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Fixed Point Results in Complex Valued Modified Intuitionistic Fuzzy Metric Space with Applications

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Abstract. The aims is to introduce complex-valued modified intuitionistic fuzzy metric spaces as a fresh perspective on complex-valued FMS and modified intuitionistic fuzzy metric spaces. Additionally, our work yields a fixed and common fixed point result on this newly introduced space. Our research outcomes are exemplified through examples that are included in this paper to help readers better grasp our findings. Our paper concludes with a discussion of how our findings can be applied to the problem of determining the existence of a unique solution for Fredholm integral equations.

1. Introduction

Fixed point theory is a powerful tool in mathematical analysis that has a wide range of applications. Ambiguous and vague situations in natural phenomena or real-life problems cannot always be expressed by mathematical models using classical set theory. To tackle this issue, Zadeh [31] established the notion of fuzzy set, where the membership of an element in a set is indicated by assigning it a value from the interval [0, 1].

Later, Atanassov [2] proposed intuitionistic fuzzy sets, which allow for the representation of degree of uncertainty when assigning membership and non-membership values to elements in a set. In 1975, Kramosil and Michalek [20] put forth FMS as a way to extend probabilistic metric spaces. The investigation into fuzzy metric FP theory was pioneered by Grabiec [11]. George and Veeramani [8] altered FMS in 1994, which resulted in the emergence of a Hausdorff topology

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on such spaces. They also proposed modifications to Grabiec's Cauchy sequence concept and demonstrated various FP outcomes on the modified spaces.

In 2004, Park [24] put forward the framework of IFMS, which expanded the scope of FMS. Since FMS came into existence, it has been generalized in several ways (see [5,6,21–23,25–29]). To explore the studies regarding FP findings in such spaces, [1,9,10] and their cited sources provide a good starting point. Complex-valued MS were brought into metric FP theory by Azam et al. [3] in 2011. After that, MIFMS introduced by Sedghi [25] and common coupled FP theorem proved by Gupta et al. [13]. While [12] advanced the theoretical foundations of fuzzy metric spaces, [19] subsequently demonstrated their economic applications, particularly in profit dynamics analysis.

Recently, Shukla et al. [28] employed this concept in the realm of FMS theory, there has been considerable research interest in exploring FP findings for contractions on CVFMS. Prominent examples of such research include the works of [7,30] and the extensive investigations conducted by Humaira et al. [14–18], which derived numerous FP and CFP outcomes, as well as their practical applications.

This manuscript presents a fixed point results in CVMIFMS with applications. This new concept generalizes both CVFMS by [28] as well as MIFMS by [25]. We present some FP outcomes for mappings subject to contractive constraints in newly defined spaces. Furthermore, we expand the fuzzy variant of BCP to intuitionistic fuzzy spaces, establishing CFP outcomes for it within CVMIFMS. We provide pratical examples and applications to demonstrate the usefulness and relevance of our results.

2. Preliminary

In the present work, the notations $\mathbb{N}$ and $\mathbb{C}$ refer to, in order, the collection of natural numbers and complex numbers. For every $z \in \mathbb{C}$, we express $z = \mathfrak{k} + iv$ by $(\mathfrak{k}, v)$, where $\mathfrak{k}$ is the real part and v is the imaginary part. Let $\mathcal{P} = \{(\mathfrak{k}, v) : 0 \leq \mathfrak{k} < \infty, 0 \leq v < \infty\} \subset \mathbb{C}$.
For $(0,0)$ and $(1,1)$ in $\mathbb{C}$, we denote them as θ and ℓ, respectively. We denote the closed unit complex interval as $I = \{(\mathfrak{k}, v) : 0 \leq \mathfrak{k} \leq 1, 0 \leq v \leq 1\}$, as well as the open unit complex interval as $I_0 = \{(\mathfrak{k}, v) : 0 < \mathfrak{k} < 1, 0 < v < 1\}$. Furthermore, $\mathcal{P}_0$ is designated as $\{(\mathfrak{k}, v) : 0 < \mathfrak{k} < \infty, 0 < v < \infty\}$.

A partial order $\preceq$ is imposed on $\mathbb{C}$, where $f_1 \preceq f_2$ if and only if $f_2 - f_1 \in \mathcal{P}$, where $f_1, f_2 \in \mathbb{C}$. We write $f_1 \prec f_2$ to express $\mathrm{Re}(f_1) < \mathrm{Re}(f_2)$ and $\mathrm{Im}(f_1) < \mathrm{Im}(f_2)$. Clearly, we have $f_1 \prec f_2$ implies and is implied by $f_2 - f_1 \in \mathcal{P}_0$. Let $\{f_n\}$ be a sequence in $\mathbb{C}$. When $f_n \preceq c_{n+1}$ or $c_{n+1} \preceq f_n$ holds for each $n \in \mathbb{N}$, the sequence $\{f_n\}$ is termed monotonic with respect to $\preceq$.

In the context of a subset K of $\mathbb{C}$, an element $\inf K \in \mathbb{C}$ is known as the infimum or greatest lower bound of K provided that it acts as a lower bound of K, meaning that $\inf K \preceq k$ for all $k \in K$ along with $\ell \preceq \inf K$ for any other lower bound ℓ of K. We introduce $\sup K$ in a similar way as the supremum or the least upper bound of K.

Remark 2.1. *[25] Given that $f_n \in \mathcal{P}$ for every $n \in \mathbb{N}$, the following statements hold:*

(1) *If $\{f_n\}$ is a monotonic sequence w.r.t $\preceq$ and there exist $\alpha, \beta \in \mathcal{P}$ such that $\alpha \preceq f_n \preceq \beta$, then a limit $c \in \mathcal{P}$ exists and $f_n \to c$ as $n \to \infty$.*
(2) *Although $\preceq$ does not provide a total ordering on $\mathbb{C}$, it creates a lattice structure on $\mathbb{C}$.*
(3) *For $K \subset \mathbb{C}$, if every $k \in K$ satisfies $\alpha \preceq k \preceq \beta$ for some $\alpha, \beta \in \mathbb{C}$, it follows that $\inf K$ and $\sup K$ exist.*

Remark 2.2. *[25] Given that $f_n, f'_n \in \mathcal{P}$ for all $n \in \mathbb{N}$, the following statements hold:*

(1) *If for every $n \in \mathbb{N}$, we have $f_n \preceq f'_n \preceq \ell$ and $\lim_{n\to\infty} f_n = \ell$, then it follows that $f'_n = \ell$.*
(2) *If $f_n \preceq y$ and $\lim_{n\to\infty} f_n = c \in \mathcal{P}$, then $c \preceq y$.*

Definition 2.1. *[25] A binary operation $*$: $I \times I \to I$ (I=closed unit complex interval) is referred to as a complex-valued t-norm when it satisfies the following conditions:*

(1) *$c * \theta = \theta$, $c * \ell = c$ for every $c \in I$;*
(2) *$*$ is associative and commutative;*
(3) *$f_1 * f_2 \preceq f_3 * f_4$ whenever $f_1 \preceq f_3$, and $f_2 \preceq f_4$ for each $f_1, f_2, f_3, f_4 \in I$ where $f_1 = (\mathfrak{k}_1, v_1)$, $f_2 = (\mathfrak{k}_2, v_2)$, $f_3 = (\mathfrak{k}_3, v_3)$, $f_4 = (\mathfrak{k}_4, v_4)$.*
For example:
   (a) *$f_1 *_m f_2 = (\min\{\mathfrak{k}_1, \mathfrak{k}_2\}, \min\{v_1, v_2\})$;*
   (b) *$f_1 *_p f_2 = (\mathfrak{k}_1\mathfrak{k}_2, v_1 v_2)$;*
   (c) *$f_1 *_L f_2 = (\max\{\mathfrak{k}_1 + \mathfrak{k}_2 - 1, 0\}, \max\{v_1 + v_2 - 1, 0\})$.*

Definition 2.2. *[25] Suppose $Z \neq \phi$, $*$ is a continuous complex-valued t-norm, and $\mathfrak{U}$ is a complex fuzzy set defined on $Z^2 \times \mathcal{P}_0$ where as the following conditions hold:*

(1) *$\mathfrak{U}(\varpi, \nu, \bar{\xi}) \succ \theta$;*
(2) *$\mathfrak{U}(\varpi, \nu, \bar{\xi}) = \mathfrak{U}(\nu, \varpi, \bar{\xi})$;*
(3) *$\mathfrak{U}(\varpi, \nu, \bar{\xi}) = \ell$ for every $c \in \mathcal{P}_0$ if and only if $\varpi = \nu$;*
(4) *$\mathfrak{U}(\varpi, \sigma, \bar{\xi} + \bar{\xi}') \succeq \mathfrak{U}(\varpi, \nu, \bar{\xi}) * \mathfrak{U}(\nu, \sigma, \bar{\xi}')$;*
(5) *$\mathfrak{U}(\varpi, \nu, \cdot) : \mathcal{P}_0 \to I$ is continuous, for every $\varpi, \nu, \sigma \in Z$ and $\bar{\xi}, \bar{\xi}' \in \mathcal{P}_0$.*

Then, $(Z, \mathfrak{U}, *)$ is referred to as a CVFMS where $\mathfrak{U}$ characterizes the degree of nearness between two points of the set Z relative to a complex parameter $c \in \mathcal{P}_0$.

