathcal{M},\mathcal{G})\}^\gamma = \{b_\infty^{r,s}(\mathcal{M},\mathcal{G})\}^\gamma = a_2^{r,s,u,v,M}$.

Proof. We provide the proof of the theorem for only part (i), the proof of other parts i.e (ii), (iii) and (iv) can be proved similarly.

Let $b = (b_n) \in w$ be given. As defined in proof of the Theorem 2.2, the sequence $z = (z_k)$ we have

$$\begin{aligned} \sum_{k=0}^{n} b_k z_k &= \sum_{k=0}^{n} M_k \left[\frac{1}{u} \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \left(-\frac{v}{u}\right)^{k-j} (-s)^{j-i}(r+s)^i r^{-j} y_i \right] b_k \\ &= \sum_{k=0}^{n} M_k \left[\frac{1}{u} \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} \left(-\frac{v}{u}\right)^{i-j} (-s)^{j-k}(r+s)^k r^{-j} b_i \right] y_k \\ &= (V^{r,s,u,v,M} y)_n \end{aligned}$$

$\forall\ n, k \in \mathbb{N}$. Then $bz = (b_n z_n) \in cs$ whenever $z = (z_k) \in b_0^{r,s}(\mathcal{M}, \mathcal{G})$ iff $V^{r,s,u,v,M} y \in c$ whenever $y \in c_0$. This outcome display us that $b = (b_k) \in \{b_0^{r,s}(\mathcal{M}, \mathcal{G})\}^\beta$ if and only if $V^{r,s,u,v,M} \in (c_0 : c)$. By merging this result and Lemma 4.3 (iii) we conclude that $b = (b_k) \in \{b_0^{r,s}(\mathcal{M}, \mathcal{G})\}^\beta$ iff

$$\sup_{n\in\mathbb{N}} \sum_k |v_{nk}^{r,s,u,v,M}| < \infty,$$

and

$$\lim_{n\to\infty}, v_{nk}^{r,s,u,v,M} \ \text{ exists}, \quad \text{ for all } \ k \in \mathbb{N},$$

thus $\{b_0^{r,s}(\mathcal{M}, \mathcal{G})\}^\beta = a_2^{r,s,u,v,M} \cap a_3^{r,s,u,v,M}$. □

5. Matrix Classes Associated to $b_c^{r,s}(\mathcal{M}, \mathcal{G})$

In this section of the paper we try to potray some matrix transformation subjected to $b_c^{r,s}(\mathcal{M}, \mathcal{G})$. Currently, we need to provide a Lemma which is used in further corollaries.

Lemma 5.1. *([8]) Let X and Y be any two sequence spaces. Let $B = (b_{nk})$ is an infinite matrix and E ba a triangle matrix. Then $B \in (X : Y_E) \Leftrightarrow EB \in (X : Y)$.*

In this particular section of the paper we use the following equalities for the sake of simplicity of notations.

$$a_{nk}^{r,s,u,v,M} = \sum_{k=0}^{n} M_k \left[\frac{1}{u} \sum_{i=k}^{\infty} \sum_{j=k}^{i} \begin{pmatrix} j \\ k \end{pmatrix} \begin{pmatrix} - & v \\ & u \end{pmatrix}^{i-j} (-s)^{j-k} (r+s)^k r^{-j} b_{ni} \right]$$

$\forall\ n, k \in \mathbb{N}$

Theorem 5.1. *$B \in b_c^{r,s}(\mathcal{M}, \mathcal{G}) : l_\infty$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}^{r,s,u,v,M}| < \infty, \tag{5.1}$$

$$a_{nk}^{r,s,u,v,M} \ \text{ exists for all } n, k \in \mathbb{N} \tag{5.2}$$

$$\sup_{m\in\mathbb{N}} \sum_{k=0}^{n} M_k \left[\left| \frac{1}{u} \sum_{i=k}^{\infty} \sum_{j=k}^{i} \begin{pmatrix} j \\ k \end{pmatrix} \begin{pmatrix} - & v \\ & u \end{pmatrix}^{i-j} (-s)^{j-k} (r+s)^k r^{-j} b_{ni} \right| \right] < \infty \quad (m \in \mathbb{N}) \tag{5.3}$$

$$\sum_{k=0}^{n} M_k \left[\lim_{m\to\infty} \frac{1}{u} \sum_{i=k}^{\infty} \sum_{j=k}^{i} \begin{pmatrix} j \\ k \end{pmatrix} \begin{pmatrix} - & v \\ & u \end{pmatrix}^{i-j} (-s)^{j-k} (r+s)^k r^{-j} b_{ni} \right] \text{ exists for all } m \in \mathbb{N}. \tag{5.4}$$