Lemma 2.1. *[25] Consider the set I^* and operation $\preceq I^*$ defined by*

$$I^* = \{(\mathfrak{k}, v) : 0 \leq |\mathfrak{k}| \leq 1, 0 \leq |v| \leq 1 \text{ and } |\mathfrak{k} + v| \leq 1\}$$

$(\mathfrak{k}_1, \mathfrak{k}_2) \preceq_{I^*} (v_1, v_2)$ if and only if $\mathfrak{k}_1 \preceq v_1$ and $\mathfrak{k}_2 \succeq v_2$, for every $(\mathfrak{k}_1, \mathfrak{k}_2), (v_1, v_2) \in I^*$. Then $(I^*, \preceq_{I^*})$ is a complete lattice.

Definition 2.3. *An IFS $A_{f,g}$ in a universe U is an object $A_{f,g} = \{(f_A(u), g_A(u)) : u \in U\}$, where $f_A(u)$ and $g_A(u) \in [0, 1]$ for all $u \in U$ are called the membership degree and non-membership degree respectively and furthermore they satisfy $f_A(u) + g_A(u) \leq 1$.*

Now, consider $z_i = (\varpi_i, \kappa_i) \in I^*$ and $c_i \in [0,1]$ such that $\sum_{j=1}^{n} c_j = 1$ then

$$f_1(\varpi_1, \kappa_1) + \cdots + f_n(\varpi_n, \kappa_n) = \left(\sum_{j=1}^{n} c_j \varpi_j, \sum_{j=1}^{n} c_j \kappa_j \right) \in I^*.$$

We represent its units as $0_{I^*} = (0,1)$ and $1_{I^*} = (1,0)$.

Definition 2.4. *[25] These definitions can be naturally extended using the lattice $(I^*, \leq_{I^*})$.*
(i) A triangular norm denoted as $*$ is a mapping that is increasing, associative, and commutative. This mapping is described by $T : [0,1]^2 \to [0,1]$, and it satisfies the condition $T(1, \varpi) = 1 * \varpi = \varpi$ for any $\varpi \in [0,1]$.
(ii) A triangular conorm denoted as $S = \oplus$ is a mapping that is increasing, commutative, and associative. This mapping is represented by $S : [0,1]^2 \to [0,1]$, and it follows the rule $S(0, \varpi) = 0 \oplus \varpi = \varpi$ for all $\varpi \in [0,1]$.

Definition 2.5. *[25] A t-norm on I^* is a mapping $R : \{I^*\}^2 \to I^*$ satisfying the following conditions:*

(1) $\forall\, \varpi \in I^*, \quad R(\varpi, 1_{I^*}) = \varpi$ *(boundary condition);*
(2) $\forall\, (\varpi, \kappa) \in I^* \times I^*, \quad R(\varpi, \kappa) = R(\kappa, \varpi)$ *(commutativity);*
(3) $\forall\, (\varpi, \kappa, z) \in I^* \times I^* \times I^*, \quad R(\varpi, R(\kappa, z)) = R(R(\varpi, \kappa), z)$ *(associativity);*
(4) $\forall\, (\varpi, \varpi_0, \kappa, \kappa_0) \in I^* \times I^* \times I^* \times I^*, \quad (\varpi \leq_{I^*} \varpi_0)$ *and* $(\kappa \leq_{I^*} \kappa_0 \Rightarrow R(\varpi, \kappa) \leq_{I^*} R(\varpi_0, \kappa_0))$ *(monotonicity).*

Definition 2.6. *[25] A mapping R defined on the interval I^* is termed continuously t-representable if and only if there exists both a continuous t-norm $*$ and a continuous t-conorm $\oplus$ defined on the closed interval $[0,1]$ such that, for all $\varpi = (\varpi_1, \varpi_2), \kappa = (\kappa_1, \kappa_2) \in I^*$, and*

$$R(\varpi, \kappa) = (\varpi_1 * \kappa_1, \varpi_2 \oplus \kappa_2).$$

Define a sequence R^n recursively in this way $R^1 = R$ and

$$R^n(\varpi^{(1)}; \ldots; \varpi^{(n+1)}) = R(R^{n-1}(\varpi^{(1)}, \ldots, \varpi^{(n)}), \varpi^{(n+1)})$$

for $n \geq 2$ and $\varpi^{(i)} \in I^*$.

Definition 2.7. *[25] An operator acting on I^* is considered a negator if it is a decreasing function denoted by $N : I^* \to I^*$, and it adheres to the conditions $N(0_{I^*}) = 1_{I^*}$ and $N(1_{I^*}) = 0_{I^*}$. If, for all $\varpi \in I^*$, the property $N(N(\varpi)) = \varpi$ holds, the negator N is termed an involutive negator.*
For the interval $[0,1]$, a negator refers to a decreasing function $N : [0,1] \to [0,1]$ that satisfies $N(0) = 1$ and $N(1) = 0$. The notation N_s is used to represent the standard negator on the interval $[0,1]$, which is defined as $N_s(\varpi) = 1 - \varpi$.

3. Main Result

Definition 3.1. *Let $\mathfrak{U}, \mathfrak{V}$ be fuzzy sets from $Z^2 \times \mathcal{P}_0$ to $[0,1]$ such that $\mathfrak{U}(\varpi, \kappa, \bar{\xi}) + \mathfrak{V}(\varpi, \kappa, \bar{\xi}) \leq 1$ and Z is a non-empty set, R is a continuous t-representable function, and $\pounds_{\mathfrak{U},\mathfrak{V}}$ is a mapping defined as $Z^2 \times \mathcal{P}_0 \to I^*$,* *where $\mathcal{P}_0$ is a specified domain. The 3-tuple $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$ is characterized as a CVMIFMS if the following conditions are met for all $\varpi, \kappa \in Z$ and $t, s > 0$*

(1) $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) \succ_{I^*} \theta_{I^*}$;
(2) $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = l_{I^*}$ *if and only if* $\varpi = \kappa$;
(3) $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = £_{\mathfrak{U},\mathfrak{V}}(\kappa, \varpi, \bar{\xi})$;
(4) $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi} + s) \succeq_{I^*} R(£_{\mathfrak{U},\mathfrak{V}}(\varpi, z, \bar{\xi}), £_{\mathfrak{U},\mathfrak{V}}(z, \kappa, s))$;
(5) $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \cdot) : \mathcal{P}_0 \to I^*$ *is continuous.*

In this case, $£_{\mathfrak{U},\mathfrak{V}}$ is called an CVMIFMS, where $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = (\mathfrak{U}(\varpi, \kappa, \bar{\xi}), \mathfrak{V}(\varpi, \kappa, \bar{\xi}))$.

Example 3.1. *Consider a metric space (Z, d). We define a function $R(\mathfrak{k}, v) = \{\mathfrak{k}_1 v_1, \min(\mathfrak{k}_2 + v_2, 1)\}$ for all $\mathfrak{k} = (\mathfrak{k}_1, \mathfrak{k}_2)$ and $v = (v_1, v_2) \in I^*$. Moreover, let $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi})$ are fuzzy set on $Z^2 \times \mathcal{P}_0$ defined as follows:*

$$£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = (\mathfrak{U}(\varpi, \kappa, \bar{\xi}), \mathfrak{V}(\varpi, \kappa, \bar{\xi})) = \left(\frac{h\bar{\xi}^n}{h\bar{\xi}^n + md(\varpi, \kappa)}, \frac{md(\varpi, \kappa)}{h\bar{\xi}^n + md(\varpi, \kappa)}\right),$$

for all $\bar{\xi}, h, m, n \in \mathcal{P}_0$. Then $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$ is an CVMIFMS.

Example 3.2. *Let $Z = \mathbb{N}$. The function $R(\mathfrak{k}, v) = \{\max(0, \mathfrak{k}_1 + v_1 - 1), \mathfrak{k}_2 + v_2 - \mathfrak{k}_2 v_2\}$ is defined for all $\mathfrak{k} = (\mathfrak{k}_1, \mathfrak{k}_2)$ and $v = (v_1, v_2) \in I^*$. Additionally, consider fuzzy set $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi})$ on $Z^2 \times \mathcal{P}_0$, which are defined as follows:*

$$£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = (\mathfrak{U}(\varpi, \kappa, \bar{\xi}), \mathfrak{V}(\varpi, \kappa, \bar{\xi})) = \begin{cases} \left(\frac{\varpi}{\kappa}, \frac{\kappa - \varpi}{\kappa}\right) \text{ if } \varpi \leq \kappa, \\ \left(\frac{\kappa}{\varpi}, \frac{\varpi - \kappa}{\varpi}\right) \text{ if } \kappa \leq \varpi, \end{cases}$$

for all $\varpi, \kappa \in Z$ and $\bar{\xi} > 0$. Then $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$ is an CVMIFMS.

Remark 3.1. *In a CVMFMS $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$, where $\mathfrak{U}(\varpi, \kappa, \bar{\xi})$ is non decreasing and $\mathfrak{V}(\varpi, \kappa, \bar{\xi})$ is non increasing then*

$$£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) \preceq_{I^*} R(\mathfrak{U}(\varpi, \kappa, \bar{\xi}), \mathfrak{V}(\varpi, \kappa, \bar{\xi}))$$

is non decreasing for all $\varpi, \kappa \in Z$ where

$$R(\mathfrak{k}, v) = (\mathfrak{k}_1 * v_1, \mathfrak{k}_2 \oplus v_2) \text{ and } \mathfrak{k} = (\mathfrak{k}_1, v_1), v = (\mathfrak{k}_2, v_2) \in I.$$

Definition 3.2. *In a CVMIFMS $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$, a sequence $\{\varpi_n\}$ is termed a Cauchy sequence if, for each $0 < e < 1$ and $\bar{\xi} > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$,*

$$£_{\mathfrak{U},\mathfrak{V}}(\varpi_n, \kappa_m, \bar{\xi}) \succ_{I^*} (N_s(e), e),$$

where N_s denotes the standard negator.

Definition 3.3. *In the CVMIFMS $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$, The sequence $\{\varpi_n\}$ is considered convergent to $\varpi \in Z$ if $£_{\mathfrak{U},\mathfrak{V}}(\varpi_n, \varpi, \bar{\xi}) \to 1_{I^*}$ as $n \to \infty$ for all $\bar{\xi} > 0$,denoted by $\varpi_n \overset{£_{\mathfrak{U},\mathfrak{V}}}{\to} \varpi$.*

Definition 3.4. *A CVMIFMS is termed complete if every Cauchy sequence within it converges.*

Lemma 3.1. *Consider $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$ as a CVMIFMS. In this context, for any $\bar{\xi} \succ 0$, the function $£_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi})$ exhibits a non-decreasing behavior with respect to $\bar{\xi}$ within the lattice $(I^*, \preceq_{I^*})$, for all $\varpi, \kappa \in Z$.*

Definition 3.5. *Let $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$ be a CVMIFMS. For any $\bar{\xi} \succ 0$, we define an open ball $B(\varpi, e, \bar{\xi})$ centered at $\varpi \in Z$ with a radius $0 < e < 1$ as*

$$B(\varpi, e, \bar{\xi}) = \{\kappa \in Z : £_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}) \succ_{I^*} (N_s(e), e)\}.$$

A subset $A \subseteq Z$ is called open if for each $\varpi \in A$, there exist $\bar{\xi} \succ 0$, $0 < e < 1$ such that $B(\varpi, e, \bar{\xi}) \subseteq A$.

Definition 3.6. *Let $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$ be an CVMIFMS. A subset A of Z is considered CVMIF-bounded if there exist $\bar{\xi} \succ 0$ and $0 < e < 1$ such that*

$$£_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}) \succeq_{I^*} (N_s(e), e) \text{ for every } \varpi, \kappa \in A.$$

Definition 3.7. *Let $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$ be an CVMIFMS. $£_{\mathfrak{U},\mathfrak{B}}$ is said to be continuous on $Z \times Z \times \mathcal{P}_0$ if*

$$\lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(\varpi_n, \kappa_n, \bar{\xi}_n) = £_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}),$$

whenever a sequence $\{(\varpi_n, \kappa_n, \bar{\xi}_n)\}$ in $Z \times Z \times \mathcal{P}_0$ converges to a point $(\varpi, \kappa, \bar{\xi}) \in Z \times Z \times \mathcal{P}_0$, i.e.,

$$\lim_n £_{\mathfrak{U},\mathfrak{B}}(\varpi_n, \varpi, \bar{\xi}) = \lim_n £_{\mathfrak{U},\mathfrak{B}}(\kappa_n, \kappa, \bar{\xi}) = 1_{I^*} \quad \text{and} \quad \lim_n £_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa_n, \bar{\xi}) = £_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}).$$

Lemma 3.2. *Let $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$ be an CVMIFMS. Then $£_{\mathfrak{U},\mathfrak{B}}$ is a continuous function on $Z \times Z \times \mathcal{P}_0$.*

Definition 3.8. *Let A and S be mappings from $(Z, £_{\mathfrak{U},\mathfrak{B}}, R) \to (Z, £_{\mathfrak{U},\mathfrak{B}}, R)$. The mappings are considered weak compatible if their coincidence point satisfies the condition that when $A\varpi = S\varpi$, implies that $AS\varpi = SA\varpi$.*

Definition 3.9. *Let A and S be mappings from $(Z, £_{\mathfrak{U},\mathfrak{B}}, R) \to (Z, £_{\mathfrak{U},\mathfrak{B}}, R)$ are said to be compatible if*

$$\lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(AS\varpi_n, SA\varpi_n, \bar{\xi}) = 1_{I^*} \quad \text{for all } \bar{\xi} \succ 0.$$

Whenever $\{\varpi_n\}$ in Z such that

$$\lim_{n\to\infty} A\varpi_n = \lim_{n\to\infty} S\varpi_n = \varpi \in Z.$$

Proposition 3.1. *In a CVMIFMS $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$, the mappings A and S are compatible, so they are weak compatible.*

The converse is not true, see the next example.

Example 3.3. *Let $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$ be an CVMIFMS, where $Z = [0, 2]$, and*

$$£_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}) = \left\{ \frac{\bar{\xi}}{\bar{\xi} + d(\varpi, \kappa)}, \frac{d(\varpi, \kappa)}{\bar{\xi} + d(\varpi, \kappa)} \right\}$$

for all $\bar{\xi} \succ 0$ and $\varpi, \kappa \in Z$. Denote $R(\mathfrak{k}, v) = (\mathfrak{k}_1 v_1, \min(\mathfrak{k}_2 + v_2, 1))$ for all $\mathfrak{k} = (\mathfrak{k}_1, \mathfrak{k}_2)$ and $v = (v_1, v_2) \in I^*$. Specify the mappings A and S on the set Z as follows:

$$A(\varpi) = \begin{cases} \frac{\varpi}{2}, & \text{if } 1 \le \varpi \le 2; \\ 2, & \text{if } 0 \le \varpi \le 1, \end{cases} \quad \text{and } S(\varpi) = \begin{cases} 2, & \text{if } \varpi = 1; \\ \frac{\varpi+3}{5}, & \text{otherwise.} \end{cases}$$

So, the value of $S(1) = A(1) = 2$ *and* $S(2) = A(2) = 1$. *Also, the value* $SA(1) = AS(1) = 1$ *and* $SA(2) = AS(2) = 2$.