Proof. Let us undertake that $B \in b_c^{r,s}(\mathcal{M}, \mathcal{G}) : l_\infty$. then Bz exists and belongs to l_∞ for each $z = (z_k) \in b_c^{r,s}(\mathcal{M}, \mathcal{G})$. This pointers to $\{b_{nk}\}_{k\in\mathbb{N}} \in \{b_0^{r,s}(\mathcal{M}, \mathcal{G})\}^\beta$ for all $n \in \mathbb{N}$. On merging this point and Theorem 4.2 (ii) we accomplish that the statements (5.2), (5.3) and (5.4) hold.
If we reflect the element that $z = \left(\frac{1-(-\frac{v}{u})^{k+1}}{u+v} \right) \in b_c^{r,s}(\mathcal{M}, \mathcal{G})$, taking $\mathcal{M} = (M_k) = (1, 1, 1.......)$ and $Bz \in l_\infty$ for all $z \in b_c^{r,s}(\mathcal{M}, \mathcal{G})$ we can see that statement (5.1) hold.

On the contrary let us undertake that the statements (5.1)-(5.4) hold. Consider an arbitrary $z = (z_k) \in b_c^{r,s}(\mathcal{M},\mathcal{G})$ and take into account the equality

$$\sum_{k=0}^{m} b_{nk}z_k = \sum_{k=0}^{m} M_k\Big[\frac{1}{u}\sum_{i=0}^{k}\sum_{j=i}^{k}\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} -\frac{v}{u} \end{pmatrix}^{k-j}(-s)^{j-i}(r+s)^i r^{-j} y_i\Big]b_{nk}$$

$$\sum_{k=0}^{m} b_{nk}z_k = \sum_{k=0}^{n} M_k\Big[\frac{1}{u}\sum_{k=0}^{m}\sum_{i=k}^{m}\Big[\sum_{j=k}^{i}\begin{pmatrix} j \\ k \end{pmatrix}\begin{pmatrix} -\frac{v}{u} \end{pmatrix}^{i-j}(-s)^{j-k}(r+s)^k r^{-j}\Big]b_{ni}y_k\Big] \tag{5.5}$$

$\forall\ m, n \in \mathbb{N}$. If we take limit as $m \to \infty$ side by side in (5.5) then we obtain that

$$\sum_k b_{nk}z_k = \sum_k a_{nk}^{r,s,u,v,M} y_k \tag{5.6}$$

$\forall\ n \in \mathbb{N}$. On taking sup-norm (5.6) side by side we get

$$\|Bz\|_\infty \le \sup_{n\in\mathbb{N}}\sum_k |a_{nk}^{r,s,u,v,M}||y_k| \le \|y\|_\infty \sup_{n\in\mathbb{N}}\sum_k |a_{nk}^{r,s,u,v,M}| < \infty.$$

Thus, $Bz \in l_\infty$, hence $B \in (b_c^{r,s}(\mathcal{M},\mathcal{G}) : l_\infty)$. □

Theorem 5.2. *$B \in (b_c^{r,s}(\mathcal{M},\mathcal{G}) : c)$ iff the statements (5.1) -(5.4) hold, and*

$$\lim_{n\to\infty}\sum_k a_{nk}^{r,s,u,v,M} = \mu, \tag{5.7}$$

$$\lim_{n\to\infty} a_{nk}^{r,s,u,v,M} = \mu_k,\ \textit{for all}\ \ k \in \mathbb{N}. \tag{5.8}$$

Proof. Take $B \in (b_c^{r,s}(\mathcal{M},\mathcal{G}) : c)$. We all identify a well-known inclusion $c \subset l_\infty$. On merging the above fact and Theorem 5.1, we get (5.1)-(5.2) hold. Bz exists and belongs to c for all $z = (z_k) \in b_c^{r,s}(\mathcal{M},\mathcal{G})$. According to this if we choose two sequence $z = \left(\frac{1-(-\frac{v}{u})^{k+1}}{u+v}\right) \in b_c^{r,s}(\mathcal{M},\mathcal{G})$, taking $\mathcal{M} = (M_k) = (1,1,1.......)$ and $z = a_{r,s,u,v,M}^{(k)}$. We (5.7) and (5.8), where $z = a_{r,s,u,v,M}^{(k)}$ is already defined in the Theorem 4.1.