Therefore, (A, S) *is becomes weak compatible. Again,*

$$A(\varpi_n) = 1 - \frac{1}{4n}, \quad S(\varpi_n) = 1 - \frac{1}{10n}.$$

Thus,

$$A(\varpi_n) \to 1, \quad S(\varpi_n) \to 1.$$

Further,

$$SA(\varpi_n) = \left(\frac{4}{5} - \frac{1}{20n}\right), \quad AS(\varpi_n) = 2.$$

Now,

$$\lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(AS(\varpi_n), SA(\varpi_n), \bar{\xi}) = \lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}\left(2, \frac{4}{5} - \frac{1}{20n}, \bar{\xi}\right)$$
$$= \left(\frac{\bar{\xi}}{\bar{\xi} + \frac{6}{5}}, \frac{\frac{6}{5}}{\bar{\xi} + \frac{6}{5}}\right) \prec_{I^*} 1_{I^*} \quad \text{for all } \bar{\xi} > 0.$$

Hence, the compatibility of (A, S) *is not established.*

Definition 3.10. *Assume that A and B are two self-mappings of the CVMIFMS* $(Z, £_{\mathfrak{U},\mathfrak{B}}, R)$. *We say that A and B meet the property (E), if there exists a sequence* $\{\varpi_n\}$ *such that*

$$\lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(A\varpi_n, z, \bar{\xi}) = \lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(B\varpi_n, z, \bar{\xi}) = 1_{I^*}$$

for some $z \in Z$ *and* $\bar{\xi} > 0$.

Example 3.4. *Let* $Z = R$ *and*

$$£_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}) = \left\{\frac{\bar{\xi}}{\bar{\xi} + |\varpi - \kappa|}, \frac{|\varpi - \kappa|}{\bar{\xi} + |\varpi - \kappa|}\right\}.$$

Let A and B be defined as,

$$A(\varpi) = 2\varpi + 1, \quad B(\varpi) = \varpi + 2.$$

And the sequence

$$\{\varpi_n\} = \left\{\frac{1}{n} + 1\right\}, n = 1, 2, \ldots$$

Thus we have,

$$\lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(A\varpi_n, 3, \bar{\xi}) = \lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(B\varpi_n, 3, \bar{\xi}) = 1_{I^*}.$$

Thus, A and B exhibit property (E).

Example 3.5. *Let* $Z = R$ *and*

$$£_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}) = \left\{\frac{\bar{\xi}}{\bar{\xi} + |\varpi - \kappa|}, \frac{|\varpi - \kappa|}{\bar{\xi} + |\varpi - \kappa|}\right\}.$$

For every $\varpi, \kappa \in Z$ *and* $\bar{\xi} > 0$, *let* $A\varpi = \varpi + 1$ *and* $B\varpi = \varpi + 2$.

If a sequence $\{\varpi_n\}$ *exists such that,* $\lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(A\varpi_n, z, \bar{\xi}) = \lim_{n\to\infty} £_{\mathfrak{U},\mathfrak{B}}(B\varpi_n, z, \bar{\xi}) = 1_{I^*}$ *for some* $z \in Z$, *therefore,*

$\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(A\varpi_n, z, \bar{\xi}) = \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_n + 1, z, \bar{\xi}) = \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_n, z-1, \bar{\xi}) = 1_{I^*}$.

$\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(B\varpi_n, z, \bar{\xi}) = \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_n + 2, z, \bar{\xi}) = \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_n, z-2, \bar{\xi}) = 1_{I^*}$.

Hence, as ϖ_n converges to $z-1$ and $z-2$, a contradiction arises. Consequently, it can be concluded that neither A nor B possesses property (E).

Lemma 3.3. *Let $(Z, \pounds_{\mathfrak{U},\mathfrak{B}}, R)$ be an CVMIFMS. Define $E_{e,\mu} : Z^2 \to \mathcal{P}$ by*

$$E_{e,\mu}(\varpi, \kappa) = \inf\left\{\bar{\xi} \succ 0 : \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi, \kappa, \bar{\xi}) \succ 1_{I^*}\,(Ns(e), e)\right\},$$

for every $0 < e < 1$ and $\varpi, \kappa \in Z$. Then,

(1) *For any $\mu \in (0,1)$, there exists e such that*

$E_{e,\mu}(\varpi_1, \varpi_n) \preceq E_{e,\mu}(\varpi_1, \varpi_2) + E_{e,\mu}(\varpi_2, \varpi_3) + \ldots + E_{e,\mu}(\varpi_{n-1}, \varpi_n)$*for any*$\varpi_1, \ldots, \varpi_n \in Z$

.

(2) *In the CVMIFMS a sequence $\{\varpi_n\}_{n\in N}$ is converges if and only if $E_{e,\mu}(\varpi_n, \varpi) \to 0$.*

Also, the sequence $\{\varpi_n\}_{n\in N}$ is a Cauchy sequence iff it is Cauchy w.r.t $E_{e,\mu}$.

Lemma 3.4. *Let $(Z, \pounds_{\mathfrak{U},\mathfrak{B}}, R)$ be an CVMIFMS. If*

$$\pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_n, \varpi_{n+1}, \bar{\xi}) \succeq_{I^*} \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_0, \varpi_1, k^n\bar{\xi})$$

then $\{\varpi_n\}$ is a Cauchy sequence for some $k > 1$ and $n \in N$.

Proof. For every $e \in (0,1)$ and $\{\varpi_n\} \in Z$, we have

$E_{e,\mu}(\varpi_{n+1}, \varpi_n) = \inf\left\{\bar{\xi} \succ 0 : \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_{n+1}, \varpi_n, \bar{\xi}) \succ_{I^*} (Ns(e), e)\right\}$

$\preceq \inf\left\{\bar{\xi} \succ 0 : \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_0, \varpi_1, k^n\bar{\xi}) \succ_{I^*} (Ns(e), e)\right\}$

$= \inf\left\{\frac{\bar{\xi}}{k^n} : \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_0, \varpi_1, \bar{\xi}) \succ_{I^*} (Ns(e), e)\right\}$

$= \frac{1}{k^n} \inf\left\{\bar{\xi} \succ 0 : \pounds_{\mathfrak{U},\mathfrak{B}}(\varpi_0, \varpi_1, \bar{\xi}) \succ_{I^*} (Ns(e), e)\right\}$

$= \frac{1}{k^n} E_{e,\mu}(\varpi_0, \varpi_1)$.

From Lemma above 3.3, for every $\mu \in (0,1)$ there exists e such that

$E_{e,\mu}(\varpi_n, \varpi_m) \preceq E_{e,\mu}(\varpi_n, \varpi_{n+1}) + E_{e,\mu}(\varpi_{n+1}, \varpi_{n+2}) + \ldots + E_{e,\mu}(\varpi_{m-1}, \varpi_m)$

$\preceq \frac{1}{k^n} E_{e,\mu}(\varpi_0, \varpi_1) + \frac{1}{k^{n+1}} E_{e,\mu}(\varpi_0, \varpi_1) + ... + \frac{1}{k^{m-1}} E_{e,\mu}(\varpi_0, \varpi_1)$.

$= E_{e,\mu}(\varpi_0, \varpi_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \to 0$.

Hence, sequence $\{\varpi_n\}$ is a Cauchy sequence. □

Definition 3.11. *Let $(Z, \pounds_{\mathfrak{U},\mathfrak{V}}, R)$ be a CVMIFMS have property (C)If it fulfills the subsequent condition:*

$$\pounds_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = C \quad \textit{for all } \bar{\xi} > 0 \textit{ implies } C = 1_{I^*}.$$

Theorem 3.1. *Let $(Z, \pounds_{\mathfrak{U},\mathfrak{V}}, R)$ be a complete CVMIFMS with property (C) , and let S and T represent two self-mappings of the set Z fulfills the subsequent condition:*

(1) $\pounds_{\mathfrak{U},\mathfrak{V}}(S\varpi, TS\kappa, \bar{\xi}) \geq \gamma(\pounds_{\mathfrak{U},\mathfrak{V}}(\varpi, S\kappa, k\bar{\xi}))$ *for some $k > 1$, where $\gamma : I^* \to I^*$ is an operator with $\gamma(\mathfrak{k}) \geq \mathfrak{k}$ for each $\mathfrak{k} \in I^*$.*

(2) *T or S is continuous.*

Then, the map S and T have a unique CFP.

Proof. Let $\varpi_0 \in Z$ be any arbitrary point, and define

$$\varpi_{2n} = S\varpi_{2n-1}, \quad n = 1, 2,$$

$$\varpi_{2n+1} = T\varpi_{2n}, \quad n = 0, 1, 2, .$$

Now, for an even integer $n = 2m$, we have

$$\pounds_{\mathfrak{U},\mathfrak{V}}(\varpi_{2m}, \varpi_{2m+1}, \bar{\xi}) = \pounds_{\mathfrak{U},\mathfrak{V}}(S\varpi_{2m-1}, T\varpi_{2m}, \bar{\xi})$$

$$= \pounds_{\mathfrak{U},\mathfrak{V}}(S\varpi_{2m-1}, TS\varpi_{2m-1}, \bar{\xi})$$

$$\geq \gamma(\pounds_{\mathfrak{U},\mathfrak{V}}(\varpi_{2m-1}, S\varpi_{2m-1}, k\bar{\xi}))$$

$$\geq \pounds_{\mathfrak{U},\mathfrak{V}}(\varpi_{2m-1}, \varpi_{2m}, k\bar{\xi})$$

$$= \pounds_{\mathfrak{U},\mathfrak{V}}(\varpi_{2m}, \varpi_{2m-1}, k\bar{\xi})...$$

$$\geq \pounds_{\mathfrak{U},\mathfrak{V}}(\varpi_0, \varpi_1, k^n\bar{\xi}).$$

By, using Lemma 3.4, the sequence $\{\varpi_n\}$ is a Cauchy sequence and completeness of Z, $\{\varpi_n\}$ converges to $\varpi \in Z$. Then,

$$\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} \varpi_{2n} = \lim_{n\to\infty} S\varpi_{2n-1} = S \lim_{n\to\infty} \varpi_{2n-1} = S\varpi = \varpi,$$
$$\lim_{n\to\infty} \varpi_{2n+1} = \lim_{n\to\infty} T\varpi_{2n} = \lim_{n\to\infty} \varpi_{2n} = \varpi.$$

Given that the map S is continuous. Then,

$$\lim_{n\to\infty} S\varpi_{2n-1} = S \lim_{n\to\infty} \varpi_{2n-1} = S\varpi = \varpi.$$

Since, $\pounds_{\mathfrak{U},\mathfrak{V}}(\varpi, T\varpi, \bar{\xi}) = \pounds_{\mathfrak{U},\mathfrak{V}}(S\varpi, TS\varpi, \bar{\xi})$

$$\geq \gamma(\pounds_{\mathfrak{U},\mathfrak{V}}(\varpi, S\varpi, k\bar{\xi}))$$

$\geq \gamma(£_{\mathfrak{U},\mathfrak{V}}(\varpi, \varpi, k\bar{\xi}))$

$\geq £_{\mathfrak{U},\mathfrak{V}}(\varpi, \varpi, \bar{\xi})$

$= 1_{I^*}$,

then $T\varpi = \varpi$.

To prove uniqueness, let us consider κ be another CFP of S and T. Then,

$£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = £_{\mathfrak{U},\mathfrak{V}}(S\varpi, TS\kappa, \bar{\xi})$

$\geq \gamma(£_{\mathfrak{U},\mathfrak{V}}(\varpi, S\kappa, k\bar{\xi}))$

$\geq £_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, k\bar{\xi}) \geq ...$

$\geq £_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, k^n\bar{\xi})$.

Now by using Lemma 3.1, we have

$$£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) \leq £_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, k^n\bar{\xi}).$$

Hence, $£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = C$ for all $\bar{\xi} > 0$, where $C = 1_{I^*}$, i.e., $\kappa = \varpi$.

Therefore, ϖ is the unique CFP. □

Example 3.6. *Let $Z = [0, 1]$. Define the mapping S and T by*

$$S\varpi = 1,$$

and

$$T\varpi = \begin{cases} 0 \text{ if } \varpi \text{ is irrational;} \\ 1 \text{ if } \varpi \text{ is rational} \end{cases} \quad \text{for every } \varpi \in [0, 1].$$

Define $\gamma : I^* \to I^*$ by

$$\gamma(\bar{\xi}_1, \bar{\xi}_2) = \begin{cases} \left(\bar{\xi}_1 + \frac{1}{2}, \bar{\xi}_2 - \frac{1}{2}\right) \text{ if } 0 \leq \bar{\xi}_1 \leq \frac{1}{2}, \frac{1}{2} \leq \bar{\xi}_2 \leq 1; \\ (1, 0) = 1 \text{ if } \bar{\xi}_1, \bar{\xi}_2 \text{ otherwise.} \end{cases}$$

Denote $R(\mathfrak{k}, v) = (\mathfrak{k}_1 v_1, \min(\mathfrak{k}_2 + v_2, 1))$ for all $\mathfrak{k} = (\mathfrak{k}_1, \mathfrak{k}_2)$ and $v = (v_1, v_2) \in I^*$. For each $\bar{\xi} \in \mathcal{P}_0$, define

$$£_{\mathfrak{U},\mathfrak{V}}(\varpi, \kappa, \bar{\xi}) = \left\{ \frac{\bar{\xi}}{\bar{\xi} + |\varpi - \kappa|}, \frac{|\varpi - \kappa|}{\bar{\xi} + |\varpi - \kappa|} \right\}.$$

Therefore, $£_{\mathfrak{U},\mathfrak{V}}(S\varpi, TS\kappa, \bar{\xi}) \geq (£_{\mathfrak{U},\mathfrak{V}}(\varpi, S\kappa, k\bar{\xi}))$.

Thus all the conditions are satisfied so 1 are the only CFP of S and T.

Theorem 3.2. *Let $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$ be a complete CVMIFMS. Suppose ϕ is the set of all continuous functions $\gamma : I^* \to I^*$ with $\gamma(\mathfrak{k}) >_{I^*} (\mathfrak{k})$ for every $\mathfrak{k} \in I \setminus \{0_{I^*}, 1_{I^*}\}$. If A, B, S, and T are four mappings from Z to Z, that they satisfy the following conditions:*

(1) $\pounds_{\mathfrak{U},\mathfrak{B}}(A\varpi, B\kappa, \bar{\xi}) \geq \phi\left\{\min(\pounds_{\mathfrak{U},\mathfrak{B}}(S\varpi, T\kappa, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(S\varpi, B\kappa, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(T\kappa, B\kappa, \bar{\xi}))\right\}$ *for every* ϖ, κ *in* Z.

(2) $A(Z) \subseteq T(Z)$, $B(Z) \subseteq S(Z)$, *also* $T(Z)$ *or* $S(Z)$ *is a closed subset of* Z.

(3) *The pairs* (A, S) *and* (B, T) *are weak compatible and* (A, S) *or* (B, T) *satisfy the property* (E).

Then A, S, B, *and* T *have a unique CFP in* Z.

Proof. Assuming the pair (B, T) exhibits property (E). Hence, there exists a sequence $\{\varpi_n\}$ such that:

$$\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(B\varpi_n, z, \bar{\xi}) = \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(T\varpi_n, z, \bar{\xi}) = 1_{I^*}$$

for some $z \in Z$ and every $\bar{\xi} > 0$. As $B(Z) \subseteq S(Z)$, there exists a sequence $\{\kappa_n\}$ such that $B\varpi_n = S\kappa_n$, Hence

$$\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(S\kappa_n, z, \bar{\xi}) = 1_{I^*}.$$

We prove that $\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(A\kappa_n, z, \bar{\xi}) = 1_{I^*}$. Since

$$\pounds_{\mathfrak{U},\mathfrak{B}}(A\kappa_n, B\varpi_n, \bar{\xi}) \geq \gamma\left(\min(\pounds_{\mathfrak{U},\mathfrak{B}}(S\kappa_n, T\varpi_n, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(S\kappa_n, B\varpi_n, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(T\varpi_n, B\varpi_n, \bar{\xi}))\right),$$

on taking $n \to \infty$ in the above inequality, we get

$$\begin{aligned}\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(A\kappa_n, B\varpi_n, \bar{\xi}) &\geq \gamma\left(\min(\pounds_{\mathfrak{U},\mathfrak{B}}(z, z, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(z, z, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(z, z, \bar{\xi}))\right) \\ &= 1_{I^*}.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(A\kappa_n, B\varpi_n, \bar{\xi}) = 1_{I^*}.$$

hence

$$\lim_{n\to\infty} A\kappa_n = z.$$

Let SZ be a CVMIFMS, then there exists $\varpi \in Z$ such that $S\varpi = z$, and

$$\begin{aligned}&\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(A\kappa_n, S\varpi, \bar{\xi}) = \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(B\varpi_n, S\varpi, \bar{\xi}) \\ &= \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(T\varpi_n, S\varpi, \bar{\xi}) \\ &= \lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(S\kappa_n, S\varpi, \bar{\xi}) \\ &= 1_{I^*}.\end{aligned}$$