Conversly, let us assume that the statements (5.1)-(5.4), (5.7) and (5.8) hold, for given $z = (z_k) \in b_c^{r,s}(\mathcal{M},\mathcal{G})$. Then by reflecting Theorem 4.2 (ii), one can say that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{b_c^{r,s}(\mathcal{M},\mathcal{G})\}^\beta$ for all $n \in \mathbb{N}$. Which shows that Bz exists. From the statements (5.1) and (5.8) we realize that

$$\sum_k^m \le \sup_{n\in\mathbb{N}}\sum_k |a_{nk}^{r,s,u,v,M}| < \infty$$

for every $m \in \mathbb{N}$. This shows that $\mu_k \in l_1$. So that the series $\sum_k \mu_k y_k$ absolute converges. Now we additionally substitute $b_{nk} - \mu_k$ instead of b_{nk} in the statement (5.6). Then we have

$$\sum_k (b_{nk}-\mu_k)z_k = \sum_{k=0}^{n} M_k\Big[\frac{1}{u}\sum_{i=k}^{\infty}\sum_{j=i}^{k}\begin{pmatrix} j \\ i \end{pmatrix}\begin{pmatrix} -\frac{v}{u} \end{pmatrix}^{k-j}(-s)^{j-i}(r+s)^i r^{-j}(b_{ni}-\mu_k)y_k\Big], \tag{5.9}$$

$\forall\ n \in \mathbb{N}$. By merging (5.9) and Lemma 4.3 (vi) we conclude that

$$\lim_{n\to\infty}(b_{nk} - \mu_k)z_k = 0 \tag{5.10}$$

Last of all, if we merge the statements (5.10) and the fact $(\mu_k y_k) \in l_1$, we conclude that $bz \in c$ or we can say $B \in (b_c^{r,s}(\mathcal{M},\mathcal{G}) : c)$. □

Corollary 5.1. *Let us consider* $T = (t_{nk})$ *instead of* $B = (b_{nk})$ *in the necessity ones in Theorem 5.1 and 5.2, where* $T = (t_{nk})$ *is defined by*

$$t_{nk} = b_{nk} - b_{n+1,k}$$

$\forall\ n,k \in \mathbb{N}$. *Then, the necessary and sufficient conditions in order for* $B = b_{nk}$ *to belong to any one of the classes* $(b_c^{r,s}(\mathcal{M},\mathcal{G}) : l_\infty(\Delta))$ *and* $(b_c^{r,s}(\mathcal{M},\mathcal{G}) : c(\Delta))$ *are obtained.*

Corollary 5.2. *Let us consider* $Q = (q_{nk}^{\sigma,\varrho})$ *instead of* $B = (b_{nk})$ *in the necessity ones in Theorem 5.1 and 5.2, where* $Q = (q_{nk}^{\sigma,\varrho})$ *is given as*

$$q_{nk}^{\sigma,\varrho} = \frac{1}{(\sigma+\varrho)^n}\sum_{j=0}^{n}\binom{n}{j}\varrho^{n-j}\sigma^j b_{jk}$$

$\forall\ n,k \in \mathbb{N}$ *where* $\sigma, \varrho \in \mathbb{R}$ *and* $\sigma, \varrho > 0$. *Then, the necessary and sufficient conditions in order for* $B = (b_{nk})$ *to belong to any one of the classes* $(b_c^{r,s}(\mathcal{M},\mathcal{G}) : b_\infty^{\sigma,\varrho})$ *and* $(b_c^{r,s}(\mathcal{M},\mathcal{G}) : b_c^{\sigma,\varrho})$ *are obtained.*

Corollary 5.3. *Let us consider* $W = (w_{nk})$ *instead of* $B = (b_{nk})$ *in the necessity ones in Theorem 5.1 and 5.2, where* $W = (w_{nk})$ *is given as*

$$w_{nk} = \sum_{j=0}^{n} b_{jk}$$

$\forall\ n,k \in \mathbb{N}$. *Then, the necessary and sufficient conditions in order for* $B = b_{nk}$ *to belong to any one of the classes* $(b_c^{r,s}(\mathcal{M},\mathcal{G}) : bs$ *and* $(b_c^{r,s}(\mathcal{M},\mathcal{G}) : cs$ *are obtained.*

6. Conclusion

From many years, great deal of work has been done on Orlicz function, double band matrix and binomial matrix. Various kind of interesting results have been studied by many researchers. In this article we intend to extend the study of Orlicz sequence spaces by the composition of both double band matrix and binomial matrix. We introduced three new sequence space $b_0^{r,s}(\mathcal{M},\mathcal{G})$, $b_c^{r,s}(\mathcal{M},\mathcal{G})$ and $b_\infty^{r,s}(\mathcal{M},\mathcal{G})$ and proved several interesting results like Schauder basis, α, β, γ duals and some inclusion relation relations. As a natural continuation of this paper, in future we will study these results with the help of Nörlund fractional difference sequence spaces.