On the other hand,

$$\pounds_{\mathfrak{U},\mathfrak{B}}(A\varpi, B\varpi_n, \bar{\xi}) \geq \gamma\left(\min(\pounds_{\mathfrak{U},\mathfrak{B}}(S\varpi, T\varpi_n, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(S\varpi, B\varpi_n, t), \pounds_{\mathfrak{U},\mathfrak{B}}(T\varpi_n, B\varpi_n, \bar{\xi}))\right)$$

therefore,

$$\lim_{n\to\infty} \pounds_{\mathfrak{U},\mathfrak{B}}(A\varpi, B\varpi_n, \bar{\xi}) \geq 1_{I^*}.$$

Hence $A\varpi = \lim_{n\to\infty} B\varpi_n = z = S\varpi$. Since A and S are weak compatible maps, that is $AS\varpi = SA\varpi$, so

$$AA\varpi = AS\varpi = SA\varpi = SS\varpi.$$

As $AZ \subseteq TZ$, there exists $v \in Z$ such that

$$A\varpi = Tv.$$

To prove $Tv = Bv$. suppose that $Tv \neq Bv$, then,

$$\pounds_{\mathfrak{U},\mathfrak{B}}(A\varpi, Bv, \bar{\xi}) \geq \gamma\left(\min(\pounds_{\mathfrak{U},\mathfrak{B}}(S\varpi, Tv, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(S\varpi, Bv, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(Tv, Bv, \bar{\xi}))\right)$$

$$\succ \pounds_{\mathfrak{U},\mathfrak{B}}(A\varpi, Bv, \bar{\xi})$$

which comes contradiction. Hence,

$$\pounds_{\mathfrak{U},\mathfrak{B}}(A\varpi, Bv, \bar{\xi}) = 1_{I^*}.$$

So, $Tv = A\varpi = S\varpi = Bv$.

Since B and T are weak compatible maps, we have,

$$BTv = TBv,$$

So,

$$TTv = TBv = BTv = BBv.$$

To prove that $A\varpi$ is a FP for A and S. i.e,

$$AA\varpi = SA\varpi = A\varpi.$$

For this let us suppose that

$$AA\varpi \neq A\varpi,$$

we have

$$\pounds_{\mathfrak{U},\mathfrak{B}}(AA\varpi, A\varpi, \bar{\xi}) = \pounds_{\mathfrak{U},\mathfrak{B}}(AA\varpi, B\varpi, \bar{\xi})$$

$$\geq \gamma\left(\min(\pounds_{\mathfrak{U},\mathfrak{B}}(SA\varpi, Tv, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(SA\varpi, Bv, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{B}}(Tv, Bv, \bar{\xi}))\right)$$

$$\succ \pounds_{\mathfrak{U},\mathfrak{B}}(AA\varpi, A\varpi, \bar{\xi})$$

which arises a contradiction. Hence,

$$A\varpi = AA\varpi = SA\varpi.$$

In the same way, Bv is a FP for T and B. Therefore $Bv = A\varpi$, so all the mappings A, B, S, and T have a CFP $A\varpi$.

For uniqueness, let, if possible, $\varpi \neq v$ be other CFP. Then

$$\pounds_{\mathfrak{U},\mathfrak{V}}(\varpi, v, \bar{\xi}) = \pounds_{\mathfrak{U},\mathfrak{V}}(A\varpi, Bv, \bar{\xi})$$

$$\geq \gamma\left(\min(\pounds_{\mathfrak{U},\mathfrak{V}}(S\varpi, Tv, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(S\varpi, Bv, \bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(Tv, Bv, \bar{\xi}))\right)$$

$$> \pounds_{\mathfrak{U},\mathfrak{V}}(\varpi, v, \bar{\xi}).$$

This comes contradiction. □

Theorem 3.3. *Let $(Z, \pounds_{\mathfrak{U},\mathfrak{V}}, R)$ be an CVMIFMS and assume $S, I, J, T : Z \to Z$ be four mappings, such that*

$$TZ \subseteq JZ, \quad SZ \subseteq IZ \tag{3.1}$$

and

$$\pounds_{\mathfrak{U},\mathfrak{V}}(T\varpi, S\kappa, \bar{\xi}) \geq a(\bar{\xi})\pounds_{\mathfrak{U},\mathfrak{V}}(I\varpi, J\kappa, \bar{\xi}) + b(\bar{\xi})\min\{\pounds_{\mathfrak{U},\mathfrak{V}}(I\varpi, T\varpi, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(J\kappa, S\kappa, k\bar{\xi})\}$$
$$+ c(\bar{\xi})\max\{\pounds_{\mathfrak{U},\mathfrak{V}}(I\varpi, T\varpi, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(J\kappa, T\kappa, k\bar{\xi})\}$$

for every $\varpi, \kappa \in Z$, and some $k > 1$, where $a, b, c : \mathcal{P} \to I$ are three continuous functions such that $a(\bar{\xi}) + b(\bar{\xi}) + c(\bar{\xi}) = 1$ for all $\bar{\xi} \succ 0$.

Suppose in addition that either

(1) *I is continuous, T, I are compatible and S, J are weak compatible, or*
(2) *J is continuous, S, J are compatible and T, I are weak compatible.*

Then S, T, J and I have a unique CFP.

Proof. Let $\varpi_0 \in Z$ be any point. Take a point $\varpi_1 \in Z$ such that $T\varpi_0 = J\varpi_1 = \kappa_1$, and a point $\varpi_2 \in Z$ such that $S\varpi_1 = T\varpi_2 = \kappa_2$.
Now, by using induction a sequence $\{\varpi_n\}$ in Z is defined as

$$I\varpi_{2n+2} = S\varpi_{2n+1} = \kappa_{2n+2}, \quad n = 0, 1, 2,$$

$$J\varpi_{2n+1} = T\varpi_{2n} = \kappa_{2n+1}, \quad n = 0, 1, 2$$

We set

$$d_n(\bar{\xi}) = \pounds_{\mathfrak{U},\mathfrak{V}}(\kappa_n, \kappa_{n+1}, \bar{\xi}), \quad n = 0, 1, 2, \dots$$

For each $n = 0, 1, 2, \dots$ we have

$$d_{2n+1}(\bar{\xi}) = \pounds_{\mathfrak{U},\mathfrak{V}}(\kappa_{2n+1}, \kappa_{2n+2}, \bar{\xi})$$

$$= \pounds_{\mathfrak{U},\mathfrak{V}}(T\varpi_{2n}, S\varpi_{2n+1}, \bar{\xi})$$

$$\geq a(\bar{\xi})\pounds_{\mathfrak{U},\mathfrak{V}}(I\varpi_{2n}, J\varpi_{2n+1}, \bar{\xi}) + b(\bar{\xi})\min\{\pounds_{\mathfrak{U},\mathfrak{V}}(I\varpi_{2n}, T\varpi_{2n}, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(J\varpi_{2n+1}, S\varpi_{2n+1}, k\bar{\xi})\}$$

$+c(\bar{\xi})\max\{£_{\mathfrak{U},\mathfrak{B}}(I\varpi_{2n},S\varpi_{2n+1},k\bar{\xi}),£_{\mathfrak{U},\mathfrak{B}}(J\varpi_{2n+1},T\varpi_{2n},k\bar{\xi})\}$

$=a(\bar{\xi})d_{2n}(\bar{\xi})+b(\bar{\xi})\min\{d_{2n}(k\bar{\xi}),d_{2n+1}(k\bar{\xi})\}+c(\bar{\xi}).$

Now, if $d_{2n+1}(k\bar{\xi})\prec d_{2n}(k\bar{\xi})$, then

$$d_{2n+1}(\bar{\xi})\succ[a(\bar{\xi})+b(\bar{\xi})+c(\bar{\xi})]\,d_{2n+1}(k\bar{\xi})$$

Hence $d_{2n+1}(\bar{\xi})\succ d_{2n+1}(k\bar{\xi})$, that is a contradiction. Therefore $d_{2n+1}(\bar{\xi})\succeq d_{2n}(k\bar{\xi})$.

Thus,

$$£_{\mathfrak{U},\mathfrak{B}}(\kappa_{2n+1},\kappa_{2n+2},\bar{\xi})\succeq £_{\mathfrak{U},\mathfrak{B}}(\kappa_{2n},\kappa_{2n+1},\bar{\xi}k).$$

So,

$£_{\mathfrak{U},\mathfrak{B}}(\kappa_{2n+1},\kappa_{2n+2},\bar{\xi})\succeq £_{\mathfrak{U},\mathfrak{B}}(\kappa_{n},\kappa_{n+1},\bar{\xi})$

$\succeq £_{\mathfrak{U},\mathfrak{B}}(\kappa_{n-1},\kappa_{n},k\bar{\xi})\succeq ...$

$\succeq £_{\mathfrak{U},\mathfrak{B}}(\kappa_{0},\kappa_{1},k^{n}\bar{\xi}).$

By Lemma 3.1, sequence $\{\kappa_n\}$ is a Cauchy sequence, therefore it converges to a point $\mathfrak{k}\in Z$.

$$i.e, \lim_{n\to\infty}\kappa_n=\mathfrak{k}=\lim_{n\to\infty}J\varpi_{2n+1}=\lim_{n\to\infty}S\varpi_{2n+1}=\lim_{n\to\infty}I\varpi_{2n+2}=\lim_{n\to\infty}T\varpi_{2n}.$$

Now (1) is satisfied. Then $I^2\varpi_{2n}\to I\mathfrak{k}$ and $IT\varpi_{2n}\to I\mathfrak{k}$. Since T and I are compatible map, i.e, $TI\varpi_{2n}\to$I.

Now, to prove that $\mathfrak{k}$ is the CFP of T,S,J,I.

Case 1. $\mathfrak{k}$ is a FP of I. Indeed, if $I\mathfrak{k}\neq\mathfrak{k}$ we have

$£_{\mathfrak{U},\mathfrak{B}}(TI\varpi_{2n},S\varpi_{2n+1},\bar{\xi})\succeq a(\bar{\xi})£_{\mathfrak{U},\mathfrak{B}}(I^2\varpi_{2n},J\varpi_{2n+1},\bar{\xi})$

$+b(\bar{\xi})\min\{£_{\mathfrak{U},\mathfrak{B}}(I^2\varpi_{2n},TI\varpi_{2n},k\bar{\xi}),£_{\mathfrak{U},\mathfrak{B}}(J\varpi_{2n+1},S\varpi_{2n+1},k\bar{\xi})\}$

$+c(\bar{\xi})\max\{£_{\mathfrak{U},\mathfrak{B}}(I^2\varpi_{2n},S\varpi_{2n+1},k\bar{\xi}),£_{\mathfrak{U},\mathfrak{B}}(J\varpi_{2n+1},TI\varpi_{2n},k\bar{\xi})\}.$

Letting $n\to\infty$, yields The expression

$£_{\mathfrak{U},\mathfrak{B}}(I\mathfrak{k},\mathfrak{k},\bar{\xi})\succeq a(\bar{\xi})£_{\mathfrak{U},\mathfrak{B}}(I\mathfrak{k},\mathfrak{k},\bar{\xi})+b(\bar{\xi})\min\{£_{\mathfrak{U},\mathfrak{B}}(I\mathfrak{k},I\mathfrak{k},k\bar{\xi}),£_{\mathfrak{U},\mathfrak{B}}(\mathfrak{k},\mathfrak{k},k\bar{\xi})\}$

$+c(\bar{\xi})\max\{£_{\mathfrak{U},\mathfrak{B}}(I\mathfrak{k},\mathfrak{k},k\bar{\xi}),£_{\mathfrak{U},\mathfrak{B}}(I\mathfrak{k},\mathfrak{k},k\bar{\xi})\}$

$$> \pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, \mathfrak{k}, k\bar{\xi})$$

leads to a contradiction, which implies that $I\mathfrak{k} = \mathfrak{k}$.

Case 2. $\mathfrak{k}$ is a FP of T. Indeed,

$$\pounds_{\mathfrak{U},\mathfrak{V}}(T\mathfrak{k}, S\varpi_{2n+1}, \bar{\xi}) \geq a(\bar{\xi})\pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, J\varpi_{2n+1}, \bar{\xi})$$

$$+b(\bar{\xi}) \min\{\pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, T\mathfrak{k}, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(J\varpi_{2n+1}, S\varpi_{2n+1}, k\bar{\xi})\}$$

$$+c(\bar{\xi}) \max\{\pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, S\varpi_{2n+1}, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(J\varpi_{2n+1}, T\mathfrak{k}, k\bar{\xi})\}$$

and letting $n \to \infty$, if $T\mathfrak{k} \neq \mathfrak{k}$ gives

$$\pounds_{\mathfrak{U},\mathfrak{V}}(T\mathfrak{k}, \mathfrak{k}, \bar{\xi}) \geq a(\bar{\xi})\pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, \mathfrak{k}, \bar{\xi}) + b(\bar{\xi}) \min\{\pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, T\mathfrak{k}, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(\mathfrak{k}, \mathfrak{k}, k\bar{\xi})\}$$

$$+c(\bar{\xi}) \max\{\pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, \mathfrak{k}, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, T\mathfrak{k}, k\bar{\xi})\}$$

$$> \pounds_{\mathfrak{U},\mathfrak{V}}(\mathfrak{k}, T\mathfrak{k}, k\bar{\xi})$$

Hence, $T\mathfrak{k} = \mathfrak{k}$.

Case 3. Since $TZ \subseteq JZ$ for all $\varpi \in Z$, there is a point $\mathfrak{b} \in Z$ such that

$$T\mathfrak{k} = \mathfrak{k} = J\mathfrak{b}.$$

We will prove that $\mathfrak{b}$ is a coincidence point for J and S. Indeed, if $T\mathfrak{k} \neq S\mathfrak{b}$ we have

$$\pounds_{\mathfrak{U},\mathfrak{V}}(T\mathfrak{k}, S\mathfrak{b}, \bar{\xi}) \geq a(\bar{\xi})\pounds_{\mathfrak{U},\mathfrak{V}}(\mathfrak{k}, J\mathfrak{b}, \bar{\xi}) + b(\bar{\xi}) \min\{\pounds_{\mathfrak{U},\mathfrak{V}}(\mathfrak{k}, T\mathfrak{k}, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(J\mathfrak{k}, S\mathfrak{k}, k\bar{\xi})\}$$

$$+c(\bar{\xi}) \max\{\pounds_{\mathfrak{U},\mathfrak{V}}(\mathfrak{k}, S\mathfrak{b}, k\bar{\xi}), \pounds_{\mathfrak{U},\mathfrak{V}}(J\mathfrak{b}, T\mathfrak{k}, k\bar{\xi})\}$$

$$> \pounds_{\mathfrak{U},\mathfrak{V}}(T\mathfrak{k}, S\mathfrak{b}, k\bar{\xi}),$$

which is a contradiction. Thus, $T\mathfrak{k} = S\mathfrak{b} = J\mathfrak{b} = \mathfrak{k}$.
Given that J and S are weakly compatible, we deduce that

$$SJ\mathfrak{b} = JS\mathfrak{b} \to S\mathfrak{k} = J\mathfrak{k}.$$

To, show that $T\mathfrak{k} = S\mathfrak{k}$. Suppose, $T\mathfrak{k} \neq S\mathfrak{k}$, we have

$£_{\mathfrak{U},\mathfrak{V}}(T\mathfrak{k}, S\mathfrak{k}, \bar{\xi}) \geq a(\bar{\xi})£_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, J\mathfrak{k}, \bar{\xi}) + b(\bar{\xi}) \min\{£_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, T\mathfrak{k}, k\bar{\xi}), £_{\mathfrak{U},\mathfrak{V}}(J\mathfrak{k}, S\mathfrak{k}, k\bar{\xi})\}$

$+c(\bar{\xi}) \max\{£_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, S\mathfrak{k}, k\bar{\xi}), £_{\mathfrak{U},\mathfrak{V}}(J\mathfrak{k}, T\mathfrak{k}, k\bar{\xi})\}$