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References

[1] B. Altay, F. Basar, On Some Euler Sequence Spaces of Nonabsolute Type, Ukr. Math. J. 57 (2005), 1–17. https://doi.org/10.1007/s11253-005-0168-9.

[2] B. Altay, F. Başar, M. Mursaleen, On the Euler Sequence Spaces Which Include the Spaces ℓ_p and ℓ_∞, Inf. Sci. 176 (2006), 1450–1462. https://doi.org/10.1016/j.ins.2005.05.008.

[3] B. Altay, H. Polat, On Some New Euler Difference Sequence Spaces, Southeast Asian Bull. Math. 30 (2006), 209–220.

[4] M.C. Bişgin, The Binomial Sequence Spaces of Nonabsolute Type, J. Ineq. Appl. 2016 (2016), 309. https://doi.org/10.1186/s13660-016-1256-0.

[5] M.C. Bişgin, The Binomial Almost Convergent and Null Sequence Spaces, Comm. Fac. Sci. Univ. Ankara Ser. A1. Math. Stat. 67 (2018), 211–224. https://doi.org/10.1501/commua1_0000000843.

[6] M.C. Bişgin, A. Sönmez, Two New Sequence Spaces Generated by the Composition of M Th Order Generalized Difference Matrix and Lambda Matrix, J. Ineq. Appl. 2014 (2014), 274. https://doi.org/10.1186/1029-242x-2014-274.

[7] M.C. Bişgin, Some Notes on the Sequence Spaces $l_p^\lambda(G^m)$ and $l_\infty^\lambda(G^m)$, GU J. Sci. 30 (2017), 381–393.

[8] F. Başar, B. Altay, On the Space of Sequences of P-Bounded Variation and Related Matrix Mappings, Ukr. Math. J. 55 (2003), 136–147. https://doi.org/10.1023/a:1025080820961.

[9] B. Choudhary, S. Nanda, Functional Analysis with Applications, Wiley, 1989.

[10] M. Et, On Some Difference Sequence Spaces, Turk. J. Math. 17 (1993), 18–24.

[11] E.E. Kara, M. Başarir, On Compact Operators and Some Euler $B^{(m)}$-Difference Sequence Spaces, J. Math. Anal. Appl. 379 (2011), 499–511. https://doi.org/10.1016/j.jmaa.2011.01.028.

[12] V. Karakaya, H. Polat, Some New Paranormed Sequence Spaces Defined by Euler and Difference Operators, Acta Sci. Math. 76 (2010), 87–100. https://doi.org/10.1007/bf03549822.

[13] M. Kirişçi, F. Başar, Some New Sequence Spaces Derived by the Domain of Generalized Difference Matrix, Comp. Math. Appl. 60 (2010), 1299–1309. https://doi.org/10.1016/j.camwa.2010.06.010.

[14] H. Kizmaz, On Certain Sequence Spaces, Can. Math. Bull. 24 (1981), 169–176. https://doi.org/10.4153/cmb-1981-027-5.

[15] J. Lindenstrauss, L. Tzafriri, On Orlicz Sequence Spaces, Israel J. Math. 10 (1971), 379–390. https://doi.org/10.1007/bf02771656.

[16] J. Meng, M. Song, Binomial Difference Sequence Spaces of Order m, Adv. Diff. Equ. 2017 (2017), 241. https://doi.org/10.1186/s13662-017-1291-2.

[17] P.N. Ng, P.Y. Lee, Cesàro Sequence Spaces of Non–absolute Type, Comment. Math. (Prace Mat.) 20 (1978), 429–433.

[18] H. Polat, F. Başar, Some Euler Spaces of Difference Sequences of Order m, Acta Math. Sci. 27 (2007), 254–266. https://doi.org/10.1016/s0252-9602(07)60024-1.

[19] M. Stieglitz, H. Tietz, Matrix Transformationen von Folgenräumen eine Ergebnisübersicht, Math. Z. 154 (1977), 1–16. https://doi.org/10.1007/bf01215107.

[20] A. Sönmez, Some New Sequence Spaces Derived by the Composition of Binomial Matrix and Double Band Matrix, J. Appl. Anal. Comp. 9 (2019), 231–244. https://doi.org/10.11948/2019.231.

[21] C.S. Wang, On Nörlund Sequence Spaces. Tamkang J. Math. 9 (1978), 269–274.

[22] A. Wilansky, Summability Throught Functional Analysis, North-Holland Mathematics Studies, Elsevier, 1984.