$\succ £_{\mathfrak{U},\mathfrak{V}}(T\mathfrak{k}, S\mathfrak{k}, k\bar{\xi})$

which is a contradiction.
Therefore,

$$S\mathfrak{k} = T\mathfrak{k} = I\mathfrak{k} = J\mathfrak{k} = \mathfrak{k}.$$

For Uniqueness, Let $\mathfrak{b} \neq \mathfrak{k}$ be another FP of I, J, T and S, then

$£_{\mathfrak{U},\mathfrak{V}}(T\mathfrak{k}, S\mathfrak{b}, \bar{\xi}) \geq a(\bar{\xi})£_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, J\mathfrak{b}, \bar{\xi}) + b(\bar{\xi}) \min\{£_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, T\mathfrak{k}, k\bar{\xi}), £_{\mathfrak{U},\mathfrak{V}}(J\mathfrak{b}, S\mathfrak{b}, k\bar{\xi})\}$

$+c(\bar{\xi}) \max\{£_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{k}, S\mathfrak{b}, k\bar{\xi}), £_{\mathfrak{U},\mathfrak{V}}(I\mathfrak{b}, T\mathfrak{k}, k\bar{\xi})\}$

$= a(\bar{\xi})£_{\mathfrak{U},\mathfrak{V}}(\mathfrak{a}, \mathfrak{b}, \bar{\xi}) + b(\bar{\xi}) \cdot 1_{I^*} + c(\bar{\xi})\mathfrak{U}(\mathfrak{k}, \mathfrak{b}, k\bar{\xi})$

$\succ £_{\mathfrak{U},\mathfrak{V}}(\mathfrak{k}, \mathfrak{b}, k\bar{\xi})$

is a contradiction. That is, $\mathfrak{k}$ is the unique CFP. □

4. Application to Fredholm Integral Equations of Second Kind

We explore in this section how Theorem 3.1 can be employed to demonstrate the existence of a unique solution for Fredholm integral equations. The set $C([0,1], \mathbb{R})$ denotes set of all continuous functions that map the interval $[0,1]$ to the real numbers. Below is an example of a second-kind non linear Fredholm integral equation:

$$\psi(\bar{\xi}) = Q(\bar{\xi}) + \gamma \int_0^1 \omega(\bar{\xi}, s)\chi(s, \psi(s))ds \tag{4.1}$$

where Q represents a real-valued function that is continuous on the interval $[0,1]$, $\omega(\bar{\xi}, s)$ represents the kernel of the integral function, $\chi(s, \psi(s))$ represents a non linear and continuous function defined on $[0,1] \times \mathbb{R}$, and $\psi(\bar{\xi})$ represents the function that we wish to be determined.

Theorem 4.1. *Consider $Z = C([0,1], \mathbb{R})$. Suppose that the conditions outlined below are met:*

(1) *An element $\alpha \in (0,1)$ can be located such that*

$$\left|\chi(s, \psi(s)) - \chi(s, \phi(s))\right| \leq \alpha \left|\psi(s) - \phi(s)\right|$$

for any $\psi, \phi \in Z$ and $s \in [0,1]$;

(2) $\int_0^1 \omega(\bar{\xi}, s)ds \leq \beta$;

(3) $\gamma^2\beta^2\alpha^2 \leq k < 1$.

Then, the integral equation (2) have a unique solution in Z.

Proof. Given a mapping $F : Z \to Z$ defined by

$$F\psi(\bar{\xi}) = Q(\bar{\xi}) + \gamma \int_0^1 \omega(\bar{\xi}, s)\chi(s, \psi(s))ds$$

for every $\psi(\bar{\xi}) \in Z$ and $\bar{\xi} \in [0, 1]$. Furthermore, $£_{\mathfrak{U},\mathfrak{V}}(\psi(\bar{\xi}), \phi(\bar{\xi}), c)$ are defined by

$$£_{\mathfrak{U},\mathfrak{V}}(\psi(\bar{\xi}), \phi(\bar{\xi}), c) = \left(\frac{\mathfrak{k} + \mathfrak{d}}{\mathfrak{k} + \mathfrak{d} + \left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2} l, \frac{\left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2}{\mathfrak{k} + \mathfrak{d} + \left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2} l \right)$$

for all $\psi, \phi \in Z, c = (\mathfrak{k}, \mathfrak{d}) > 0$, and $\bar{\xi} \in [0, 1]$. The fact that $(Z, £_{\mathfrak{U},\mathfrak{V}}, R)$ is a complete CVMIFMS can be established without much effort.

For all $\psi, \phi \in Z$ and $\bar{\xi} \in [0, 1]$, it follows that

$$\left|F\psi(\bar{\xi}) - F\phi(\bar{\xi})\right|^2 = \left|Q(\bar{\xi}) + \gamma \int_0^1 \omega(\bar{\xi}, s)\chi(s, \psi(s))ds - Q(\bar{\xi}) - \gamma \int_0^1 \omega(\bar{\xi}, s)\chi(s, \phi(s))ds\right|^2$$

$$= \gamma^2 \left| \int_0^1 \omega(\bar{\xi}, s)\chi(s, \psi(s))ds - \int_0^1 \omega(\bar{\xi}, s)\chi(s, \phi(s))ds \right|^2$$

$$\leq \gamma^2 \left(\int_0^1 \omega(\bar{\xi}, s)ds \right)^2 \left|\chi(s, \psi(s)) - \chi(s, \phi(s))\right|^2$$

$$\leq \gamma^2\beta^2\alpha^2 \left|\psi(s) - \phi(s)\right|^2 \leq k \left|\psi(s) - \phi(s)\right|^2$$

Now, for all $\psi, \phi \in Z$ and $c \in \mathcal{P}_0$, it can be seen that

$$£_{\mathfrak{U},\mathfrak{V}}(F\psi(\bar{\xi}), F\phi(\bar{\xi}), kc) = \left(\frac{k(\mathfrak{k} + \mathfrak{d})}{k(\mathfrak{k} + \mathfrak{d}) + \left|F\psi(\bar{\xi}) - F\phi(\bar{\xi})\right|^2} l, \frac{\left|F\psi(\bar{\xi}) - F\phi(\bar{\xi})\right|^2}{k(\mathfrak{k} + \mathfrak{d}) + \left|F\psi(\bar{\xi}) - F\phi(\bar{\xi})\right|^2} l \right)$$

$$= \left(\frac{k(\mathfrak{k} + \mathfrak{d})}{k(\mathfrak{k} + \mathfrak{d}) + \left|F\psi(\bar{\xi}) - F\phi(\bar{\xi})\right|^2} l, 1 - \frac{k(\mathfrak{k} + \mathfrak{d})}{k(\mathfrak{k} + \mathfrak{d}) + \left|F\psi(\bar{\xi}) - F\phi(\bar{\xi})\right|^2} l \right)$$

$$\geq \left(\frac{k(\mathfrak{k} + \mathfrak{d})}{k(\mathfrak{k} + \mathfrak{d}) + k\left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2} l, 1 - \frac{k(\mathfrak{k} + \mathfrak{d})}{k(\mathfrak{k} + \mathfrak{d}) + k\left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2} l \right)$$

$$= \left(\frac{\mathfrak{k} + \mathfrak{d}}{\mathfrak{k} + \mathfrak{d} + \left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2} l, \frac{\left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2}{\mathfrak{k} + \mathfrak{d} + \left|\psi(\bar{\xi}) - \phi(\bar{\xi})\right|^2} l \right)$$

$$= £_{\mathfrak{U},\mathfrak{V}}(\psi(\bar{\xi}), \phi(\bar{\xi}), c).$$

As a consequence, all the requirements of Theorem 3.1 are met, which indicates F possesses a unique FP in Z. □

Abbrevation

FP - fixed point.
MS - metric space.
FMS - fuzzy metric space.
CFP - common fixed point.
BCP - Banach contraction principle.
MIFMS - modified intuitionistic fuzzy metric space.
CVMS - complex valued metric space.
CVFMS - complex valued fuzzy metric space.
CVFbMS - complex valued fuzzy b-metric space.
CVIFMS - complex valued intuitionistic fuzzy metric space.
CVMIFMS - complex valued modified intuitionistic fuzzy metric space.

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